Modular Constraints on Conformal Field Theories with Currents

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Introduction

Conformal Field Theories in Two dimension

Governed by the Virasoro algebra : L_n and $\bar{L}_n$, $n \in \mathbb{Z}$

h, $\bar{h}$ ($\Delta = h + \bar{h}$ and $\ell = |h - \bar{h}|$)

$c < 1$

- Unitary rep defined only for

$$c = 1 - \frac{6}{m(m+1)},$$

$$h_{r,s} = \frac{[(m+1)r - ms]^2 - 1}{4m(m+1)}$$

- Completely solved (integrable)

$c > 1$

- Finiteness of primary states

Rational CFT / Irrational CFT

- Conserved currents
- AdS dual?

(e.g. extremal CFTs)

Goal of this project :

Investigate how the **modular constraint** act on the partition function $Z(\tau, \bar{\tau}) = Z(-\frac{1}{\tau}, -\frac{1}{\bar{\tau}})$

→ Reproduce known partition function

→ Construct unknown partition function

- The pure quantum gravity in AdS_3

• Brown-Henneaux : Two copies of the Virasoro algebra appear at asymptotic infinity(with $c = \frac{3\ell}{2G_N}$), they acts on the physical Hilbert space.

$$Z(\tau, \bar{\tau}) \stackrel{?}{=} q^{-\frac{c}{24}} \prod_{n=2}^{\infty} \frac{1}{1-q^n} \qquad \text{Vacuum} \bigoplus \text{descendants}$$

It cannot be right, as it does not invariant under the **modular transformation**.

• The BTZ black holes are three dimensional spacetime with metric

$$ds^2 = -\frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^2\ell^2} dt^2 + \frac{r^2\ell^2}{(r^2 - r_+^2)(r^2 - r_-^2)} dr^2 + r^2 (d\phi^2 - \frac{r_+ r_-}{r^2 \ell} dt)^2$$

with mass and angular momentum are given by $M = \frac{r_+^2 + r_-^2}{8G_N\ell^2}$ and $J = \frac{2r_+ r_-}{8G_N \ell}$.
In extremal case $(r_+ = r_-)$, $M\ell = J$.

• The AdS/CFT correspondence says that the BTZ black holes are *thermal states* in boundary 2d CFT where the mass and angular momentum are identified as

$$L_0 - \frac{c}{24} = M\ell - J = 2\bar{h}, \quad \bar{L}_0 - \frac{c}{24} = M\ell + J = 2h$$

- The Extremal Conformal Field Theory ($c = 24k, k \in \mathbb{Z}$)
  - The entropy of the BTZ black hole is given by $S = 4\pi\sqrt{k}(\sqrt{L_0} + \sqrt{\bar{L}_0})$ with $\frac{\ell}{16G_N} = k$. To have non-trivial configuration, we require : $L_0 \geq 1$.

$$Z(\tau, \bar{\tau}) = q^{-\frac{c}{24}} \prod_{n=2}^{\infty} \frac{1}{1-q^n} + \mathcal{O}(q^1)$$

when the primary contribution start from q^1, we call those theory as **extremal CFT**.

- The partition function of $c = 24$ and $c = 48$ extremal CFT can be written in terms of the Klein j-invariant.

$$Z_{c=24}(\tau, \bar{\tau}) = (j(q) - 744)(j(\bar{q}) - 744)$$
$$Z_{c=48}(\tau, \bar{\tau}) = ((j(q) - 744)^2 - 393767)((j(\bar{q}) - 744)^2 - 393767)$$
$$j(q) - 744 = q^{-1} + \underbrace{196884}_{1+196883} q + \underbrace{21493760}_{1+196883+21296876} q^2 + \cdots.$$

- Comments on the 196883

Entropy : $S = \text{Log}(196883) = 12.1904 \overset{?}{\sim} 12.5664 = 4\pi$.

Dimension of irreps of the Monster group (automorphism of the moonshine module).

Settings and Numerical Results

- The Character Decomposition
  - The (Virasoro) vacuum characters and primary characters are defined by

$$\chi_0(\tau) = \frac{1}{\eta(\tau)} q^{-\frac{c-1}{24}}(1-q), \quad \chi_h(\tau) = \frac{1}{\eta(\tau)} q^{h-\frac{c-1}{24}}$$

The torus partition function of unitary CFT admit the **character decomposition**,

$$Z(\tau,\bar{\tau}) = \chi_0(\tau)\bar{\chi}_0(\bar{\tau}) + \sum_{h,\bar{h}} d(h,\bar{h})\chi_h(\tau)\bar{\chi}_{\bar{h}}(\bar{\tau}) + \sum_{j=1} \left[d(j)\chi_j(\tau)\bar{\chi}_0(\bar{\tau}) + \tilde{d}(j)\chi_0(\tau)\bar{\chi}_j(\bar{\tau}) \right],$$

where the degeneracies $d(h,\bar{h})$, $d(j)$ and $\tilde{d}(j)$ are positive integers.

- The constraints from $SL(2,\mathbb{Z})$
  - T- transformation : All states should have **integer spin**.
  - S- transformation : $Z(\tau,\bar{\tau}) = Z(-\frac{1}{\tau}, -\frac{1}{\bar{\tau}})$

$$\boxed{\mathcal{Z}_0(\tau,\bar{\tau}) + \sum_{h,\bar{h}} d(h,\bar{h})\mathcal{Z}_{h,\bar{h}}(\tau,\bar{\tau}) + \sum_{j=1} \left[d(j)\mathcal{Z}_j(\tau,\bar{\tau}) + \tilde{d}(j)\mathcal{Z}_{\bar{j}}(\tau,\bar{\tau}) \right] = 0}$$

where the function $\mathcal{Z}_\lambda(\tau,\bar{\tau})$ is defined as $\chi_\lambda(\tau)\bar{\chi}_\lambda(\bar{\tau}) - \chi_\lambda(-\frac{1}{\tau})\bar{\chi}_\lambda(-\frac{1}{\bar{\tau}})$.

Modular Bootstrap - Basic Strategy [Rattazzi, Rychkov, Tonni, Vichi 08], [Poland, Simmons-Duffin 10]

- In the computation, we mainly use the reduced character for convenience.

$$\hat{\chi}_0(\tau) = \tau^{\frac{1}{4}}\eta(\tau)\chi_0(\tau), \quad \hat{\chi}_h(\tau) = \tau^{\frac{1}{4}}\eta(\tau)\chi_h(\tau)$$

- Apply the linear functional $\alpha\left[\hat{\mathcal{Z}}(z,\bar{z})\right] \equiv \sum_{m,n}^{m+n=N} \alpha_{m,n}\partial_z^m \partial_{\bar{z}}^n \hat{\mathcal{Z}}(z,\bar{z})$ to the modular bootstrap equation. ($\tau \equiv ie^z$, the crossing point at $z=0$)

$$\alpha\left[\hat{\mathcal{Z}}_0(z,\bar{z})\right] + \sum_{j=1}^{j_{max}}\left(d(j)\,\alpha\left[\hat{\mathcal{Z}}^j(z,\bar{z})\right] + \bar{d}(j)\,\alpha\left[\hat{\mathcal{Z}}^{\bar{j}}(z,\bar{z})\right]\right) + \sum_{h,\bar{h}\in\mathcal{P}} d(h,\bar{h})\,\alpha\left[\hat{\mathcal{Z}}^{h,\bar{h}}(z,\bar{z})\right] = 0.$$

- Find $\alpha_{m,n}$ such that,

$$\alpha\left[\hat{\mathcal{Z}}_0(z,\bar{z})\right] > 0,$$

$$\text{and} \quad \alpha\left[\hat{\mathcal{Z}}^j(z,\bar{z})\right] \geq 0, \; \alpha\left[\hat{\mathcal{Z}}^{\bar{j}}(z,\bar{z})\right] \geq 0 \quad \text{for} \quad j \in \mathbb{Z},$$

$$\text{and} \quad \alpha\left[\hat{\mathcal{Z}}^{h,\bar{h}}(z,\bar{z})\right] \geq 0 \quad \text{for} \quad (h,\bar{h}) \in \mathcal{P}$$

If we find such $\alpha_{m,n}$, then **we conclude that no modular invariant partition function can be exist**. This problem can be converted to the semi-definite programming.

- Assumptions on the spectrum [Collier, Lin, Yin 16]

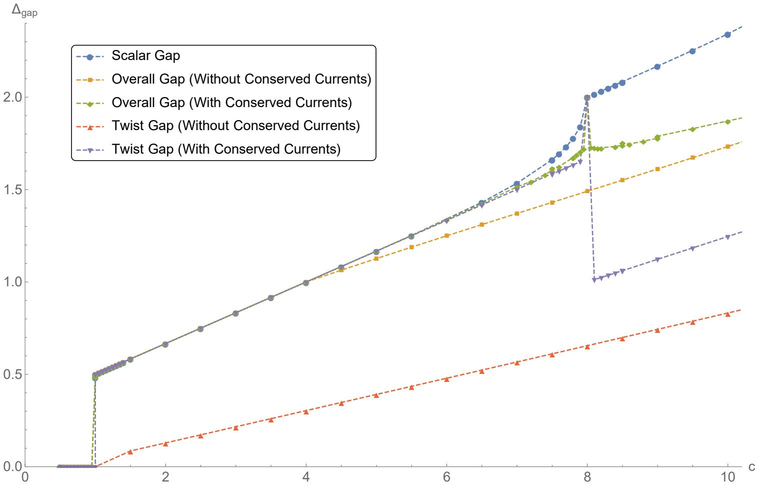
• In the modular bootstrap equation, we sum the primaries $(h, \bar{h}) \in \mathcal{P}$. We can make three different assumptions on $\mathcal{P}$.

Scalar Gap Problem	*Overall Gap Problem*	*Twist Gap Problem*
In this problem, we impose a gap Δ_s only to the scalar operator. $\Delta \geq \Delta_s$ for $j = 0$, $\Delta \geq j$ for $j \neq 0$.	In this problem, we impose a gap Δ_o to the certain low-spin operators. $\Delta \geq \mathrm{Max}(j, \Delta_o)$	In this problem, we impose a gap Δ_t to the twist, defined as $t \equiv \Delta - j$. $\Delta \geq j + \Delta_t$

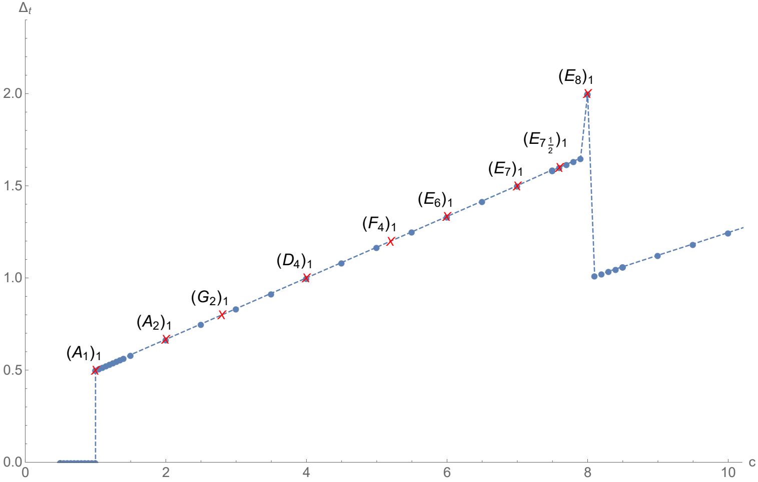
- Additionally, **we impose the contribution of conserved currents** in the modular bootstrap equation.

- For a given c and Δ_{gap} (Δ_s or Δ_o or Δ_t), examine if one can find the numerical solution($\alpha_{m,n}$) to the semi-definite programming or not. The results of this scanning process can be summarized on the two-dimensional plot.

- The Numerical Results ($c \leq 8$)



- The Numerical Results ($c \leq 8$), Focus on the Twist Gap



- Expected CFTs on the numerical bound (Twist Gap)
  - For the Wess-Zumino-Witten model with affine Lie algebra $\hat{\mathfrak{g}}$ and level-k,

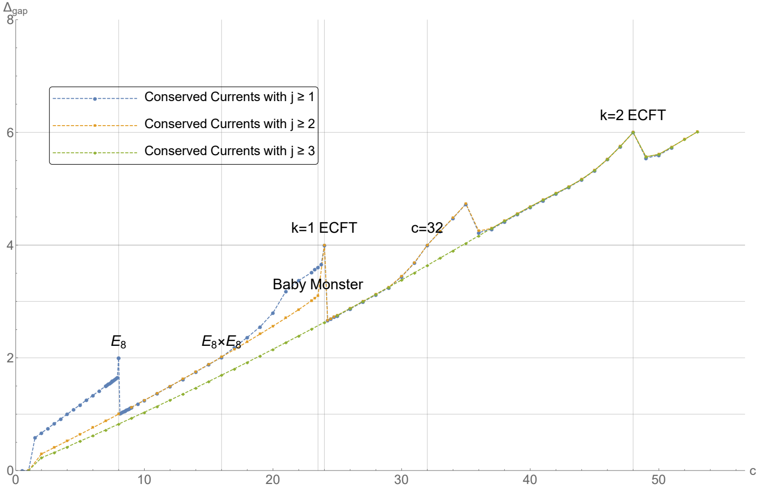
$$c = \frac{k\ \dim\hat{\mathfrak{g}}}{k+h^\vee}, \qquad h_\lambda = \frac{(\lambda, \lambda+2\rho)}{2(k+h^\vee)}$$

  - The above formulae suggest that the **twist gap problem realize level-1 WZW models on the numerical boundary!**($c \leq 8$)

Central Charge	Lowest Primary	Expected CFT
$c=1$	$\Delta_t = 1/2$	$SU(2)_1$ WZW model
$c=2$	$\Delta_t = 2/3$	$SU(3)_1$ WZW model
$c=14/5$	$\Delta_t = 4/5$	$(G_2)_1$ WZW model
$c=4$	$\Delta_t = 1$	$SO(8)_1$ WZW model
$c=26/5$	$\Delta_t = 6/5$	$(F_4)_1$ WZW model
$c=6$	$\Delta_t = 4/3$	$(E_6)_1$ WZW model
$c=7$	$\Delta_t = 3/2$	$(E_7)_1$ WZW model
$c=8$	$\Delta_t = 2$	$(E_8)_1$ WZW model

  - The eight simple Lie group $A_1, A_2, G_2, D_4, F_4, E_6, E_7$ and E_8 are referred to as Deligne's exceptional series.

- The Numerical Result (Twist Gap, $c \leq 54$)



- The Numerical Result (Twist Gap, $c \leq 54$)
  - When the holomorphic currents are included from $j=1$, the following four classes are further realized on the numerical boundary.

Central Charge	Lowest Primary	Expected CFT
$c=16$	$\Delta_t=2$	$(E_8\times E_8)_1$ WZW model

  - When the holomorphic currents are included from $j=2$,

Central Charge	Lowest Primary	Expected CFT
$c=24$	$\Delta_t=4$	Monster CFT
$c=48$	$\Delta_t=6$	"$c=48$ ECFT"
$c=8$	$\Delta_t=1$	CFT with $O_{10}^+(2)$
$c=16$	$\Delta_t=2$	CFT with $O_{10}^+(2)$
$c=47/2$	$\Delta_t=3$	Baby Monster CFT

  - For instance, the unique modular invariant partition function at $c=24$ is,

$$\begin{aligned} Z_{k=1}(q,\bar{q}) &= (j(q)-744)(\bar{J}(\bar{q})-744) \\ &= (1+196884q^2+\cdots)(1+196884\bar{q}^2+\cdots) \end{aligned}$$

- The Modular Differential Equation(MDE)
  - Idea : n characters of rational conformal field theory(RCFT) are the solutions to the n-th order modular differential equation, [Mathur, Muhki, Sen 88]

$$D_\tau^n \chi(\tau) + \sum_{k=0}^{n-1} \phi_k(\tau) D_\tau^k \chi(\tau) = 0,$$

with $D_\tau f(\tau) \equiv \partial_\tau f(\tau) - \frac{\pi i r}{6} f(\tau)$. ($r$ is the modular weight of the test function $f(\tau)$)

- Second Order Modular Differential Equation
  - To get the vacuum character, solve the second order differential equation,

$$D_\tau^2 \chi(\tau) + \hat{\mu} E_4(\tau) \chi(\tau) = 0,$$

with an ansatz $\chi_{\hat{\lambda}}(q) = q^\alpha (a_0 + a_1 q + a_2 q^2 + a_3 q^3 + a_4 q^4 + \cdots)$.

  - The coefficients $\{a_0, a_1, a_2, \cdots\}$ are **positive integer** only for [Mathur, Muhki, Sen 88], [Tuite 08]

$$\boxed{c \in \{\frac{2}{5}, 1, 2, \frac{14}{5}, 4, \frac{26}{5}, 6, 7, \frac{38}{5}, 8\}}.$$

- Third Order Modular Differntial Equation
  - To get the vacuum character, solve the third order differential equation,

$$D_\tau^3\chi(\tau) + \mu_1 E_4(\tau) D_\tau \chi(\tau) + \mu_2 E_6(\tau)\chi(\tau) = 0,$$

with an ansatz $\chi_{\hat{\lambda}}(q) = q^\alpha(a_0 + a_2 q^2 + a_3 q^3 + a_4 q^4 + \cdots)$.

  - The coefficients $\{a_0, a_2, a_3, \cdots\}$ are **positive integer** only for [Mathur, Muhki, Sen 88], [Tuite 08]

$$\boxed{c \in \{-\frac{44}{5}, 8, 16, \frac{47}{2}, 24, 32, \frac{164}{5}, \frac{236}{7}, 40\}}.$$

  - The primary characters have the form of

$$\chi_{h_\pm}(\tau) = q^{h_\pm - \frac{c}{24}}\left[b_0 + b_1 q + b_2 q^2 + \cdots\right]$$

with $h_\pm(c) = \frac{c+4}{16} \pm \frac{\sqrt{368+24c-c^2}}{16\sqrt{31}}$.

  - The coefficients in the primary characters are **not completely fixed** from the modular differential equation.

Spectral Analysis

- Finding the degeneracy bound [Rattazzi, Rychkov, Vichi 10]
  - Rewrite the modular bootstrap equation as

$$\alpha\left[\hat{\mathcal{Z}}_0(z,\bar{z})\right] + d(h^*,\bar{h}^*)\ \alpha\left[\hat{\mathcal{Z}}^{h^*,\bar{h}^*}(z,\bar{z})\right] + \alpha\left[\hat{\mathcal{Z}}^{rest}(z,\bar{z})\right] = 0,$$

$$\alpha\left[\hat{\mathcal{Z}}^{rest}(z,\bar{z})\right] \equiv \sum_{j=j_{min}}^{j_{max}} \left(d(j)\ \alpha\left[\hat{\mathcal{Z}}^{j}(z,\bar{z})\right] + \bar{d}(j)\ \alpha\left[\hat{\mathcal{Z}}^{\bar{j}}(z,\bar{z})\right]\right) + \sum_{h,\bar{h}\in\mathcal{P}} d(h,\bar{h})\ \alpha\left[\hat{\mathcal{Z}}^{h,\bar{h}}(z,\bar{z})\right],$$

  and solve the following problem via the semi-definite programming.

$$\text{Maximize} \quad \alpha\left[\hat{\mathcal{Z}}_0(z,\bar{z})\right], \quad \text{such that} \quad \alpha\left[\hat{\mathcal{Z}}^{h^*,\bar{h}^*}(z,\bar{z})\right] = 1$$

$$\text{and} \quad \alpha\left[\hat{\mathcal{Z}}^{j}(z,\bar{z})\right] \geq 0,\ \alpha\left[\hat{\mathcal{Z}}^{\bar{j}}(z,\bar{z})\right] \geq 0 \quad \text{for} \quad j\in\mathbb{Z},$$

$$\text{and} \quad \alpha\left[\hat{\mathcal{Z}}^{h,\bar{h}}(z,\bar{z})\right] \geq 0 \quad \text{for} \quad (h,\bar{h})\in\mathcal{P}$$

  - This gives **the maximum bound of the degeneracy** of the state with $(h^*,\bar{h}^*)$.

$$\boxed{d(h^*,\bar{h}^*) \leq -\alpha\left[\hat{\mathcal{Z}}_0(z,\bar{z})\right]}$$

- Extremal Functional Method [Paulos, El-Showk 14]

  • Suppose the degeneracies of all primaries saturated the maximum bound. Then, the modular bootstrap equation is reduced to the below form.

$$\sum_{j=j_{min}}^{j_{max}} \left(d(j)\ \beta^* \left[\hat{\mathcal{Z}}^{j}(z,\bar{z}) \right] + \bar{d}(j)\ \beta^* \left[\hat{\mathcal{Z}}^{\bar{j}}(z,\bar{z}) \right] \right) + \sum_{h,\bar{h}\in\mathcal{P}} d(h,\bar{h})\ \beta^* \left[\hat{\mathcal{Z}}^{h,\bar{h}}(z,\bar{z}) \right] = 0$$

  • For the primaries, the above reduced equation forces :

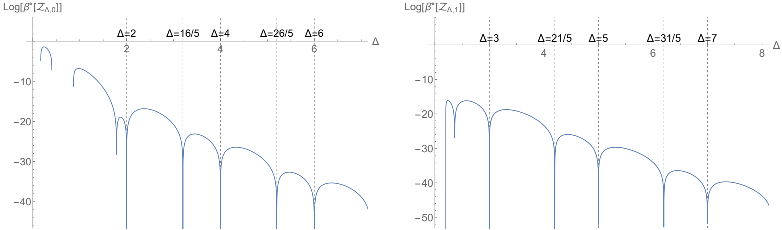
$$d(h,\bar{h})\ \beta^* \left[\hat{\mathcal{Z}}^{h,h}(z,\bar{z}) \right] = 0, \quad \text{for} \quad {}^{\forall}(h,\bar{h}) \in \mathcal{P}.$$

Idea : **Find the states such that** $\beta^* \left[\hat{\mathcal{Z}}^{h,h}(z,\bar{z}) \right] = 0$**!** (Otherwise, $d(h,\bar{h}) = 0$.)

- Spectrum Analysis

  1. Apply the EFM and find the states such that make $\beta^* \left[\hat{\mathcal{Z}}^{h,h}(z,\bar{z}) \right] = 0$.
  2. For those states, find the corresponding maximal degeneracies.
  3. *Assuming every primaries hit the maximal degeneracies*, find the consistent modular invariant partition function.

- F_4 example
  - The EFM analysis applied to the hypothetical CFT with $c = \frac{26}{5}$,



  - From the EFM analysis, the data of spin-0 and spin-1 low-lying primaries are,

$$\Delta_{j=0} \in \{\frac{6}{5} + 2n, 2 + 2n \Big| n \in \mathbb{Z}_{\geq 0}\}, \quad \Delta_{j=1} \in \{\frac{11}{5} + 2n, 3 + 2n \Big| n \in \mathbb{Z}_{\geq 0}\}.$$

  - The solutions to the second order MDE with $c = \frac{26}{5}$ gives :

$$f_0^{c=26/5}(q) = q^{-\frac{13}{60}} \left(1 + 52q + 377q^2 + 1976q^3 + 7852q^4 + \cdots\right),$$
$$f_1^{c=26/5}(q) = q^{\frac{3}{5} - \frac{13}{60}} \left(26 + 299q + 1702q^2 + 7475q^3 + 27300q^4 + \cdots\right).$$

- F_4 example (continued)
  - For each low-lying primaries, the maximum degeneracies are,

$(h,\bar{h})$	Max. Deg	$(h,\bar{h})$	Max. Deg	$(h,\bar{h})$	Max. Deg
$(\frac{3}{5},\frac{3}{5})$	676.0000	$(1,1)$	2704.0000	$(1,0)$	52.00028
$(\frac{3}{5},\frac{8}{5})$	7098.0001	$(2,1)$	16848.001	$(2,0)$	324.0007
$(\frac{3}{5},\frac{13}{5})$	35802.002	$(3,1)$	80444.061	$(3,0)$	1547.0091
$(\frac{8}{5},\frac{8}{5})$	74529.0001	$(2,2)$	104976.005	$(4,0)$	5499.0126

  - The relation between *partition function* and *reduced partition function* is given by,

$$\hat{Z}_{F_4}(q,\bar{q}) = |\tau|^{\frac{1}{2}}\eta(\tau)^2\bar{\eta}(\bar{\tau})^2 Z_{F_4}(q,\bar{q}) - \underbrace{(1-q)(1-\bar{q})}_{\text{Vaccum contribution}}.$$

  - The partition function of $(F_4)_1$ WZW model is known :

$$\boxed{Z_{F_4}(q,\bar{q}) = |f_0^{c=26/5}(q)|^2 + |f_1^{c=26/5}(q)|^2}$$

This perfectly agree with the numerical result.

- The Result Summary ($c \leq 8$)

  • In case of $(G_2)_1, (F_4)_1$ and $(E_7)_1$ WZW model, its modular invariant partition function is known. In terms of the solutions to the second order MDE, they are written as [Gannon 92]

$$Z_{G_2}(q,\bar{q}) = |f_0^{c=14/5}(q)|^2 + |f_1^{c=14/5}(q)|^2$$
$$Z_{F_4}(q,\bar{q}) = |f_0^{c=26/5}(q)|^2 + |f_1^{c=26/5}(q)|^2$$
$$Z_{E_7}(q,\bar{q}) = |f_0^{c=7}(q)|^2 + |f_1^{c=7}(q)|^2$$

  and in case of $(E_6)_1$ WZW model,

$$Z_{E_6}(q,\bar{q}) = f_0^{c=6}(q)\bar{f}_0^{c=6}(\bar{q}) + 2f_1^{c=6}(q)\bar{f}_1^{c=6}(\bar{q})$$

  For them, we checked the spectral analysis successfully reproduce the known partition function.

  • The $(A_1)_1, (A_2)_1, (G_2)_1, (D_4)_1$ and $(E_8)_1$ WZW models are realized by the scalar gap problem (Collier, Lin, Yin 16), it turns out that it also realized by the **twist gap problem**.

- $(E_{7,1/2})_1$ WZW model?
  - $E_{7,1/2}$ is non-simple Lie algebra, its subalgebra is E_7. It splits into $E_7 \oplus 56 \oplus \mathbb{R}$.
  - The degeneracy analysis at $c = \frac{38}{5}$ gives, [Cohen, Man de 96], [Landsberg, Manivel 06]

$(h, \bar{h})$	Max. Deg	$(h, \bar{h})$	Max. Deg	$(h, \bar{h})$	Max. Deg
$(\frac{4}{5}, \frac{4}{5})$	3249.0004	$(1, 1)$	36100.000	$(1, 0)$	190.00412
$(\frac{4}{5}, \frac{9}{5})$	59565.012	$(2, 1)$	501600.00	$(2, 0)$	2640.0481
$(\frac{9}{5}, \frac{9}{5})$	1092025.06	$(2, 2)$	6969600.01	$(3, 0)$	19285.021

- The solutions to the second order MDE with $c = 38/5$ are given by,

$$f_0^{c=38/5}(q) = q^{-\frac{19}{60}} \left(1 + 190q + 2831q^2 + 22306q^3 + 129276q^4 + \cdots\right),$$
$$f_1^{c=38/5}(q) = q^{\frac{4}{5} - \frac{19}{60}} \left(57 + 1102q + 9367q^2 + 57362q^3 + 280459q^4 + \cdots\right).$$

- If there is $(E_{7,1/2})_1$ WZW model, the modular invariant partition function may have the following diagonal form.

$$Z_{E_{7,1/2}}(q, \bar{q}) = |f_0^{c=\frac{38}{5}}(q)|^2 + |f_1^{c=\frac{38}{5}}(q)|^2$$

- Problems of the $(E_{7,1/2})_1$ WZW model [Mathur, Mukhi, Sen 89]

• In case of the two-character RCFT, the modular differential equation admit the exact solution :

$$\chi_0 = \left(\frac{1}{16}\lambda(1-\lambda)\right)^{-\frac{c}{12}} {}_2F_1\left(\frac{1}{3}-\frac{c}{12}, -\frac{c}{4}; \frac{2}{3}-\frac{c}{6}; \lambda\right), \quad \chi_1 = N\left(\frac{1}{16}\lambda(1-\lambda)\right)^{\frac{1}{3}+\frac{c}{12}} {}_2F_1\left(\frac{2}{3}+\frac{c}{12}, 1+\frac{c}{4}; \frac{4}{3}+\frac{c}{6}; \lambda\right)$$

where $\lambda \equiv \frac{\vartheta_2(\tau)^4}{\vartheta_3(\tau)^4}$ and is transform $\lambda \to 1-\lambda$ under S-transformation.

• The S-matrix is defined by $\begin{pmatrix} \chi_0(1-\lambda) \\ \chi_1(1-\lambda) \end{pmatrix} = S \begin{pmatrix} \chi_0(\lambda) \\ \chi_1(\lambda) \end{pmatrix}$. The fusion rule coefficients are can be defined by S-matrix.

$$\mathcal{N}_{ijk} = \sum_n \frac{S_{in}S_{jn}S_{kn}}{S_{0n}}$$

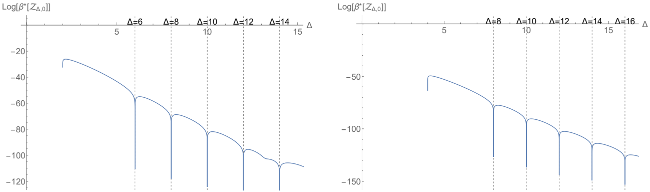
• It turns out that $c = \frac{38}{5}$ and $N = 57$ gives negative fusion rule coefficient($\mathcal{N}_{111}$). To circumvent it, we consider effective theory of $c = -\frac{58}{5}$: switch the role of the vacuum and primary character.

• After that, we have positive fusion rule. But is it reasonable to have 57-fold vacuum states?

- Examine ECFTs via the modular bootstrap
  - CLAIM : Twist gap problem **realize the ECFTs with** $\mathbf{c = 24, 48}$ on the boundary.
  - The partition function of $c = 24$ ECFT is obatined by the solutions to the third order MDE, while the $c = 48$ partition function is realized by the fourth order MDE.

$$c = 24 : Z_{c=24}(q, \bar{q}) = J(q)\bar{J}(\bar{q})$$
$$c = 48 : Z_{c=48}(q, \bar{q}) = (J(q)^2 - 393767)(\bar{J}(\bar{q})^2 - 393767)$$

  - The EFM analysis suggests that all of them have the states with integer Δ.



  - We find that the results of the numerical analysis perfectly matched to the modular invariant partition functions.

- CFTs without Kac-Moody symmetry
  - In the mathematics, the corresponding vertex operator algebra was constructed.

Exceptional Vertex Operator Algebras and the Virasoro Algebra

Michael P. Tuite

$C = 8, d_2 = 155$: This can be realized as the fixed point free lattice VOA V_L^+ (fixed under the automorphism lifted from the reflection isometry of the lattice L) for the rank 8 even lattice $L = \sqrt{2}E_8$. The automorphism group is $O_{10}^+(2).2$ [**G**].

$C = 16, d_2 = 2295$: The VOA V_L^+ for the rank 16 Barnes-Wall even lattice $L = \Lambda_{16}$ whose automorphism group is $2^{16}.O_{10}^+(2)$ [**S**].

$C = 23\frac{1}{2}, d_2 = 96255$: This can be realized as the integrally graded subVOA of Höhn's Baby Monster Super VOA $VB^\natural$ whose automorphism group is the Baby Monster group $\mathbb{B}$ [**Ho2**].

- With an ansatz $f_0(q) = q^\alpha(a_0 + a_2q^2 + a_3q^3 + a_4q^4 + \cdots)$, the solutions to the third order MDE with $c = 8$, $c = 16$ and $c = \frac{47}{2}$ are given by,

$$f_0^{c=8}(q) = q^{-1/3}\left(1 + 156q^2 + 1024q^3 + 6790q^4 + 32768q^5 + \cdots\right)$$

$$f_0^{c=16}(q) = q^{-2/3}\left(1 + 2296q^2 + 65536q^3 + 1085468q^4 + \cdots\right)$$

$$f_0^{c=47/2}(q) = q^{-47/48}\left(1 + 96256q^2 + 9646891q^3 + 366845011q^4 + \cdots\right)$$

- The partition function of $c=8$ CFT without Kac-Moody symmetry
  - The degeneracy analysis without conserved current of $j=1$ gives,

$(h,\bar{h})$	Max. Deg	$(h,\bar{h})$	Max. Deg	$(h,\bar{h})$	Max. Deg
$(\frac{1}{2},\frac{1}{2})$	496.0000000	$(1,1)$	33728.00000	$(2,0)$	155.000000
$(\frac{1}{2},\frac{3}{2})$	17360.00000	$(2,1)$	505920.0000	$(3,0)$	868.000000
$(\frac{3}{2},\frac{3}{2})$	607600.0009	$(2,2)$	7612825.000	$(4,0)$	5610.00000

  - The other two solutions to the third order MDE with $c=8$ are :

$$f_{h=1/2}(\tau)=a_0 q^{1/6}\Big(1+36q+394q^2+2776q^3+15155q^4+\cdots\Big),$$
$$f_{h=1}(\tau)=a_1 q^{2/3}\Big(1+16q+136q^2+832q^3+4132q^4+\cdots\Big)$$

  - Our numerical results suggest that the modular invariant partition function reads,

$$Z_{c=8}=f^{c=8}_{h=0}(\tau)\bar{f}^{c=8}_{h=0}(\bar{\tau})+496f^{c=8}_{h=1/2}(\tau)\bar{f}^{c=8}_{h=1/2}(\bar{\tau})|_{a_0=1}+33728f^{c=8}_{h=1}(\tau)\bar{f}^{c=8}_{h=1}(\bar{\tau})|_{a_1=1}.$$
$$=1+\underbrace{496}_{1+155+340}q^{\frac{1}{2}}\bar{q}^{\frac{1}{2}}+\underbrace{17856}_{2\times155+2\times868+15810}q^{\frac{3}{2}}\bar{q}^{\frac{1}{2}}+\underbrace{33728}_{2108+31620}q\bar{q}+\underbrace{539648}_{539648}q^2\bar{q}+\cdots$$

- The partition function of $c = 16$ CFT without Kac-Moody symmetry
  - The degeneracy analysis without conserved current of $j = 1$ gives, gives,

$(h, \bar{h})$	Max. Deg	$(h, \bar{h})$	Max. Deg	$(h, \bar{h})$	Max. Deg
$(\frac{3}{2}, \frac{3}{2})$	32505856.0032	$(1, 1)$	134912.0000	$(2, 0)$	2295.00000
$(\frac{3}{2}, \frac{5}{2})$	1657798656.0001	$(2, 1)$	18213120.00	$(3, 0)$	63240.0000
$(\frac{3}{2}, \frac{7}{2})$	34228666368.005	$(2, 2)$	2464038225.003	$(4, 0)$	1017636.00

  - The other two solutions to the third order MDE with $c = 16$ are :

$$f_{h=1} = b_0 q^{1/3}\Big(1 + 136q + 4132q^2 + 67712q^3 + 770442q^4 + \cdots\Big),$$
$$f_{h=3/2} = b_1 q^{5/6}\Big(1 + 52q + 1106q^2 + 14808q^3 + 147239q^4 + \cdots\Big)$$

  - Our numerical results suggest that the modular invariant partition function reads,

$$Z_{c=16} = f^{c=16}_{h=0}(\tau)\bar{f}^{c=16}_{h=0}(\bar{\tau}) + 134912 f^{c=16}_{h=1}(\tau)\bar{f}^{c=16}_{h=1}(\bar{\tau})|_{b_0=1} + 32505856 f^{c=16}_{h=3/2}(\tau)\bar{f}^{c=16}_{h=3/2}(\bar{\tau})|_{b_1=1}.$$
$$= 1 + \underbrace{2296}_{2\times 1+186+2108} q^2 + \underbrace{65536}_{2\times 1+186+14756+50592} q^3 + \underbrace{134912}_{186+340+868+22858+110670} q\bar{q} + \cdots$$

- Baby Monster CFT [Höhn 07]
  - The degeneracies with $c = \frac{47}{2}$ reads,

$(h, \bar{h})$	Max. Deg $(h, \bar{h})$	$(h, \bar{h})$	Max. Deg
$(\frac{3}{2}, \frac{3}{2})$	19105641.026984403127	$(\frac{5}{2}, \frac{5}{2})$	1298173112605.3499336
$(2, 2)$	9265025041.322733803	$(\frac{31}{16}, \frac{31}{16})$	9265217540.6086142750
$(\frac{5}{2}, \frac{3}{2})$	4980203754.2560961756	$(\frac{47}{16}, \frac{31}{16})$	1011288637613.8107313

  - The three solutions to the third order MDE with $c = \frac{47}{2}$ are given by,

$$f^{c=47/2}_{h=3/2} = q^{25/48} a_1 \left(1 + \frac{785}{3} q + \frac{44393}{3} q^2 + 418441 q^3 + \frac{23301881}{3} q^4 + \cdots \right)$$
$$f^{c=47/2}_{h=31/16} = q^{23/24} a_2 \left(1 + \frac{5177}{47} q + 4372 q^2 + 100627 q^3 + 1625207 q^4 + \cdots \right)$$

  - Corresponding modular invariant partition function reads,

$$Z_{c=47/2} = f^{c=47/2}_{h=0}(\tau) \bar{f}^{c=47/2}_{h=0}(\bar{\tau}) + f^{c=47/2}_{h=3/2}(\tau) \bar{f}^{c=47/2}_{h=3/2}(\bar{\tau}) \Big|_{a_1=\sqrt{4371}} + f^{c=47/2}_{h=31/16}(\tau) \bar{f}^{c=47/2}_{h=31/16}(\bar{\tau}) \Big|_{a_2=\sqrt{96256}}$$
$$= 1 + \underbrace{96256}_{1+96255} q^2 + \underbrace{9646891 q^3}_{2\times 1 - 4371 + 2\times 96255 + 9458750} + \underbrace{19105641 q^{3/2} \bar{q}^{3/2}}_{1+96255+9458750+9550635} + \cdots .$$

- Comments on the "dual" CFT description
  - Ising model versus Babymonster CFT

	c	h_1	h_2
Ising model	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{16}$
Baby monster CFT	$\frac{47}{2}$	$\frac{3}{2}$	$\frac{31}{16}$
Sum	24	2	2

Ising model and Baby monster CFT are related via bilinear relation : [Hampapura, Mukhi 16]

$$j(\tau) - 744 = \chi_{\mathrm{VB}^\natural_{(0)}}(\tau)\chi^{\mathrm{Ising}}_{\mathrm{vac}}(\tau) + \chi_{\mathrm{VB}^\natural_{(1)}}(\tau)\chi^{\mathrm{Ising}}_{h=\frac{1}{2}}(\tau) + \chi_{\mathrm{VB}^\natural_{(3)}}(\tau)\chi^{\mathrm{Ising}}_{h=\frac{1}{16}}(\tau)$$

- $c = 8$ and $c = 16$ CFTs without Kac-Moody symmetry are related by bilinear relation :

$$j(\tau) - 744 = \left(f^{c=8}_{h=0}\right)\left(f^{c=16}_{h=0}\right) + \left(f^{c=8}_{h=1}\Big|_{a_1=\sqrt{33728}}\right)\left(f_{h=1}\Big|^{c=16}_{b_0=\sqrt{134912}}\right)$$
$$+ \left(f^{c=8}_{h=1/2}\Big|_{a_0=\sqrt{496}}\right)\left(f^{c=16}_{h=3/2}\Big|_{b_1=\sqrt{32505856}}\right)$$

Application to the $\mathcal{W}$-algebra cases

- Bootstrapping with $\mathcal{W}$-algebra

• In case of the $\mathcal{W}(2,3)$-algebra, we have spin-3 generator W_n. The corresponding fugacity $p = e^{2\pi i z}$ should be introduced in the character.

$$\chi_{(h,w;c)}(\tau, z) = \mathrm{Tr}_{h,w}(q^{L_0 - \frac{c}{24}} p^{W_0})$$
$$= \mathrm{Tr}_{h,w}(q^{L_0 - \frac{c}{24}}) + \alpha_1 \mathrm{Tr}_{h,w}(q^{L_0 - \frac{c}{24}} W_0) + \alpha_2 \mathrm{Tr}_{h,w}(q^{L_0 - \frac{c}{24}} W_0^2) + \cdots$$

The modular transformation property is only known up to W_0^2 order. [Iles, Watts 14]
We will focus on the **unrefined character** which means W_0-zeroth order character.

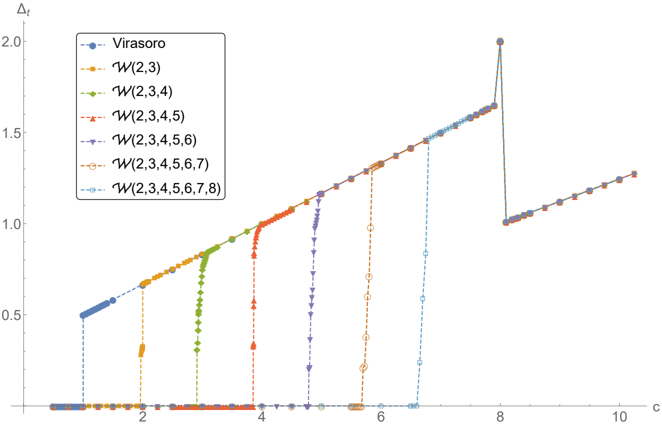
$$\chi_0(\tau) = \frac{q^{-\frac{c-2}{24}}(1-q)^3(1+q)}{\eta(\tau)^2}, \qquad \chi(\tau) = \frac{q^{h-\frac{c-2}{24}}}{\eta(\tau)^2}$$

because of the null states $\langle 0 | L_1 L_{-1} | 0 \rangle = 0$, $\langle 0 | W_1 W_{-1} | 0 \rangle = 0$ and $\langle 0 | W_2 W_{-2} | 0 \rangle = 0$.

• Assuming the non-vacuum module is non-degenerate, the unrefined character of rank-r $\mathcal{W}(d_1, d_2, \cdots d_r)$-algebra is given by,

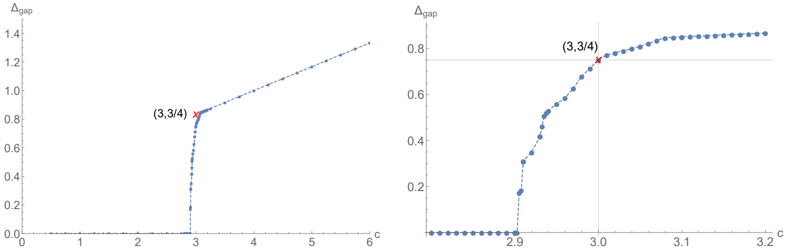
$$\chi_0(\tau) = \frac{q^{-\frac{c-N+1}{24}}}{\eta(\tau)^{N-1}} \prod_{j=1}^{r} \prod_{i=1}^{f_j - 1} (1-q^i), \quad \chi(\tau) = \frac{q^{h-\frac{c-N+1}{24}}}{\eta(\tau)^{N-1}}.$$

- The Numerical Bounds(Twist Gap)



- The numerical bound at $c \geq r$ is identical to the one obtained from the Virasoro character. This results suggest that the unitary irreducible representations of $\mathcal{W}(d_1, d_2, \cdots, d_r)$-algebra do not contain any nontrivial null states when $c \geq r$.

- Numerical bound with Rank-3 $\mathcal{W}$-algebra



• $(c = 3, \Delta = \frac{3}{4})$ sits on the numerical boundary that obtained using the unrefined character of rank-3 $\mathcal{W}(2,3,4)$ algebra. Note that $c = 3$ is not in the list of the two-character RCFTs.

• The hypothetical CFT with $c = 3$ can be identified to the $(A_3)_1$ WZW model. This theory is realized by third order MDE, with Kac-Moody symmtery.

• CLAIM : The twist gap problem with $\mathcal{W}_{2,3,4}$ algebra EXCLUSIVELY realize $(A_3)_1$ WZW model on the numerical bound!

- Spectral Analysis on $(A_3)_1$ WZW model
  - The maximal degeneracies are :

$(h, \bar{h})$	Max. Deg	$(h, \bar{h})$	Max. Deg
$(\frac{3}{8}, \frac{3}{8})$	32.00000	$(\frac{1}{2}, \frac{1}{2})$	36.000000
$(\frac{3}{8}, \frac{11}{8})$	96.00000	$(\frac{1}{2}, \frac{3}{2})$	48.00000
$(\frac{11}{8}, \frac{11}{8})$	288.01585	$(\frac{3}{2}, \frac{3}{2})$	64.11818

  - The characters of (A_3) WZW model reads,

$$\chi^{A_3}_{[0]}(q) = q^{-\frac{1}{8}}\left(1 + 15q + 51q^2 + 172q^3 + 453q^4 + 1128q^5 + \cdots\right),$$
$$\chi^{A_3}_{[1]}(q) = q^{\frac{3}{8}-\frac{1}{8}}\left(4 + 24q + 84q^2 + 248q^3 + 648q^4 + 1536q^5 + \cdots\right),$$
$$\chi^{A_3}_{[2]}(q) = q^{\frac{1}{2}-\frac{1}{8}}\left(6 + 26q + 102q^2 + 276q^3 + 728q^4 + 1698q^5 + \cdots\right),$$

  - We find the maximal degeneracies are perfectly agree with the degeneracies in the below partition function.

$$\boxed{Z_{A_3}(q, \bar{q}) = |\chi^{A_3}_{[0]}(q)|^2 + 2|\chi^{A_3}_{[1]}(q)|^2 + |\chi^{A_3}_{[2]}(q)|^2}$$

- Conclusion and Outlook

• The twist gap problem with holomorphic currents $(j \geq 1)$ successfully realize two-character RCFTs and three-character RCFTs on the numerical bound. The various RCFTS include level-one WZW models and extremal conformal field theories.

• When the holomorphic currents are included from $j = 2$, the CFTs without Kac-Moody symmetry are realized on the numerical boundary. It include $c = 8$, $c = 16$ CFTs and baby monster CFT. **We suggest the modular invariant partition function of those theories** based on the numerical results.

The coefficients in partition function can be decomposed by the dimension of irrpes of the $O^+_{10}(2)$ or baby monster group. They are expected to be a underlying symmetry of the three special theories.

• The numerical analysis extended to the $\mathcal{W}$-algebra cases, using the unrefined character. The numerical bounds suggest the absence of the degenerate states in unitary irreducible representation when $c \geq r$.

• Application to the supersymmetric cases : Can we examine the super WZW models, super extremal conformal field theory? Unexpected super-RCFTs?