Nuclear matter
from strong coupling lattice QCD

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PhD thesis of Michael Fromm (ETH)

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and in progress
Physics of color singlets

- “One-body” physics: confinement
  hadron masses
  form factors, etc..

\[ \beta = 0 \text{ LQCD} \]
Scope of lattice QCD simulations

Physics of color singlets

- “One-body” physics: confinement
  - hadron masses
  - form factors, etc..

- “Two-body” physics: nuclear interactions
  - pioneers: Hatsuda et al, Savage et al

hard-core + pion exchange?
QCD phase diagram according to Wikipedia

- **many-body** physics: hadron ↔ nuclear matter transition
- **two-body**: $T = 0$ nuclear interactions
A different approach to the sign problem

\[ Z = \int \mathcal{D} A \mathcal{D} \bar{\psi} \mathcal{D} \psi \exp \left( -\frac{1}{4} F_{\mu \nu} F_{\mu \nu} + \sum_{i=1}^{N_f} \bar{\psi}_i (\not{D} + m + \mu \gamma_0) \psi_i \right) \]

\( \det(\not{D} + m + \mu \gamma_0) \) complex \( \rightarrow \) try integrating over the gauge field first!

- Problem: \(-\frac{1}{4} F_{\mu \nu} F_{\mu \nu} \rightarrow \beta_{\text{gauge}} \text{Tr} U_{\text{Plaquette}}, \text{ie. 4-link interaction} \)

- Solution: set \( \beta_{\text{gauge}} = \frac{2 N_c}{g^2} \) to zero, ie. \( g = \infty \), strong coupling limit

- Then integral over gauge links factorizes: \( \sim \int \prod dU \exp(\bar{\psi}_x U_{x,\hat{\mu}} \psi_{x+\hat{\mu}}) \)
  - analytic 1-link integral \( \rightarrow \) only color singlets survive
  - perform Grassmann integration last \( \rightarrow \) hopping of color singlets
  \( \rightarrow \) **hadron worldlines**

  - sample gas of worldlines by Monte Carlo

Note: when \( \beta_{\text{gauge}} = 0 \), quarks are *always* confined \( \forall (\mu, T) \), ie. **nuclear matter**
The price to pay: not continuum QCD

Strong coupling LQCD: why bother?

Asymptotic freedom: \( a(\beta_{\text{gauge}}) \propto \exp\left(-\frac{\beta_{\text{gauge}}}{4N_c b_0}\right) \)

ie. \( a \to 0 \) when \( \beta_{\text{gauge}} \equiv \frac{2N_c}{g^2} \to +\infty \). Here \( \beta_{\text{gauge}} = 0 \)

- Lattice “infinitely coarse”
- Physics not universal

Nevertheless:
- Properties similar to QCD: confinement and \( \chi_{\text{SB}} \)
- Include (perhaps) next term in strong coupling expansion, ie. \( \beta_{\text{gauge}} > 0 \)

When \( \beta_{\text{gauge}} = 0 \), sign problem is manageable \( \to \) complete solution

Valuable insight?

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\( \beta = 0 \) LQCD
Further motivation

- 25+ years of analytic predictions:
  - 80’s: Kluberg-Stern et al., Kawamoto-Smit, Damgaard-Kawamoto
    \[ \mu_c(T = 0) = 0.66, \quad T_c(\mu = 0) = 5/3 \]
  - 90’s: Petersson et al., 1/g^2 corrections
  - 00’s: detailed \((\mu, T)\) phase diagram: Nishida, Kawamoto,...

  now: Ohnishi et al. \(O(\beta)\) & \(O(\beta^2)\), Münster & Philipsen,...

  How accurate is mean-field \((1/d)\) approximation?

- Almost no Monte Carlo crosschecks:
  - 89: Karsch-Mütter \(\rightarrow\) MDP formalism \(\rightarrow\) \(\mu_c(T = 0) \sim 0.63\)
  - 92: Karsch et al. \(T_c(\mu = 0) \approx 1.40\)
  - 99: Azcoiti et al., MDP ergodicity ??

  06: PdF-Kim, HMC \(\rightarrow\) hadron spectrum \(\sim 2\%\) of mean-field

Can one trust the details of analytic phase-diagram predictions?
Phase diagram from Nishida (2004, mean field, cf. Fukushima)

- Very similar to conjectured phase diagram of $N_f = 2$ QCD
- But no deconfinement here: high density phase is nuclear matter
- Baryon mass $= M_{\text{proton}} \Rightarrow$ lattice spacing $a \sim 0.6$ fm not universal
Strong coupling $SU(3)$ with staggered quarks

$$Z = \int \mathcal{D}U \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp(-\bar{\psi}(\mathcal{D}(U) + m_q)\psi), \text{ no plaquette term (\(\beta_{\text{gauge}} = 0\))}$$

- **One** complex colored fermion field per site (no Dirac indices, spinless)
- $\mathcal{D}(U) = \frac{1}{2} \sum_{x,\nu} \eta_\nu(x)(U_\nu(x) - U_\nu^\dagger(x - \hat{\nu}))$, $\eta_\nu(x) = (-)^{x_1 + \ldots + x_{\nu-1}}$

$U(1)_V \times U(1)_A \text{ symmetry}$ when $m = 0$:

\[
\begin{align*}
\psi(x) &\rightarrow e^{i\theta}\psi(x) \\
\bar{\psi}(x) &\rightarrow e^{-i\theta}\bar{\psi}(x)
\end{align*}
\]

unbroken $\Rightarrow$ quark number $\Rightarrow$ chem. pot.

\[
\begin{align*}
\psi(x) &\rightarrow e^{i\gamma_5(x)\theta}\psi(x) \\
\bar{\psi}(x) &\rightarrow e^{i\gamma_5(x)\theta}\bar{\psi}(x)
\end{align*}
\]

spont. broken ($m = 0$) $\Rightarrow$ quark condensate

$$\gamma_5(x) = (-)^{x_1 + x_2 + x_3 + x_4}$$

($N_f = 1 \longrightarrow U(1) \text{ chiral symmetry}$)
Strong coupling $SU(3)$ with staggered quarks

\[ Z = \int D U D \bar{\psi} D \psi \exp(-\bar{\psi}(\slashed{D}(U) + m_q)\psi), \] no plaquette term ($\beta_{\text{gauge}} = 0$)

- **One** complex colored fermion field per site (no Dirac indices, spinless)
- $\slashed{D}(U) = \frac{1}{2} \sum_{x,\nu} \eta_\nu(x)(U_\nu(x) - U_\nu^\dag(x - \hat{\nu})))$, $\eta_\nu(x) = (-)^{x_1 + \ldots + x_{\nu - 1}}$
- Chemical potential $\mu \rightarrow \exp(\pm a\mu) U_{\pm 4}$
- $D U = \prod dU$ factorizes $\rightarrow$ integrate over links Rossi & Wolff

**Color singlet** degrees of freedom:

- **Meson** $\bar{\psi}\psi$: *monomer*, $M(x) \in \{0, 1, 2, 3\}$
- **Meson hopping**: *dimer*, non-oriented $n_\nu(x) \in \{0, 1, 2, 3\}$
- **Baryon hopping**: oriented $\bar{B}B_\nu(x) \in \{0, 1\}$ $\rightarrow$ *self-avoiding loops* $C$

Point-like, hard-core baryons in pion bath

No $\pi NN$ vertex

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$\beta = 0$ LQCD
MDP Monte Carlo

\[ Z(m_q, \mu) = \sum_{\{M, n_\nu, C\}} \prod_x \frac{m_q^M(x)}{M(x)!} \prod_{x, \nu} \frac{(3 - n_\nu(x))!}{n_\nu(x)!} \prod_{\text{loops } C} \rho(C) \]

with constraint \( (M + \sum_{\pm \nu} n_\nu)(x) = 3 \ \forall x \notin \{C\} \)

Constraint: 3 blue symbols or a baryon loop at every site
$Z(m_q, \mu) = \sum_{\{M, n_\nu, C\}} \prod_x \frac{m_q^M(x)}{M(x)!} \prod_{x, \nu} \frac{(3 - n_\nu(x))!}{n_\nu(x)!} \prod_{\text{loops } C} \rho(C)$

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The dense (crystalline) phase: 1 baryon per site
MDP Monte Carlo

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with constraint \((M + \sum_{\nu} n_{\nu})(x) = 3 \quad \forall x \notin \{C\}\)

Remaining difficulties:

- Baryons are fermions: mild sign problem from \(\rho(C)\)  
  \(\rightarrow \) volumes up to \(16^3 \times 4 \quad \forall \mu\)
- tight-packing constraint \(\rightarrow \) local update inefficient, esp. as \(m \rightarrow 0\)
  Solved with worm algorithm  
  (Prokof’ev & Svistunov 1998)
  Efficient even when \(m_q = 0\)

Local Metropolis, \(4^3 \times 2\) at \(\mu_c, m_q = 0.025\)

Worm, same parameter set

\(\beta = 0\) LQCD
$(\mu, T)$ phase diagram in the chiral limit $m_q = 0$, and for $m_q \neq 0$

- Phase boundary for breaking/restoration of $U(1)$ chiral symmetry
- 2nd order at $\mu = 0$: 3d $O(2)$ universality class
- 1st order at $T = 0$: $\rho_B$ jumps from 0 to 1 baryon per site $\Rightarrow$ tricrit. pt. TCP

Finite-size scaling: $(\mu, T)_{TCP} = (0.33(3), 0.94(7))$ vs $(0.577, 0.866)$ (mean-field)

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- $m_q \neq 0$: liquid-gas transition $T_{CEP} \sim 200$MeV – traj. of CEP obeys tricrit. scaling
Nuclear matter: spectroscopy

- Can compare masses of differently shaped “isotopes”
- $E(B=2) - 2E(B=1) \sim -0.4$, ie. “deuteron” binding energy ca. 120 MeV
- $am(A) \sim a\mu_B^{\text{crit}} A + (36\pi)^{1/3} \sigma a^2 A^{2/3}$, ie. (bulk + surface tension)
  - Bethe-Weizsäcker parameter-free ($\mu_B^{\text{crit}}$ and $\sigma$ measured separately)
- “Magic numbers” with increased stability: $A = 4, 8, 12$ (reduced area)
Nuclear potential: more than hard core

- Nucleons are point-like $\rightarrow$ no ambiguity with definition of static potential
- Nearest-neighbour attraction $\sim 120$ MeV at distance $\sim 0.5$ fm: cf. real world
- Baryon worldlines self-avoiding $\rightarrow$ no direct meson exchange (just hard core)
  Attraction due to bath of neutral pions: cf. Casimir effect (see next)
How the nucleon got its mass

- Point-like nucleon **distorts pion bath** cf. Casimir

![Graphs showing vacuum, static baryon, and effect on pions]

- Energy = nb. time-like pion lines
  - Constraint: 3 pion lines per site \((m_q = 0)\) → energy density = \(3/4\) in vacuum
  - No spatial pion lines connecting to site occupied by nucleon → energy increase

**Steric effect**
- \(am_B \approx 2.88 = (3 - 0.75) + \Delta E_\pi\), ie. "valence"(78%) + "pion cloud"(22%)
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$$<n(R) \approx 3/4$$

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So, in fact, nucleon is *not* point-like

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Static baryon prevents monomers = static ($t$-invariant) monomer “source”

Linear response $\propto$ Green’s fct. of lightest $t$-invariant meson, ie. rho/omega

(pion has factor $(-1)^t$)
So, in fact, nucleon is *not* point-like

Point-like “bag” of 3 valence quarks → **macroscopic** disturbance in pion vacuum

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(pion has factor \((-1)^t\))

\[
\langle n_t(R) \rangle - \frac{3}{4} \propto \frac{\exp(-m_\rho/\omega r)}{r} \times (-1)^x y z
\]

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\(\beta = 0\) LQCD
Nuclear interaction via pion clouds (thanks W. Weise)

- Here, baryons make self-avoiding loops $\rightarrow$ no direct meson exchange
- Interaction comes because of pion clouds

The two pion clouds can interpenetrate at $\approx$ constant energy (2nd order effect)
But each set of valence quarks disturbs pion cloud of other baryon

$$V_{NN}(R) \approx -2 \times \Delta E\pi(R) \propto \left(\frac{\mathrm{exp}\left(-m_p/\omega R\right)}{R}\right) \times (-1)^{x+y+z}$$
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\[
V_{NN}(R) \approx -2 \times \Delta E_{\pi}(R) \propto \exp\left(\frac{-m_\rho/\omega R}{R}\right) \times (-1)^{x+y+z}
\]

Meson exchange potential without meson exchange!
Is pion bath essential? Classical hard spheres

$g(r) \equiv \langle \rho(0)\rho(r) \rangle$ relaxes to $\langle \rho \rangle^2$ with damped oscillations $\rightarrow$ liquid
Is pion bath essential? Classical hard spheres

“Potential of mean force” $V_{\text{eff}}(r) \equiv -\log(g(r))$ is hard-core + damped oscillatory
Is pion bath essential? Classical hard spheres

"Potential of mean force" \( V_{\text{eff}}(r) \equiv -\log(g(r)) \) is hard-core + damped oscillatory

Consistent with Yukawa form \( \frac{\exp(-mr)}{r} \times \cos(\Gamma r) \)
Is pion bath essential? Classical hard spheres

\[ \log(|V_{\text{eff}}(r)|) + \text{Yukawa fit} \]

Perfect fit at large distance

Hard-sphere “potential of mean force” is of Yukawa form

\[ V_{\text{eff}}(r) = \text{Re} \left[ \frac{e^{-(m+i\Gamma)r}}{r} \right] \]
Recap & speculation

- Baryons are not point-like: pion cloud $\sim \exp(-m_\rho/\omega r)$

- Nuclear potential:
  - Hard-core from Pauli principle
  - Yukawa potential (times $(-1)^r$) from the two pion clouds

- Exactly like a **classical hard-sphere fluid**:
  - “Pion cloud” from ripples around tagged sphere
  - Density-density correlation $\leftrightarrow V_{\text{eff}}(r) \sim \exp(-(m + i\Gamma)r)/r$

- **Note**: at high density (packing fraction $\eta \in [0.4945, 0.6802]$),

  **hard-sphere system in solid phase**  
  cf. Kepler, microsphere exp. @ ISS

  $\rho_0 = 0.16/fm^3$ and $r \sim 0.5$ fm $\rightarrow \eta \sim 0.08 \sim \frac{1}{6} \eta_{\text{crit}}$

- **Speculation**: IF baryons similar to hard [enough] billiard balls,
  THEN expect solid phase at high enough density ($\sim 6\rho_0$)

Solid phase due to (close packing + hard core) $\Rightarrow$ robust w.r.t. details of potential
Conclusions

Summary

- Phase diagram: take mean-field results with a grain of salt
- [Crude, crystalline] nuclear matter from QCD: tabletop simulations of first-principles nuclear physics
- Nucleon: point-like “bag” ($\rightarrow$ hard core) + large pion cloud ($\rightarrow$ Yukawa)
- Hard core $\Longrightarrow$ solid phase at high density

Outlook

- Include second quark species $\rightarrow$ isospin
- Include $O(\beta)$ effects?