Fermionic partial transpose and non-local order parameters for symmetry protected topological (SPT) phases of fermions

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Main Ref:

KS-Shapourian-Gomi-Ryu, arXiv:1710.01886

Related refs:

Shapourian-KS-Ryu, arXiv:1607.03896

KS-Ryu, arXiv:1607.06504

KS-Shapourian-Ryu, arXiv:1609.05970

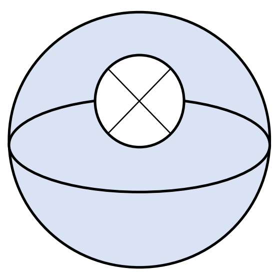
Shapourian-KS-Ryu, 1611.07536

Take-home message

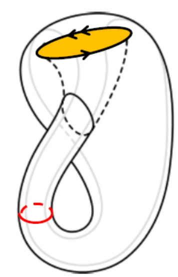
- Transpose $\sim$ time-reversal (TR) transformation for the imaginary time path-integral.

$$(\mathcal{O}_n \ldots \mathcal{O}_2 \mathcal{O}_1)^{tr} = \mathcal{O}_1^{tr} \mathcal{O}_2^{tr} \cdots \mathcal{O}_n^{tr}$$

- The partial transpose enable us to simulate the partition function over non-orientable manifolds in the operator formalism.



Real projective plane



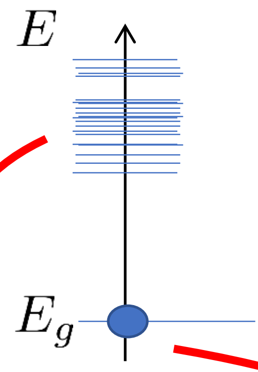
Klein bottle

- Developed the partial transpose for fermions in the operator formalism

$$A^{tr_1} = \sum_{k_1, k_2, k_1+k_2 \in \text{even}} A_{p_1 \cdots p_{k_1}, q_1 \cdots q_{k_2}} i^{k_1} \underbrace{a_{p_1} \cdots a_{p_{k_1}}}_{I_1} \underbrace{b_{q_1} \cdots b_{q_{k_2}}}_{I_2}$$

Motivation

- How to detect symmetry protected topological (SPT) phases ($\sim$ gapped phases without ground state degeneracy)



- The classification of symmetry-protected topological phases
$\sim$ the classification of U(1)-valued topological partition functions
[Kapustin, Freed-Hopkins, $\cdots$]

$$Z[M] = \big|Z[M]\big| \times e^{iS_{\rm top}[M]}$$

Euclidian spacetime manifold

Excited states

Characterizes SPT phases

Motivation

- In SPT phases with time-reversal (TR) symmetry, the spacetime manifold M to detect nontrivial SPT phases is sometimes non-orientable.
  - ✓ Ex: Haldane chain phase, topological insulator/superconductor, $\cdots$
- The partition function over a suitable non-orientable manifold is the "order parameter" of SPT phases.
- Ex: (1+1)d bosonic SPT phases with TR sym.
  - ✓ "Order parameter" = real projective plane RP^2

$$e^{iS_{\mathrm{top}}(M)} = e^{i\pi\nu \int_M w_2(TM)}, \quad (\nu = 0, 1)$$

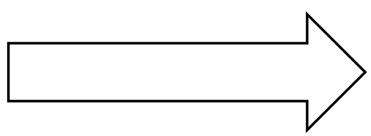
$$\Rightarrow e^{iS_{\mathrm{top}}(RP^2)} = (-1)^{\nu}.$$

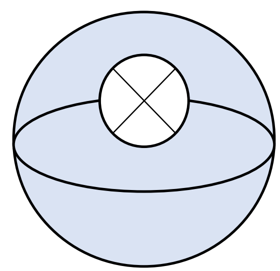
  - ✓ If the partition function over RP^2 is negative, the theory is in nontrivial SPT phases.

Motivation

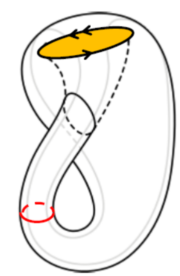
- Non-orientable manifold???
- In cond-mat, we have only
  - ✓ Hamiltonian H or the ground state $|\psi\rangle$, and
  - ✓ TR operator T.
- How to make non-orientable manifold from a set of a ground state wave function and a TR operator?

$$|GS\rangle\ ,\ T$$





Real projective plane



Klein bottle

- An answer is to use the partial transpose.

TRS ↔ Transpose

TRS $\leftrightarrow$ Transpose

- Given a TR operator, how to get the transpose?

- Let's consider the expectation value of the TR operator.

$$\langle\psi|T|\psi\rangle$$

- This is ill-defined, because T is anti-linear.

- However, the amplitude is well-defined.

- Some calculation:

$$
\begin{aligned}
|\langle\psi|T|\psi\rangle|^2 &= \langle\psi| U |\psi\rangle^* \langle\psi|^* U^\dagger |\psi\rangle \\
&= \mathrm{tr}[|\psi\rangle \langle\psi| U |\psi\rangle^* \langle\psi|^* U^\dagger] \\
&= \mathrm{tr}[\rho U \rho^* U^\dagger] \\
&= \mathrm{tr}[\rho U \rho^{tr} U^\dagger],
\end{aligned}
$$

Complex conjugate

Transpose in the operator algebra

- Here, $\rho = |\psi\rangle\langle\psi|$ and U is the unitary part of the TR operator, i.e. $T = UK$ with K the complex conjugate.

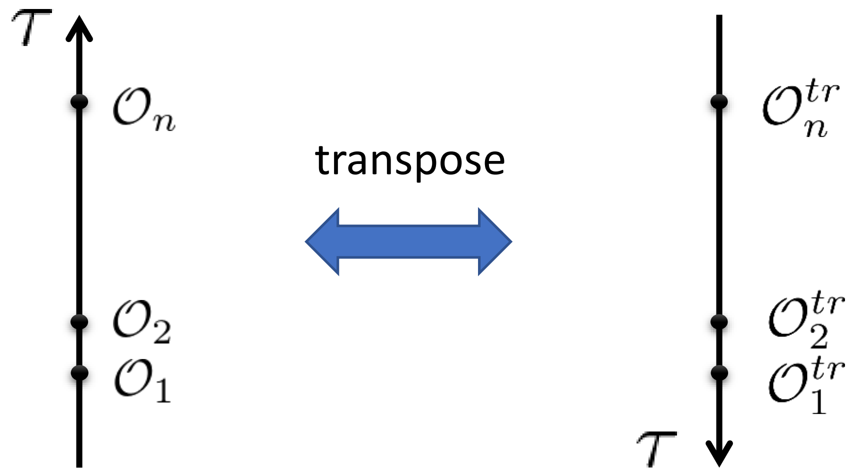
- We used the Hermiticity $\rho^\dagger = \rho$.

- In this way, a TR operator T induces a sort of the transpose in the operator algebra.

$$T = UK \quad \Rightarrow \quad U\rho^{tr}U^\dagger$$

- The transpose is understood as the time-reversal transformation in the imaginary time path-integral.

$$(\mathcal{O}_n \ldots \mathcal{O}_2 \mathcal{O}_1)^{tr} = \mathcal{O}_1^{tr} \mathcal{O}_2^{tr} \cdots \mathcal{O}_n^{tr}$$



- Therefore, it is expected that the transpose can be used to "simulate" non-orientable manifolds.

- Advantage: the transpose operation is linear, so it can be applied to a subsystem of the real space.

$\Rightarrow$ Partial transpose

Partial transpose
and
non-orientable manifolds
in
bosonic systems

Bosonic transpose

- In bosonic (spin) systems, the operator algebra is the matrix algebra.
- The transpose is the matric transpose

$$\left(|i\rangle\langle j|\right)^{tr} = |j\rangle\langle i| \,.$$

- Given operator

$$A = \sum_{i,j} A_{i,j} |i\rangle\langle j| \,,$$

the transposed operator is given by

$$A^{tr} = \sum_{i,j} A_{i,j} |{\color{red}j}\rangle\langle {\color{red}i}| \,.$$

Bosonic partial transpose

- Divide the Hilbert space into two parts.

- Given a operator:

$$A = \sum_{ij,kl} A_{ij,kl} \left| i \in I_1, j \in I_2 \right\rangle \left\langle k \in I_1, l \in I_2 \right|$$

- The partial transpose on the subsystem I_1 is defined as the matrix transpose on I_1.

$$A^{tr_1} = \sum_{ij,kl} A_{ij,kl} \left| k \in I_1, j \in I_2 \right\rangle \left\langle i \in I_1, l \in I_2 \right|$$

An application: the non-local order parameter for the Haldane chain phase

- A model Hamiltonian: (1+1)d antiferromagnetic Heisenberg model

$$H = \sum_j \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1}.$$

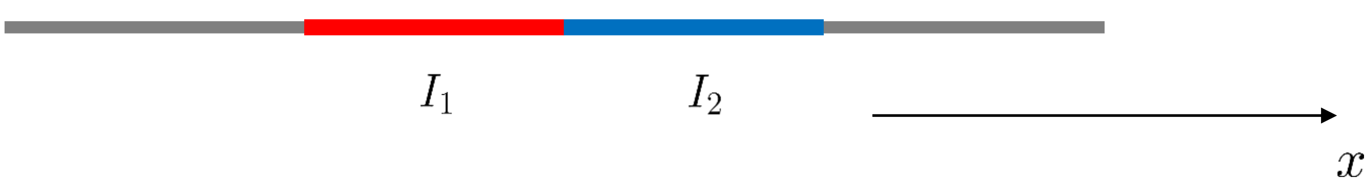
- TR symmetry $T = \bigotimes_j e^{i\pi S_j^y} K$

- The classification of SPT phases is known to be Z_2.

- For the S=1 spin system, the ground state is nontrivial SPT phase. (the Haldane chain).

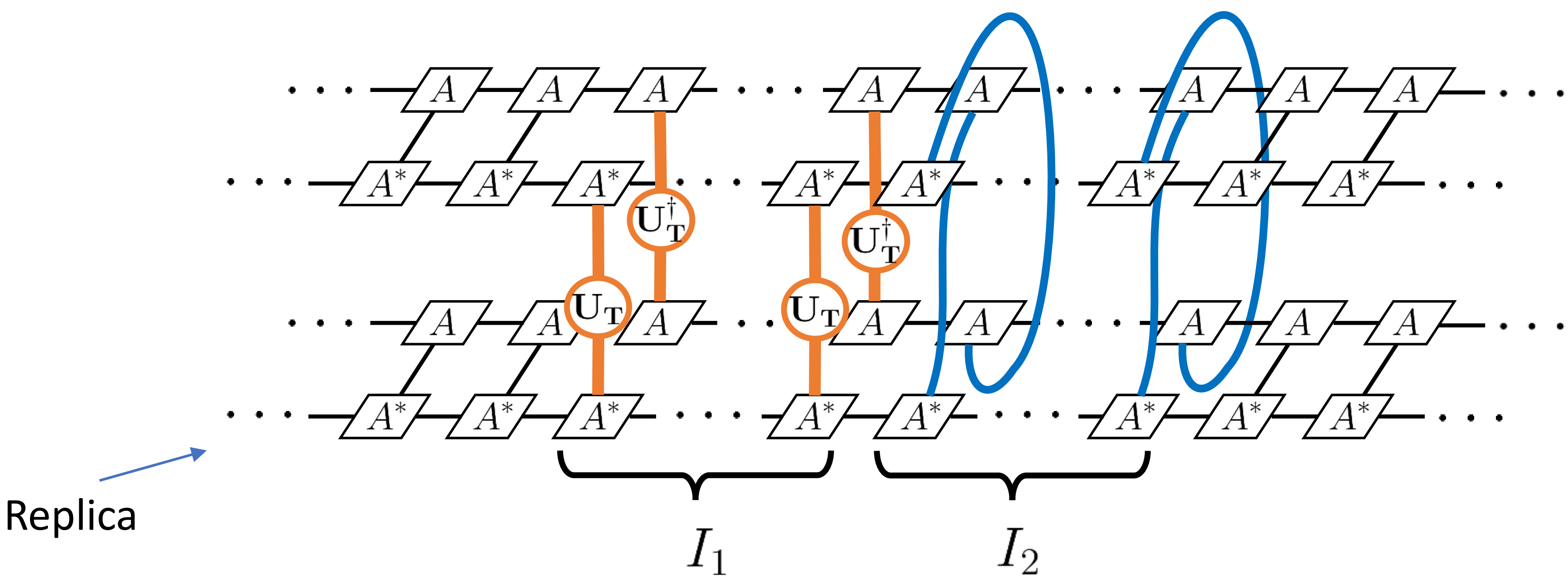
- The Z_2 "order parameter" is the partition function on RP^2 (real projective plane).

- Let's construct the Z_2 "order parameter" in the operator formalism.
- The rule of this game is:
  - ✓ Input data
    - Pure state (ground state) $|\psi\rangle$
    - TR operator $T = \bigotimes_j e^{i\pi S_j^y} K$
  - ✓ Out put = Z_2 order parameter
- Pollmann and Turner discovered that the Z_2 order parameter is the "partial transpose" on two adjacent intervals. [Pollmann-Turner '12]



- The partial transpose on two adjacent intervals. [Pollmann-Turner '12]

$$Z_{PT} := \mathrm{tr}\left[\rho_{I_1\cup I_2}\left(\prod_{j\in I_1} e^{i\pi S_j^y}\right)(\rho_{I_1\cup I_2})^{tr_1}\left(\prod_{j\in I_1} e^{-i\pi S_j^y}\right)\right]$$

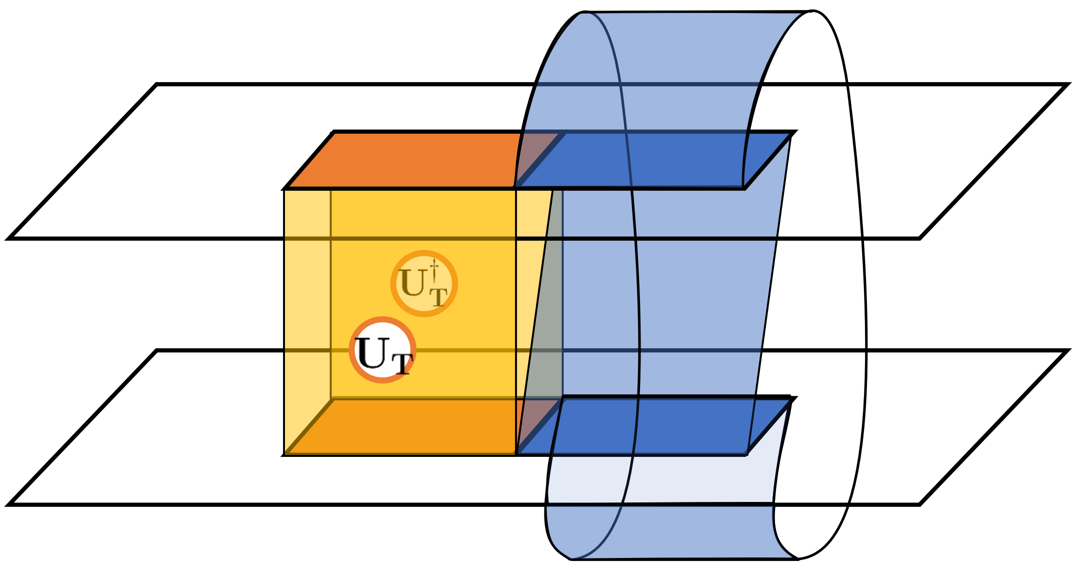


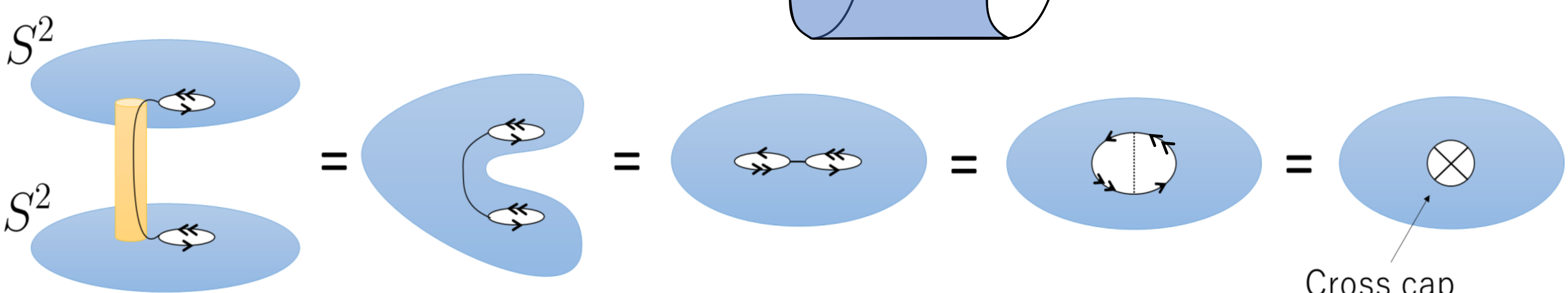
- Using the MPS, one can prove that the U(1) phase of Z_{PT} is quantized if intervals are large enough compared to the correlation length.

$$Z_{PT}/|Z_{PT}| \to \pm 1, \qquad |I_1|, |I_2| \gg \xi.$$

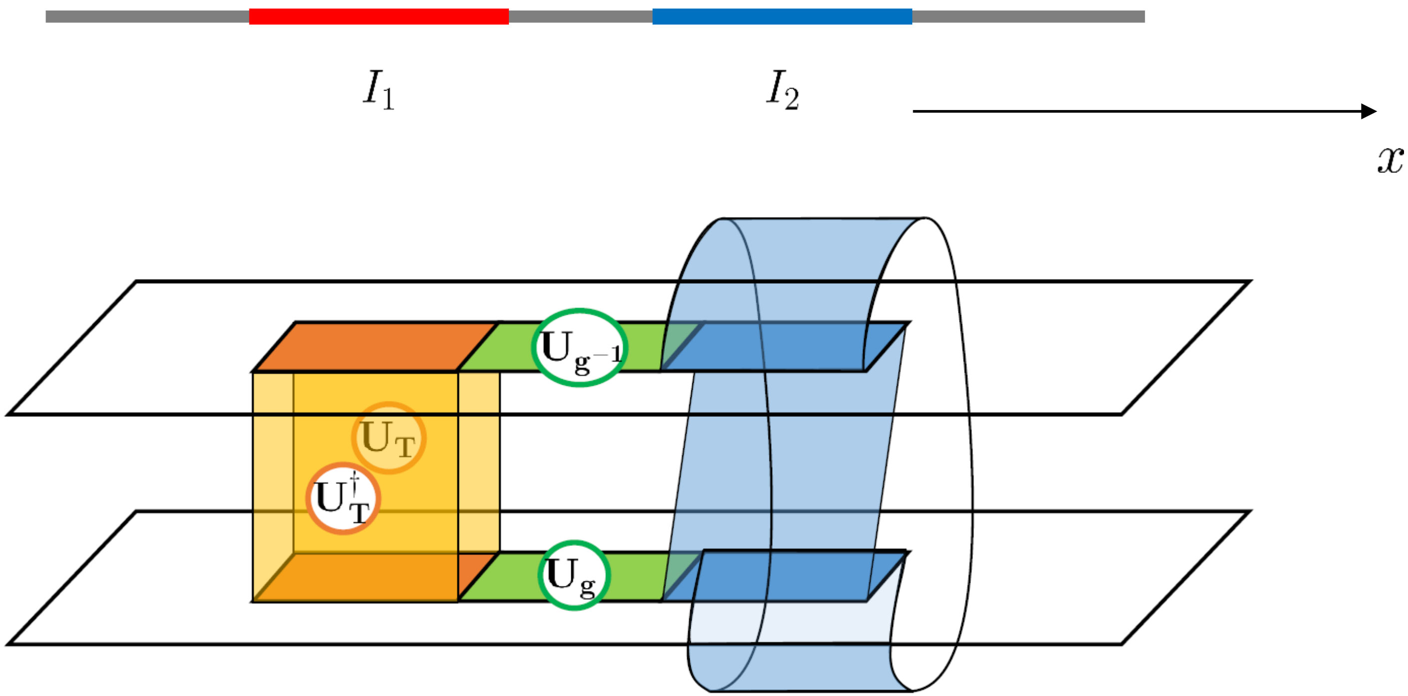
- The Pollmann-Turner invariant Z_{PT} is topologically equivalent to the partition function over RP². [KS-Ryu, '16]

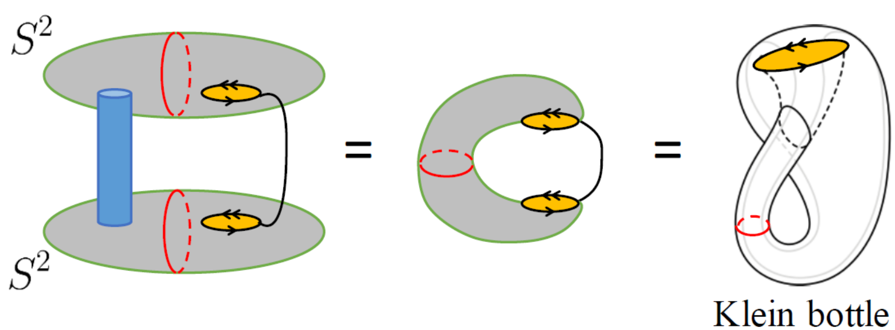
$$Z := \mathrm{tr}\left[\rho_{I_1\cup I_2}\left(\prod_{j\in I_1} e^{i\pi S_j^y}\right)\left(\rho_{I_1\cup I_2}\right)^{tr_1}\left(\prod_{j\in I_1} e^{-i\pi S_j^y}\right)\right]$$





- In the same way, the partial transpose for disjoint two intervals is topologically equivalent to the Klein bottle partition function.
[Calabrese-Cardy-Tonni '12]





Partial transpose
and
non-orientable manifolds
in
fermionic systems

Transpose in the fermionic operator algebra

- Every operator can be expanded by Majorana fermions

$$A = \sum_{k=1}^{2N} \sum_{p_1 < p_2 \cdots < p_k} A_{p_1 \cdots p_k} c_{p_1} \cdots c_{p_k},$$

$$c_j^\dagger = c_j, \quad \{c_j, c_k\} = 2\delta_{jk}.$$

- Operator algebra = the algebra generated by the Majorana fermions (Clifford algebra).

- The transpose (linear anti-automorphism) would be defined as reversing the order of Majorana fermions.

$$(c_{p_1} c_{p_2} \cdots c_{p_k})^{tr} := c_{p_k} \cdots c_{p_2} c_{p_1}$$

$$(\alpha A + \beta B)^{tr} = \alpha A^{tr} + \beta B^{tr}, \quad (AB)^{tr} = B^{tr} A^{tr}.$$

- This transpose is “canonical” in the sense that the transpose is compatible with the basis change

$$V c_j V^\dagger = [\mathcal{O}_V]_{jk} c_k, \quad \mathcal{O}_V \in O(2N).$$

$$\begin{array}{ccc} A & \xrightarrow{V} & VAV^\dagger \\ {\scriptstyle tr}\downarrow & \circlearrowleft & {\scriptstyle tr}\downarrow \\ A^{tr} & \xrightarrow{V} & VA^{tr}V^\dagger \end{array}$$

- We got the full transpose for fermions.

- A remark: we did not use a TR transformation to define the transpose of fermions. This can be compared with the transpose of bosons where there is no canonical transpose in the absence of a TR transformation.

Fermionic transpose and Grassmannian

- Does the fermionic transpose give the TR transformation in the imaginary time path-integral?

- Yes. Let us consider a simple TR transformation for complex fermions f_j as

$$T f_j^\dagger T^{-1} = f_j^\dagger, \qquad T\left|vac\right\rangle = \left|vac\right\rangle.$$

- The unitary part of the TR transformation T is found to be the particle-hole transformation

$$C_T f_j^\dagger C_T^{-1} = f_j, \qquad C_T\left|vac\right\rangle = \left|full\right\rangle.$$

- The transpose operation corresponding to the TR transformation T is

$$A \mapsto C_T A^{tr} C_T^\dagger$$

- For coherent state basis

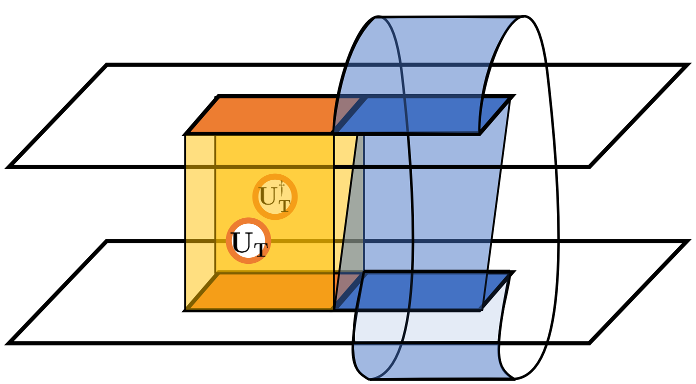
$$|\{\xi_j\}\rangle = e^{-\sum_j \xi_j f_j^\dagger} |vac\rangle ,$$

the transpose with the particle-hole transformation reads

$$C_T \Big(|\{\xi_j\}\rangle \langle\{\chi_j\}| \Big)^{tr} C_T^\dagger = |\{i\chi_j\}\rangle \langle\{i\xi_j\}| \, .$$

- This is the desired TR transformation for the path-integral.

- Therefore, the partial transpose in fermions is expected to be used to simulate the real projective plane and the Klein bottle, as in the cases of bosons.



= RP^2

Fermionic partial transpose

[KS-Shapourian-Gomi-Ryu, 1710.01886,
cf. Shapourian-KS-Ryu, 1607.03896;
Shapourian-Ryu, 1804.08637]

- What is the partial transpose for fermions?

- Introduce two subsystems of degrees of freedom.

$$A = \sum_{k_1, k_2} A_{p_1 \cdots p_{k_1}, q_1 \cdots q_{k_2}} \underbrace{a_{p_1} \cdots a_{p_{k_1}}}_{I_1} \underbrace{b_{q_1} \cdots b_{q_{k_2}}}_{I_2}$$

- We want to define the partial transpose "A^{tr_1}“ on the subsystem I_1.

- It is natural to impose the following three good properties on the partial transpose :

1. Preserve the identity:

$$(\mathrm{Id})^{tr_1} = \mathrm{Id}$$

2. The successive partial transposes on I_1 and I_2 goes back to the full transpose:

$$(A^{tr_1})^{tr_2} = A^{tr}$$

3. The partial transpose is compatible with the basis changes preserving subsysems $I_1 \cup I_2$.

$$(VAV^\dagger)^{tr_1} = VA^{tr_1}V^\dagger,$$

$$Va_jV^\dagger = [\mathcal{O}_{I_1}]_{jk}a_k, \quad Vb_jV^\dagger = [\mathcal{O}_{I_2}]_{jk}b_k.$$

- For generic operators there is no solution that meets the above three conditions.

- For operators preserving the total fermion parity (like the density matrix)

$$A = \sum_{k_1, k_2, k_1+k_2 \in \text{even}} A_{p_1 \cdots p_{k_1}, q_1 \cdots q_{k_2}} \underbrace{a_{p_1} \cdots a_{p_{k_1}}}_{I_1} \underbrace{b_{q_1} \cdots b_{q_{k_2}}}_{I_2},$$

  there is the unique solution.

- The partial transpose for the subsystem I_1 is the scalar multiplication by i^{k_1} which depends on the number of the Majorana fermions in the subspace I_1.

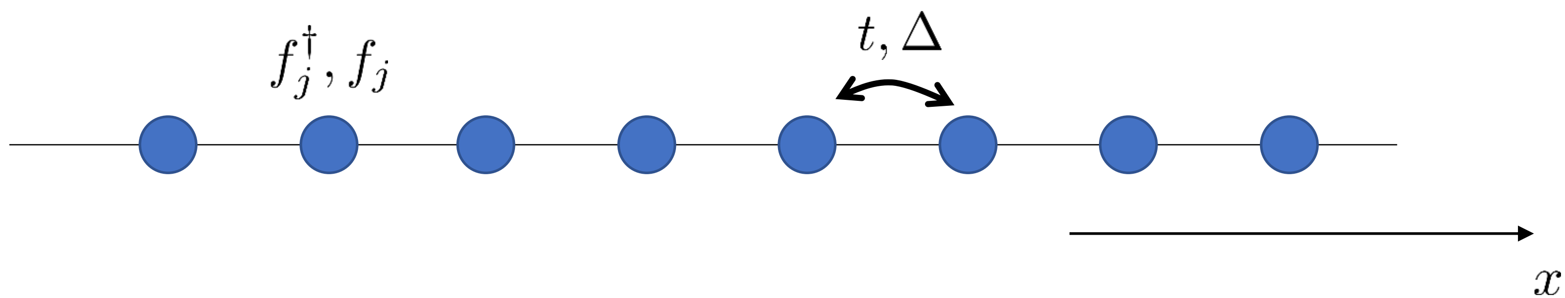
$$A^{tr_1} = \sum_{k_1, k_2, k_1+k_2 \in \text{even}} A_{p_1 \cdots p_{k_1}, q_1 \cdots q_{k_2}} i^{k_1} \underbrace{a_{p_1} \cdots a_{p_{k_1}}}_{I_1} \underbrace{b_{q_1} \cdots b_{q_{k_2}}}_{I_2}$$

Shapourian-KS-Ryu, 1607.03896.; KS-Shapourian-Gomi-Ryu, 1710.01886.

An application: the non-local order parameter for 1d superconductors with TR symmetry

- A model Hamiltonian: (1+1)d p-wave superconductor (Kitaev chain)

$$H = \sum_j \left[-t f_{j+1}^\dagger f_j + \Delta f_{j+1}^\dagger f_j^\dagger + h.c. \right] - \mu \sum_j f_j^\dagger f_j$$



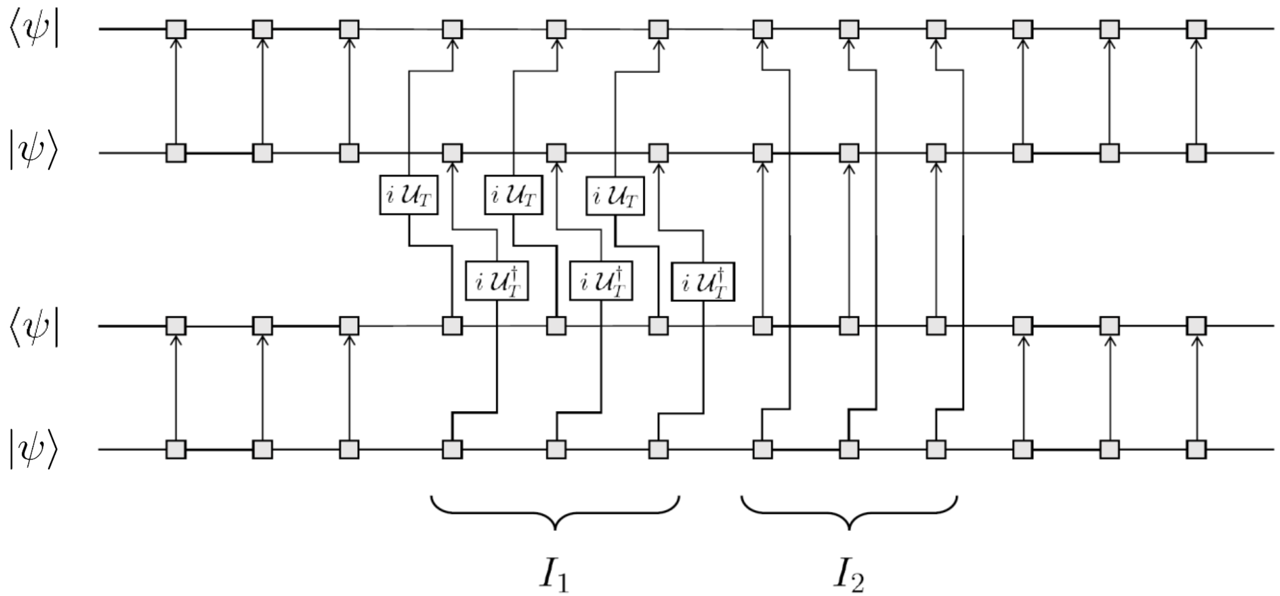
- TR symmetry $\quad T f_j^\dagger T^{-1} = f_j^\dagger, \qquad T \left| vac \right\rangle = \left| vac \right\rangle .$

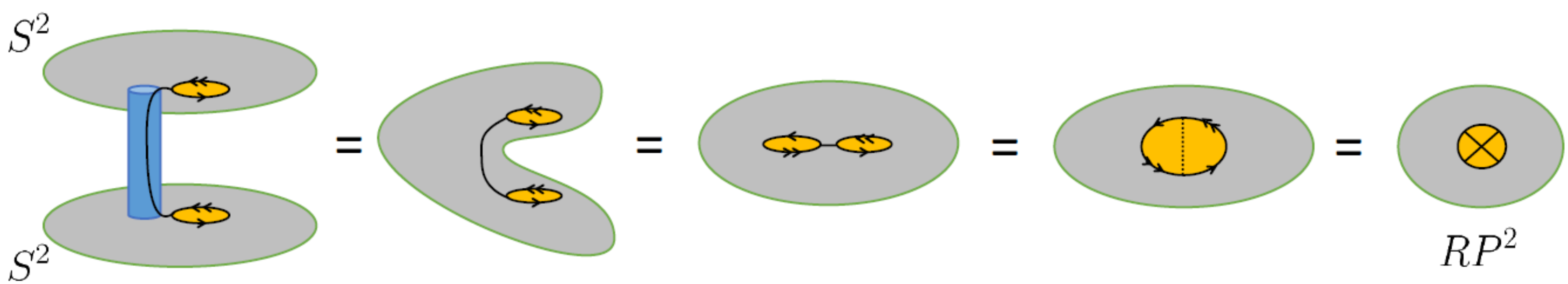
- The classification of SPT phases is known to be Z_8. [Fidkowski-Kitaev '10]

- The Z_8 "order parameter" is the partition function on RP^2 (real projective plane). $\quad \Omega_2^{Pin_-}(pt) = \mathbb{Z}_8$

- The Z_8 order parameter in operator formalism is the partial transpose on adjacent two intervals

$$Z = \mathrm{tr}_{I_1 \cup I_2}\left[\rho_{I_1\cup I_2} C_T^{I_1} \rho_{I_1\cup I_2}^{tr_1} [C_T^{I_1}]^\dagger\right]$$

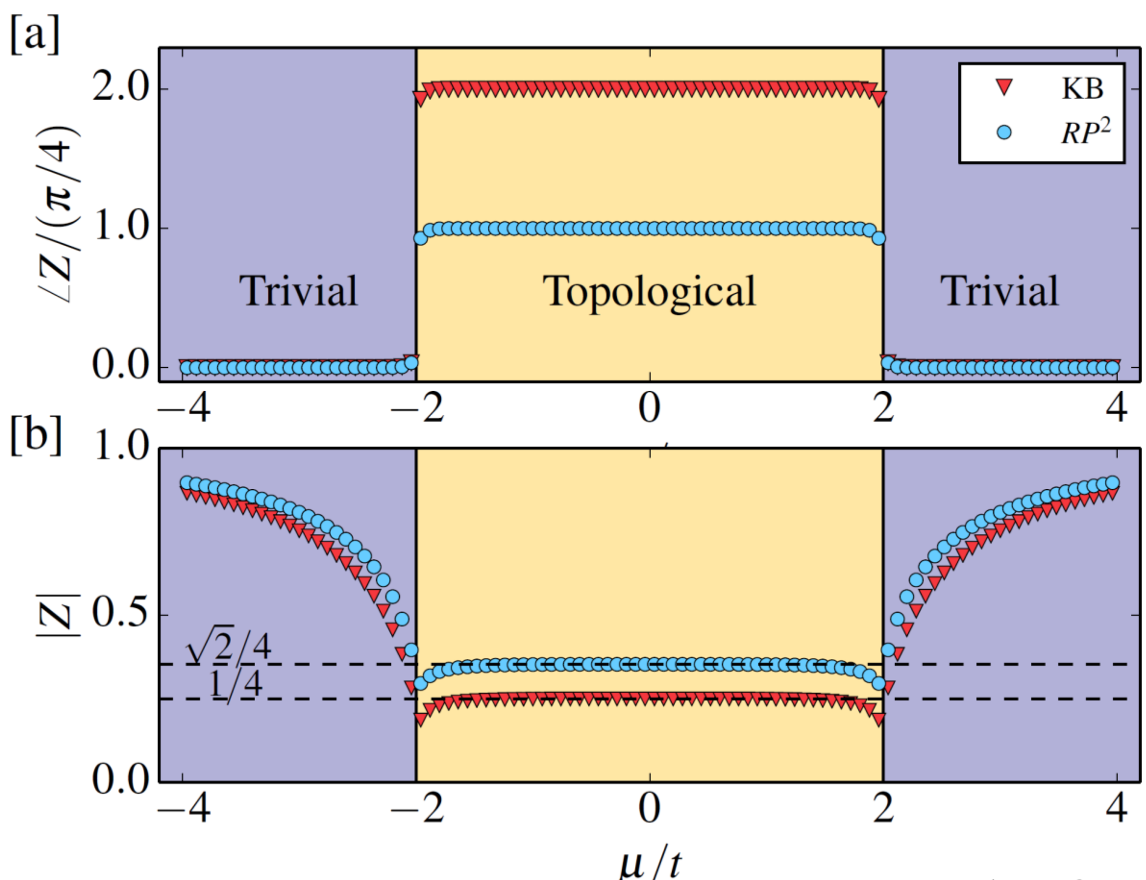




- Numerical calculation [arXiv:1607.03896]

$$H = -t\sum_j \left[f_j^\dagger f_j - \Delta f_{j+1}^\dagger f_j^\dagger + h.c.\right] - \mu \sum_j f_j^\dagger f_j$$

$$Z = \mathrm{tr}_{I_1\cup I_2}\left[\rho_{I_1\cup I_2} C_T^{I_1} \rho_{I_1\cup I_2}^{tr_1} [C_T^{I_1}]^\dagger\right]$$



Δ is fixed to be $\Delta = t$.

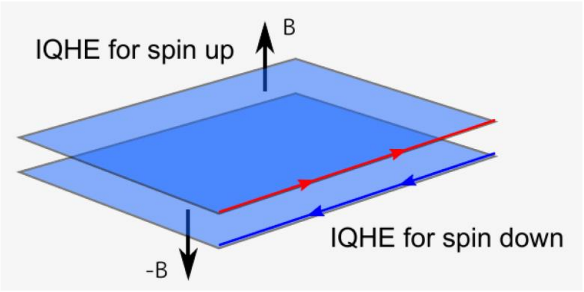
Manybody Z_2 Kane-Mele invariant

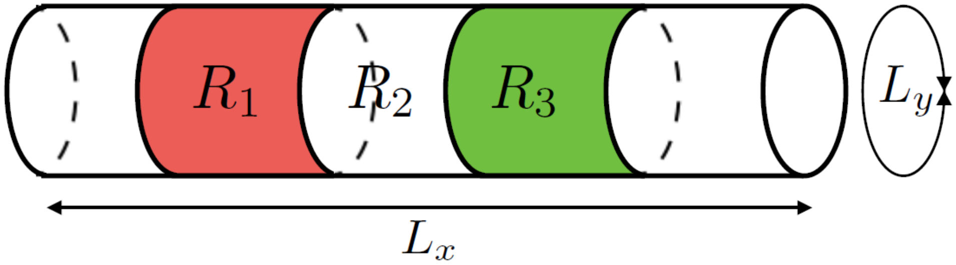
- (2+1)d topological insulator (TR symmetry with Kramers)

- The generating manifold is the Klein bottle × S[1] with a unit magnetic flux. [Witten '16]

- Combine two technics:

  ✓ Disjoint partial transpose -> Klein bottle
  ✓ Twist operator -> a unit magnetic flux





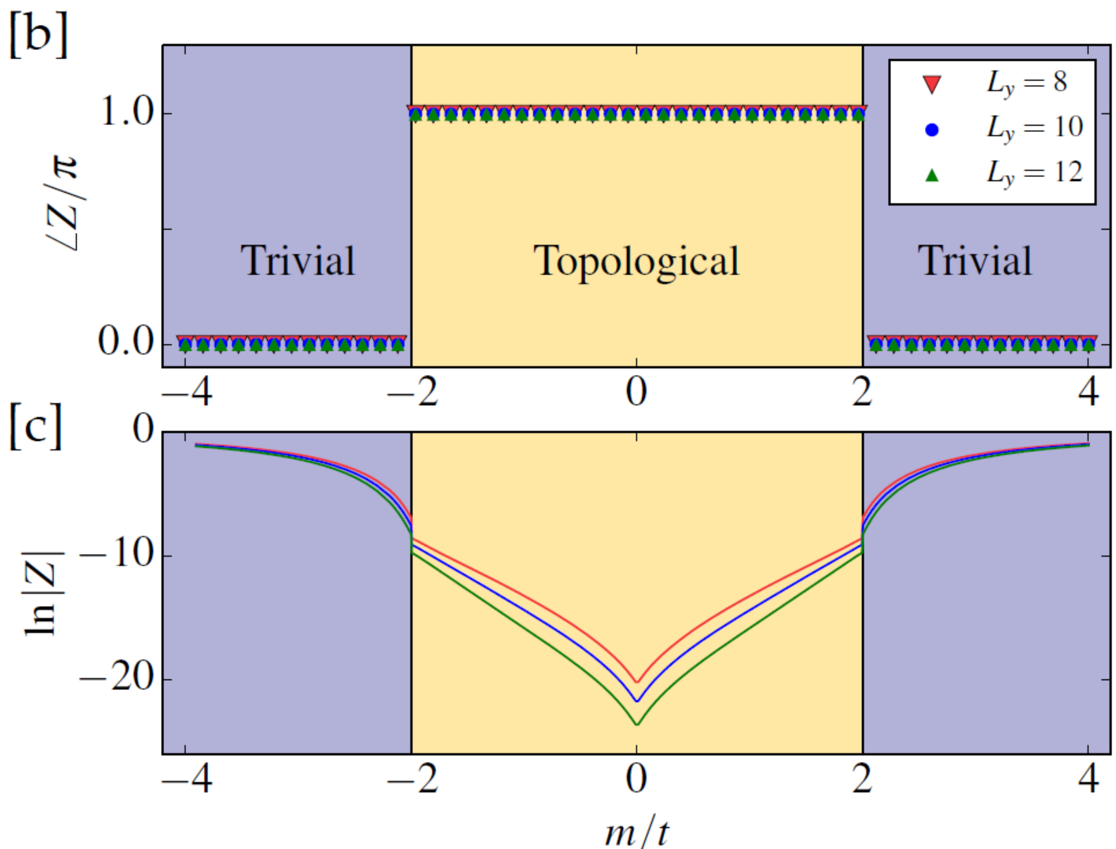
$$\rho^{\pm}_{R_1\cup R_3} := \mathrm{Tr}_{\overline{R_1\cup R_3}}\left[e^{\pm\sum_{(x,y)\in R_2}\frac{2\pi i y}{L_y}\hat{n}(x,y)}\left|GS\right\rangle\left\langle GS\right|\right],$$

$$Z := \mathrm{Tr}_{R_1\cup R_3}\left[\rho^{+}_{R_1\cup R_3}C_T^{I_1}[\rho^{-}_{R_1\cup R_3}]^{tr_1}[C_T^{I_1}]^{\dagger}\right].$$

Manybody Z_2 Kane-Mele invariant

- Numerical calculation for a free fermion model

$$Z := \mathrm{Tr}_{R_1\cup R_3}\left[\rho^{+}_{R_1\cup R_3} C_T^{I_1} [\rho^{-}_{R_1\cup R_3}]^{tr_1} [C_T^{I_1}]^{\dagger}\right].$$

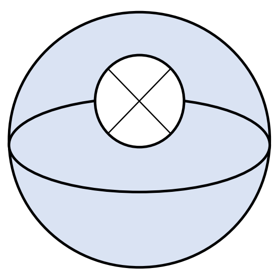


Take-home message

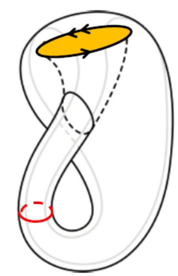
- Transpose $\sim$ time-reversal (TR) transformation for the imaginary time path-integral.

$$(\mathcal{O}_n \ldots \mathcal{O}_2 \mathcal{O}_1)^{tr} = \mathcal{O}_1^{tr} \mathcal{O}_2^{tr} \cdots \mathcal{O}_n^{tr}$$

- The partial transpose enable us to simulate the partition function over non-orientable manifolds in the operator formalism.



Real projective plane



Klein bottle

- Developed the partial transpose for fermions in the operator formalism

$$A^{tr_1} = \sum_{k_1, k_2, k_1+k_2 \in \text{even}} A_{p_1 \cdots p_{k_1}, q_1 \cdots q_{k_2}} i^{k_1} \underbrace{a_{p_1} \cdots a_{p_{k_1}}}_{I_1} \underbrace{b_{q_1} \cdots b_{q_{k_2}}}_{I_2}$$

Plan

- Why non-orientable manifolds?
- Time-reversal operation = a sort of transpose in the operator algebra
- Partial transpose and non-orientable manifolds in bosons
- Partial transpose and non-orientable manifolds in fermions

- Applications of the fermionic partial transpose

  - ✓ To simulate the partition function on unoriented manifolds in the operator formalism

  - ✓ Manybody SPT invariant for Kitaev chain
    [Shapourian-KS-Ryu, arXiv:1607.03896]

  - ✓ Fermionic entanglement negativity
    [Shapourian-KS-Ryu, arXiv:1611.07536]

- The emergence of the matrix transpose can be also understood as follows: In the matrix algebra, every linear anti-automorphism

$$\mathcal{O} \mapsto \phi(\mathcal{O}), \qquad \phi(\alpha\mathcal{O}) = \alpha\phi(\mathcal{O}), \quad \phi(\mathcal{O}_1\mathcal{O}_2) = \phi(\mathcal{O}_2)\phi(\mathcal{O}_1),$$

  can be written in a form

$$\phi(\mathcal{O}) = U\mathcal{O}^{tr}U^{\dagger}$$

  with *U* a unitary matrix.

- Under a basis change, the linear anti-automorphism *Φ* is changed as

$$\phi(\mathcal{O}) \mapsto \phi(V^{\dagger}\mathcal{O}V) = U(V^{\dagger}\mathcal{O}V)^{tr}U^{\dagger} = V^{\dagger}(VUV^{tr})\mathcal{O}^{tr}(VUV^{tr})^{\dagger}V$$

- Hence, the unitary matrix *U* for Φ is changed as

$$U \mapsto VUV^{tr}$$

- This is nothing but the basis change of the unitary part of TRS.

$$T = UK \mapsto VUKV^{\dagger} = VUV^{tr}K$$

Comment (1)

- It should be noted that the matrix transpose is **basis-dependent**: under a basis change, the matrix transpose is changed as

$$\mathcal{O}^{tr} \mapsto (V^\dagger \mathcal{O} V)^{tr} = V^\dagger (V V^{tr}) \mathcal{O}^{tr} (V V^{tr})^\dagger V$$

- In general, VV^{tr} is not the identity, implying the absence of a "canonical" transpose in the operator algebra of spin systems.

- The transpose is well-defined only in the presence of a TRS T.

- Let’s consider a spin system.
- The Hilbert space is the tensor product of local Hilbert spaces.

$$\mathcal{H} = \otimes_x \mathcal{H}_x, \quad \mathcal{H}_x \cong \mathbb{C}^N.$$

- The matrix transpose is well-defined.

$$A \mapsto A^{tr}$$

Fermionic Fock space

- Let f_j be complex fermions.

$$\{f_j, f_k^\dagger\} = \delta_{jk}, \qquad \{f_j, f_k\} = \{f_j^\dagger, f_k^\dagger\} = 0.$$

- The Fock space $\mathcal{F}$ is spanned or defined by the occupation basis

$$|n_1 n_2 \ldots n_N\rangle = |\{n_j\}\rangle := (f_1^\dagger)^{n_1} (f_2^\dagger)^{n_2} \cdots (f_N^\dagger)^{n_N} |\mathrm{vac}\rangle .$$

- We always assume the fermion parity symmetry.

$$(-1)^F := \prod_{j=1}^{N} (-1)^{f_j^\dagger f_j}, \qquad (-1)^F |\mathrm{vac}\rangle = |\mathrm{vac}\rangle .$$

- Summary of the definition of fermionic partial transpose:

  - ✓ A two-subdivision of the Fock space (per complex fermions)

  - ✓ The fermionic partial transpose is defined only on operators preserving the fermion parity.

$$A = \sum_{k_1, k_2, k_1+k_2=\mathrm{even}} A_{p_1 \cdots p_{k_1}, q_1 \cdots q_{k_2}} \underbrace{a_{p_1} \cdots a_{p_{k_1}}}_{I_1} \underbrace{b_{q_1} \cdots b_{q_{k_2}}}_{I_2}.$$

$$A^{tr_1} := \sum_{k_1, k_2, k_1+k_2=\mathrm{even}} A_{p_1 \cdots p_{k_1}, q_1 \cdots q_{k_2}} \, i^{k_1} \, a_{p_1} \cdots a_{p_{k_1}} b_{q_1} \cdots b_{q_{k_2}}.$$

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- From the Schur's lemma, the condition 3 leads to that the partial transpose is a scalar multiplication which may depend on the number of the Majorana fermions in the subspace I_1.

$$(a_{p_1} \cdots a_{p_{k_1}} b_{q_1} \cdots b_{q_{k_2}})^{tr_1} = z_{k_1} a_{p_1} \cdots a_{p_{k_1}} b_{q_1} \cdots b_{q_{k_2}}, \qquad z_{k_1} \in \mathbb{C}.$$

- The conditions 1. and 2. reads

$$z_0 = 1, \qquad z_{k_1} z_{k_2} = \begin{cases} -1 & (k_1 + k_2 = 2 \mod 4), \\ 1 & (k_1 + k_2 = 0 \mod 4). \end{cases}$$

- There are two solutions

$$z_k = (\pm i)^k, (k = 0, 1 \dots),$$

which are related by the fermion parity. I employ the convention

$$z_k = i^k, (k = 0, 1 \dots).$$

- If we includes $k_1 + k_2$ = odd, there is no solution.

Fermionic partial TR transformation

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- Combining the fermionic partial transpose and the unitary part C_T of a given TR operator T, one can introduce the fermionic partial TR transformation:

- Def. (Femrionic partial TR transformation)

  ✓ Let A be an operator preserving the fermion parity defined on the two intervals $I_1 \cup I_2$.

  ✓ Let $C_T^{I_1}$ be the unitary part of T on the subsystem I_1.

  ✓ The partial TR transformation on I_1 is defined by

$$A \mapsto C_T^{I_1} A^{tr_1} [C_T^{I_1}]^\dagger$$

- In the coherent state basis

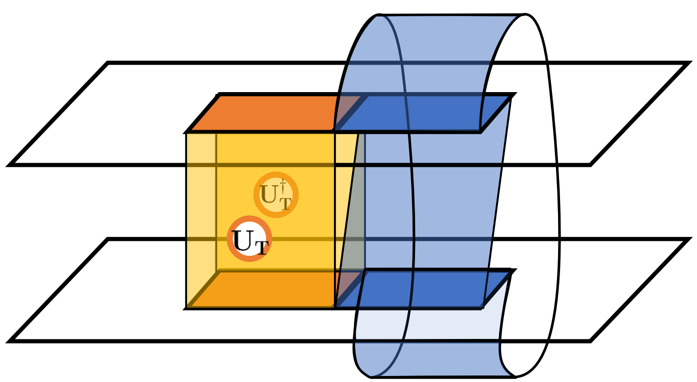
$$|\{\xi_{i\in I_1}\},\{\xi_{i\in I_2}\}\rangle = e^{-\sum_{i\in I_1}\xi_i f_i^\dagger - \sum_{i\in I_2}\xi_i f_i^\dagger}|\mathrm{vac}\rangle\,,$$

the partial TR transformation reads as

$$C_T^{I_1}\left(|\{\xi_j\}_{j\in I_1},\{\xi_j\}_{j\in I_2}\rangle\langle\{\chi_j\}_{j\in I_1},\{\chi_j\}_{j\in I_2}|\right)^{tr_1}[C_T^{I_1}]^\dagger$$
$$= |\{i[\mathcal{U}_T]_{jk}\chi_k\}_{j\in I_1},\{\xi_j\}_{j\in I_2}\rangle\langle\{i\xi_k[\mathcal{U}_T^\dagger]_{kj}\}_{j\in I_1},\{\chi_j\}_{j\in I_2}|\,.$$

- This is the same as the TR transformation on the subsystem I_1 in the imaginary time path-integral.

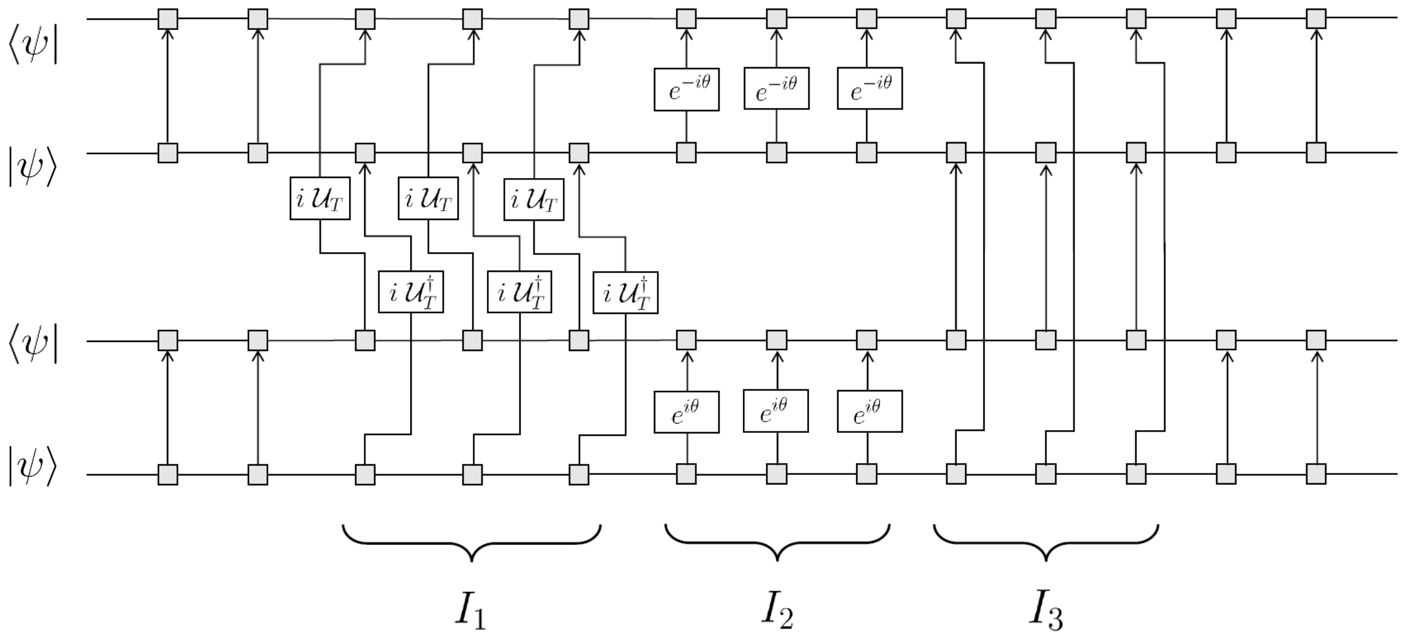
- Therefore, the partial TR transformation serves to simulate the real projective plane and the Klein bottle.

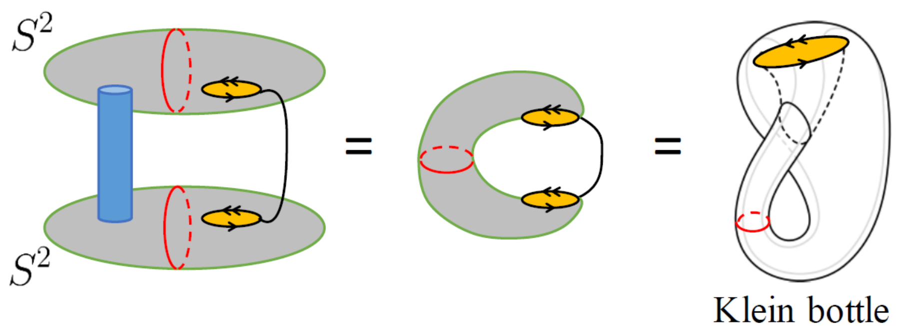


= RP^2

- Cf. Network rep. for the Klein bottle (detect the Z4 subgroup)

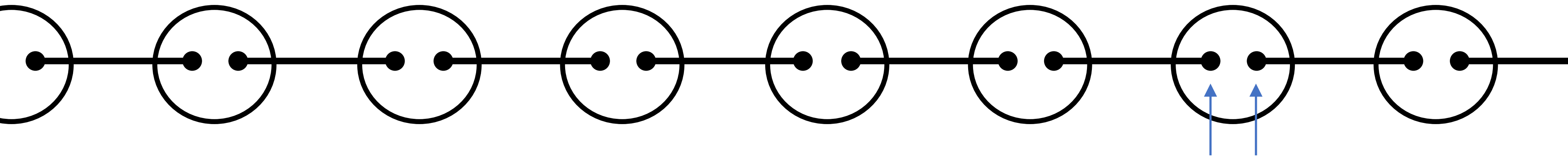
$$Z = \mathrm{tr}_{I_1 \cup I_3}\left[\rho_{I_1\cup I_3} C_T^{I_1} \rho_{I_1\cup I_3}^{tr_1} [C_T^{I_1}]^\dagger\right]$$





Z8 invariant of the Kitaev Chain

- (1+1)d class BDI superconductors $T^2 = 1$



- Classification = Z8 [Fidkowski-Kitaev].
- Background structure = pin- structrue
- Topological action = eta invariant (see Kapustin-Thorngren-Turzillo-Wang)

$$e^{iS_{\text{top}}[M,A]} = \int D\psi D\bar{\psi} e^{-S_M[\psi,\bar{\psi},A]} = e^{2\pi i \eta(M,A)/8}$$

Pin- str.

Z8 valued:
$\eta(M,A) \in \{0,1,\ldots,7\}$

- For M= RP^2, the eta invariant takes the smallest value ±1.

$$\eta(RP^2, A) = \pm 1$$

- This means that the partition function on RP^2 is the Z8 order parameter of the Kitaev chian with TRS, as for the Haldane chain.

Summary

- The TQFT description of SPT phases w/ TR symmetry suggest using unoriented manifolds.

- The problem is how to obtain unoriented manifolds from the TR operator.

- The (fermionic) partial transpose can simulate the partition function over (i) the real projective plane and (ii) the Klein bottle.

- We defined the fermionic analog of the partial transpose, and our definition correctly simulate the partition function over unoriented manifolds in fermionic systems.

- Various non-local order parameters for fermionic SPT phases are constructed in this way. Please see the list in [arXiv:1710.01886].

- Another topic: our definition of fernionic partial transpose can be used to define a fermionic analog of entanglement negativity. [arXiv:1611.07536]