「集団運動モデル」数式集

1. 回転系準粒子シェルモデル

回転座標系での準粒子ハミルトニアン

$$H = \sum_i (e_i - \lambda) c_i^\dagger c_i - \Delta {\sum_i}' (c_i^\dagger c_{\bar{i}}^\dagger + c_{\bar{i}} c_i) - \omega_{\rm rot} \sum_{i,j} < i|J_x|j > c_i^\dagger c_j \quad (1)$$

$$= \sum_\mu E_\mu a_\mu^\dagger a_\mu + \sum_{\bar{\mu}} E_{\bar{\mu}} a_{\bar{\mu}}^\dagger a_{\bar{\mu}} \quad (2)$$

回転座標系でのエネルギー、整列角運動量、励起エネルギー

$$E' = E - \omega_{\rm rot} I \quad (3)$$

$$i(\omega) = I(\omega) - I_g(\omega) \quad (4)$$

$$e'(\omega) = E'(\omega) - E_g(\omega) \quad (5)$$

運動学的 (kinematical) 慣性モーメントと動力学的 (dynamical) 慣性モーメント

$$\mathcal{J}^{(1)} = \frac{I}{\omega_{\rm rot}} = \left(\frac{1}{I}\frac{dE}{dI}\right)^{-1} = -\frac{1}{\omega}\frac{dE'}{d\omega} \simeq \frac{2I}{E_\gamma} \quad (6)$$

(7)

$$\mathcal{J}^{(2)} = \frac{dI}{d\omega_{\rm rot}} = \left(\frac{d^2E}{dI^2}\right)^{-1} = -\frac{d^2E'}{d\omega^2} \simeq \frac{4}{\Delta E_\gamma} \quad (8)$$

(9)

回転角速度と回転エネルギーの関係

$$\hbar\omega_{\rm rot}(I) \simeq \frac{\partial E_{\rm rot}}{\partial I} \quad (10)$$

$$= \frac{1}{2}\{E_{\rm rot}(I+1) - E_{\rm rot}(I-1)\} \quad (11)$$

$$= \frac{1}{2}E_\gamma \quad (12)$$

(13)

回転系へのユニタリー変換

$$|\phi(\theta, I)\rangle = U(\theta, I)|\phi_0\rangle, \quad (14)$$

$$U(\theta, I) = e^{-i\theta J_x} e^{iG(I)}, \quad (15)$$

$$|\phi(\theta, I)\rangle = e^{-i\theta J_x}|\phi_{\rm intr}(I)\rangle \quad (16)$$

内部状態

$$|\phi_{\rm intr}(I)\rangle = e^{iG(I)}|\phi_0\rangle \quad (17)$$

時間依存変分原理

$$\delta\langle\phi(\theta,I)|i\frac{\partial}{\partial t}-H|\phi(\theta,I)\rangle=0 \tag{18}$$

$$\begin{aligned}\dot{\theta} &= \frac{\partial\mathcal{H}}{\partial I}=\omega_{\mathrm{rot}} &(19)\\ \dot{I} &= -\frac{\partial\mathcal{H}}{\partial\theta}=0 &(20)\end{aligned}$$

$$\mathcal{H}(I)\equiv\langle\phi(\theta,I)|H|\phi(\theta,I)\rangle=\langle\phi_{\mathrm{intr}}(I)|H|\phi_{\mathrm{intr}}(I)\rangle \tag{21}$$

$$\delta\langle\phi_{\mathrm{intr}}(I)|H-\omega_{\mathrm{rot}}J_x|\phi_{\mathrm{intr}}(I)\rangle=0 \tag{22}$$

$$H'=H-\omega_{\mathrm{rot}}J_x \tag{23}$$

シグネチャーの固有状態

$$\begin{aligned}d_i^\dagger &= (c_{\bar{i}}^\dagger+c_i^\dagger)/\sqrt{2} &(24)\\ d_{\bar{i}}^\dagger &= (c_{\bar{i}}^\dagger-c_i^\dagger)/\sqrt{2} &(25)\end{aligned}$$

$$e^{-i\pi J_x}\begin{pmatrix} d_i^\dagger \\ d_{\bar{i}}^\dagger\end{pmatrix}e^{i\pi J_x}=\mp i\begin{pmatrix} d_i^\dagger \\ d_{\bar{i}}^\dagger\end{pmatrix} \tag{26}$$

一般化された Bogoliubov 変換

$$\begin{aligned}a_\mu^\dagger &= \sum_{i>0}(U_{i\mu}d_i^\dagger+V_{i\mu}d_{\bar{i}}), &(27)\\ a_{\bar{\mu}}^\dagger &= \sum_{i>0}(\bar{U}_{i\mu}d_{\bar{i}}^\dagger+\bar{V}_{i\mu}d_i) &(28)\end{aligned}$$

$$a_\mu|\phi_{\mathrm{intr}}(I)\rangle=a_{\bar{\mu}}|\phi_{\mathrm{intr}}(I)\rangle=0 \tag{29}$$

回転系での準粒子に対する固有方程式

$$\begin{pmatrix} h-\omega_{\mathrm{rot}}j_x & -\Delta \\ -\Delta & -(h-\omega_{\mathrm{rot}}j_x)\end{pmatrix}\begin{pmatrix} U \\ V\end{pmatrix}=E\begin{pmatrix} U \\ V\end{pmatrix} \tag{30}$$

2. Bohr-Mottelson 集団モデル概要

集団運動と 1 粒子運動の統一的記述

$$H = H_{\mathrm{coll}}(\boldsymbol{\alpha},\boldsymbol{\pi})+H_{\mathrm{part}}(\boldsymbol{r},\boldsymbol{\alpha}) \tag{31}$$

振動運動と回転運動に対する古典ハミルトニアン

$$
\begin{aligned}
H_{\text{coll}} &= T_{\text{coll}}(\boldsymbol{\alpha}, \boldsymbol{\pi}) + V_{\text{coll}}(\boldsymbol{\alpha}) && (32) \\
T_{\text{coll}} &= T_{\text{rot}}(\boldsymbol{\alpha}, \boldsymbol{\pi}) + T_{\text{vib}}(\boldsymbol{\alpha}, \boldsymbol{\pi}) \\
H_{\text{coll}}^{(\text{classical})} &= \frac{1}{2}D_{\beta\beta}(\beta,\gamma)\dot{\beta}^2 + \frac{1}{2}D_{\beta\gamma}(\beta,\gamma)\dot{\beta}\dot{\gamma} + \frac{1}{2}D_{\gamma\gamma}(\beta,\gamma)\dot{\gamma}^2 && (33) \\
&+ \frac{1}{2}\sum_{\kappa=1,2,3}\mathcal{J}_\kappa(\beta,\gamma)\omega_\kappa^2 + V_{\text{coll}}(\beta,\gamma) && (34)
\end{aligned}
$$

振動運動を無視すると (粒子-回転モデル)

$$\boldsymbol{\alpha} \Longrightarrow \boldsymbol{\alpha}_0 \tag{35}$$

$$T_{\text{vib}}(\boldsymbol{\alpha}, \boldsymbol{\pi}) \Longrightarrow 0 \tag{36}$$

$$H_{\text{part-rot}} = T_{\text{rot}}(\boldsymbol{\alpha}_0, \boldsymbol{\pi}) + H_{\text{part}}(\boldsymbol{r}, \boldsymbol{\alpha}_0) \tag{37}$$

回転運動を無視すると (粒子-振動モデル)

$$T_{\text{rot}}(\boldsymbol{\alpha}, \boldsymbol{\pi}) \Longrightarrow 0 \tag{38}$$

$$H_{\text{part-vib}} = T_{\text{vib}}(\boldsymbol{\alpha}, \boldsymbol{\pi}) + H_{\text{part}}(\boldsymbol{r}, \boldsymbol{\alpha}) \tag{39}$$

5 次元集団ハミルトニアンの量子化

$$T = T_{\text{vib}} + T_{\text{rot}} \tag{40}$$

$$T_{\text{vib}} = \frac{1}{2}D_{\beta\beta}(\beta,\gamma)\dot{\beta}^2 + D_{\beta\gamma}(\beta,\gamma)\dot{\beta}\dot{\gamma} + \frac{1}{2}D_{\gamma\gamma}(\beta,\gamma)\dot{\gamma}^2 \tag{41}$$

$$T_{\text{rot}} = \sum_k \frac{1}{2}\mathcal{J}_k(\beta,\gamma)\omega_k^2 \tag{42}$$

慣性モーメント

$$
\begin{aligned}
\mathcal{J}_k &= 4\beta^2 D_k(\beta,\gamma)\sin^2\gamma_k && (43) \\
\gamma_k &= \gamma - (2\pi k)/3 && (44)
\end{aligned}
$$

$$\hat{H} = \hat{T}_{\text{vib}} + \hat{T}_{\text{rot}} + V(\gamma) \tag{45}$$

$$\hat{T}_{\text{rot}} = \sum_k \frac{\hat{I}_k^2}{2\mathcal{J}_k} \tag{46}$$

量子化された振動ハミルトニアン

$$T_{\mathrm{vib}} = \frac{-\hbar^2}{2\sqrt{WR}}\left\{\frac{1}{\beta^3}\left[\frac{\partial}{\partial\beta}\left(\beta^3\sqrt{\frac{R}{W}}D_{\gamma\gamma}\frac{\partial}{\partial\beta}\right) - \frac{\partial}{\partial\beta}\left(\beta^3\sqrt{\frac{R}{W}}D_{\beta\gamma}\frac{\partial}{\partial\gamma}\right)\right]\right. \tag{47}$$

$$\left. + \frac{1}{\sin 3\gamma}\left[-\frac{\partial}{\partial\gamma}\left(\sqrt{\frac{R}{W}}\sin 3\gamma D_{\beta\gamma}\frac{\partial}{\partial\beta}\right) + \frac{\partial}{\partial\gamma}\left(\sqrt{\frac{R}{W}}\sin 3\gamma D_{\beta\beta}\frac{\partial}{\partial\gamma}\right)\right]\right\} \tag{48}$$

$$W = D_{\beta\beta}(\beta,\gamma)D_{\gamma\gamma}(\beta,\gamma) - D_{\beta\gamma}^2(\beta,\gamma) \tag{49}$$

$$R = D_1(\beta,\gamma)D_2(\beta,\gamma)D_3(\beta,\gamma) \tag{50}$$

すべての集団質量 (慣性関数) $D_{\beta\beta}, D_{\gamma\gamma}, D_{\beta\gamma}, D_1, D_2, D_3$ を定数 D と近似すると

$$T_{\mathrm{vib}} = -\frac{\hbar^2}{2D}\left(\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{1}{\beta^2\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma}\right) \tag{51}$$

（この近似は球形まわりの小振幅振動に対してのみ正当化される）

集団波動関数

$$\Psi_{IMk}(\gamma,\Omega) = \sum_{K=0}^{I}\Phi_{IKk}(\gamma)\langle\Omega|IMK\rangle, \tag{52}$$

$$\langle\Omega|IMK\rangle = \sqrt{\frac{2I+1}{16\pi^2(1+\delta_{K0})}}(\mathcal{D}_{MK}^I(\Omega) + (-)^I\mathcal{D}_{M-K}^I(\Omega)) \tag{53}$$

規格化条件

$$\int_{\gamma=0^\circ}^{\gamma=60^\circ}\sin 3\gamma d\gamma\sum_{K=0}^{I}\Phi_{IKk}^*(\gamma)\Phi_{IKk'}(\gamma) = \delta_{kk'}. \tag{54}$$

振動の波動関数に対する固有方程式

$$\left(\hat{T}_{\mathrm{vib}} + V(\gamma)\right)\Phi_{IKk}(\gamma) + \sum_{K'=0}^{I}\langle IMK|\hat{T}_{\mathrm{rot}}|IMK'\rangle\Phi_{IK'k}(\gamma) = E_{I,k}\Phi_{IKk}(\gamma), \tag{55}$$

3. 曲がった空間での量子化

線素の二乗

$$ds^2 = \sum_{i,j}g_{ij}(x)dx_i dx_j \tag{56}$$

運動エネルギー

$$T = \frac{1}{2}\left(\frac{ds}{dt}\right)^2 \tag{57}$$

$$= \frac{1}{2}\sum_{i,j}g_{ij}(x)\frac{dx_i}{dt}\frac{dx_j}{dt} \tag{58}$$

$$\tag{59}$$

その量子化

$$\hat{T} = -\frac{\hbar^2}{2}\frac{1}{\sqrt{g}}\sum_{i,j}\frac{\partial}{\partial x_i}\sqrt{g}g^{ij}\frac{\partial}{\partial x_j} \tag{60}$$

3 次元極座標の例

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2 \tag{61}$$

$$(g_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix} \tag{62}$$

$$\det(g_{ij}) \equiv g = r^4 \sin^2\theta \tag{63}$$

$$(g^{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1/r^2\sin^2\theta \end{pmatrix} \tag{64}$$

$$\begin{aligned} \hat{T} &= -\frac{\hbar^2}{2}\frac{1}{\sqrt{g}}\left(\frac{\partial}{\partial r}\sqrt{g}\frac{\partial}{\partial r} + \frac{\partial}{\partial\theta}\frac{\sqrt{g}}{r^2}\frac{\partial}{\partial\theta} + \frac{\partial}{\partial\varphi}\frac{\sqrt{g}}{r^2\sin^2\theta}\frac{\partial}{\partial\varphi}\right) \qquad (65) \\ &= -\frac{\hbar^2}{2}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin\theta}\frac{\partial^2}{\partial\varphi^2}\right) \qquad (66) \end{aligned}$$

4. Wobbling モード

回転運動エネルギー

$$H = \frac{I_x^2}{2\mathcal{J}_x} + \frac{I_y^2}{2\mathcal{J}_y} + \frac{I_z^2}{2\mathcal{J}_z}, \qquad I^2 = I_x^2 + I_y^2 + I_z^2$$

$$I_\pm = I_y \pm i I_z$$

角運動量が大きくて、y, z 方向の成分が x 方向に比べて小さい場合

$$[I_-,\ I_+] = 2I_x \approx 2I, \qquad \text{for } I_x \gg I_y,\ I_z$$

ボソン演算子の導入

$$b = \frac{I_-}{\sqrt{2I}}, \qquad b^\dagger = \frac{I_+}{\sqrt{2I}}$$

$$\left[b,\ b^\dagger\right] = 1$$

ユニタリー変換

$$B^\dagger = xb^\dagger - yb$$

対角化されたハミルトニアン

$$H = \frac{I^2}{2\mathcal{J}_x} + \hbar\omega(B^\dagger B + \frac{1}{2})$$

Wobbling 振動数

$$\hbar\omega = I\sqrt{(\frac{1}{\mathcal{J}_y} - \frac{1}{\mathcal{J}_x})(\frac{1}{\mathcal{J}_z} - \frac{1}{\mathcal{J}_x})}$$

回転スペクトル

$$E(n, I) = \frac{I(I+1)}{2\mathcal{J}_x} + \hbar\omega(n + \frac{1}{2})$$

(重要) このモードが存在するためには $\mathcal{J}_x \geq \mathcal{J}_y,\ \mathcal{J}_x$ が必要

$$\hbar\omega = I\sqrt{(\frac{1}{\mathcal{J}_y} - \frac{1}{\mathcal{J}_x})(\frac{1}{\mathcal{J}_z} - \frac{1}{\mathcal{J}_x})}$$

5. シェル構造の半古典論（周期軌道理論）

1粒子シュレーディンガー方程式

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V(\beta)\right)\psi(\boldsymbol{r}) = e(\beta)\psi(\boldsymbol{r}) \tag{67}$$

可積分系に対するＥＢＫ量子化　（action-angle 変数を導入できる）

$$\dot{\boldsymbol{\theta}} = \frac{\partial h}{\partial \boldsymbol{I}} = \boldsymbol{\omega}(\boldsymbol{I}) \tag{68}$$

$$e(\boldsymbol{n}) = h(\boldsymbol{I}) \qquad \text{with} \quad \boldsymbol{I} = \hbar\,(\boldsymbol{n} + \tfrac{1}{4}\boldsymbol{\alpha}), \tag{69}$$

1粒子エネルギーが縮退する条件

$$\begin{aligned} e(\boldsymbol{n} + \Delta\boldsymbol{n}) - e(\boldsymbol{n}) &= h(\boldsymbol{I} + \Delta\boldsymbol{I}) - h(\boldsymbol{I}) \\ &\simeq \frac{\partial h}{\partial \boldsymbol{I}}\Delta\boldsymbol{I} \\ &= \hbar\boldsymbol{\omega}\cdot\Delta\boldsymbol{n} \\ &= \hbar\omega_1\Delta n_1 + \hbar\omega_2\Delta n_2 + \hbar\omega_3\Delta n_3 \\ &= 0, \end{aligned} \tag{70}$$

非可積分系における1粒子準位密度に対する半古典論　（トレース公式）

$$\begin{aligned} g(e) &= \sum_i \delta(e - e_i) \\ &\simeq \bar{g}(e) + \delta g(e) \\ &= \bar{g}(e) + \sum_\gamma A_\gamma \cos\left(\frac{1}{\hbar} S_\gamma(e) - \frac{\pi}{2}\mu_\gamma\right), \end{aligned} \tag{71}$$

作用積分

$$S_\gamma = \oint_\gamma \boldsymbol{p} \cdot d\boldsymbol{x} \tag{72}$$