# Holographic Transport and Black Hole Horizons

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In holography some very special quantities of the dual CFT are captured, universally, by the black hole horizon:

Equilibrium quantities:

- The temperature of the CFT is given by the surface gravity

$$T_H = \frac{\kappa}{2\pi}$$

- The entropy of the CFT is given by the area of the horizon

$$S_{BH} = \frac{A}{4G}$$

- Conserved charges

$$d * F = 0 \quad \Rightarrow \quad \int_\infty *F = \int_H *F$$

In seeking applications of holography to real materials, thermoelectric conductivities are important observables

$$\begin{pmatrix} J^i \\ Q^i \end{pmatrix} = \begin{pmatrix} \sigma^{ij} & T\alpha^{ij} \\ T\bar{\alpha}^{ij} & T\bar{\kappa}^{ij} \end{pmatrix} \begin{pmatrix} E_j \\ \zeta_j \end{pmatrix}$$

$\zeta \leftrightarrow -(\nabla T)/T$

DC conductivity matrix is universally obtained exactly by solving generalised, linearised Navier-Stokes equations for an incompressible charged fluid on the curved black hole horizon

[Donos,JPG]

- Not a special case of fluid-gravity correspondence
- Can view as an exact version of old membrane paradigm
  [Damour][McDonald,Price,Thorne][...]

$$\boxed{\begin{pmatrix} J^i \\ Q^i \end{pmatrix} = \begin{pmatrix} \sigma^{ij} & T\alpha^{ij} \\ T\bar{\alpha}^{ij} & T\bar{\kappa}^{ij} \end{pmatrix} \begin{pmatrix} E_j \\ \zeta_j \end{pmatrix}}$$

$\zeta \leftrightarrow -(\nabla T)/T$

Naively the DC conductivity is an IR observable but it depends on UV physics

e.g. CFT on Minkowski space is described by the AdS Schwarzschild black hole

Translational invariance $\Rightarrow$ momentum conserved $\Rightarrow \bar{\kappa}_{DC} = \infty$

More precisely, $Re[\bar{\kappa}(\omega)] \sim \delta(\omega)$

General framework for dissipating momentum:

## Holographic Lattices

CFT with a deformation by an operator that explicitly breaks translation invariance.

[Horowitz, Santos, Tong] [.........]

- For example, consider D=4 black holes with bulk scalar field

$$\phi(r,x^i) \to \frac{\phi_s(x^i)}{r} + \frac{v(x^i)}{r^2} + \ldots \qquad r \to \infty$$

Corresponds to a deformation of the d=3 CFT:

$$L_{CFT} \to L_{CFT} + \int dx \phi_s(x) \mathcal{O}(x)$$

# Holographic lattices are interesting

The lattice deformation can lead to:

- 'Coherent metals' aka Drude physics - smoothed out $\delta(\omega)$

  [Hartnoll,Hoffman][Horowitz, Santos,Tong][…]

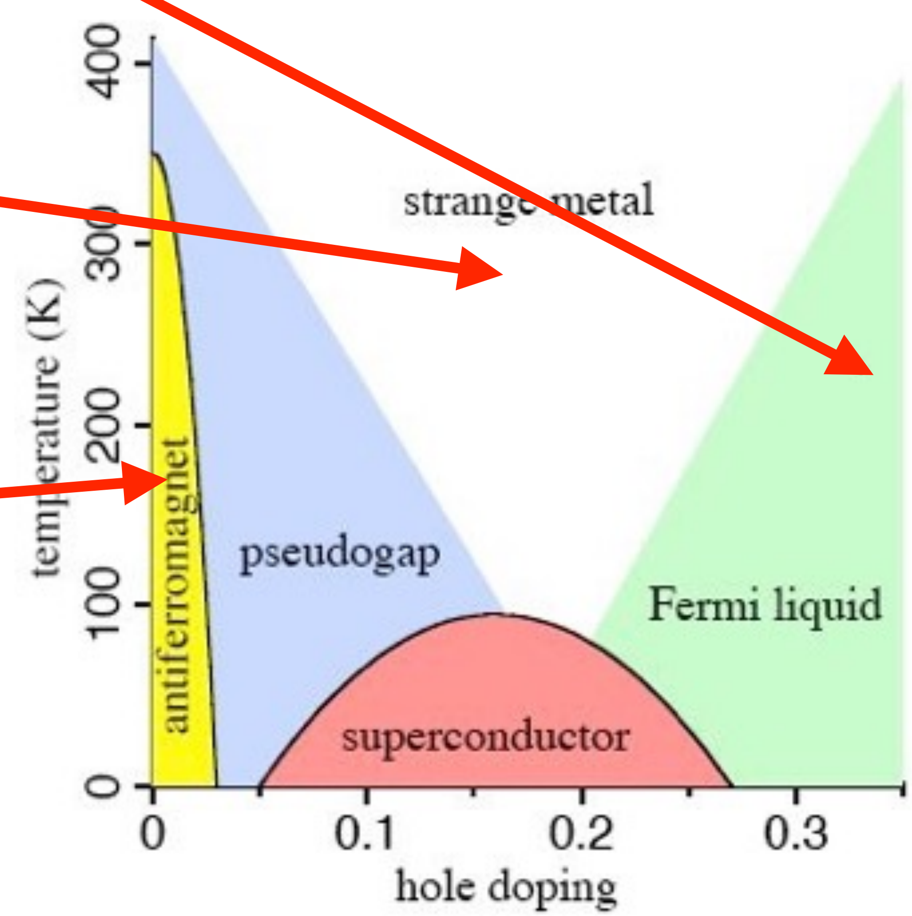
  Arises when momentum is nearly conserved.

- Novel 'incoherent' metals

  [Donos,JPG][Gouteraux][…..]

- Insulators and M-I transitions

  [Donos,Hartnoll][Donos,JPG][……]



## Plan of talk

- DC conductivity matrix by solving Stokes equations on the curved black hole horizon

- Comments and generalisations

- What else can we get from the horizon?

Dispersion relation for hydrodynamic modes associated with diffusion of charge - derive generalised Einstein relation from Einstein equations

Note recent work on studying diffusion and relations to quantum chaos: $D \sim v_B^2 \tau_L$

[Hartnoll][Blake][Sachdev,Lucas…..]

# DC Conductivity and Stokes equations

Illustrate with D=4 Einstein-Maxwell Theory

$$S=\int d^4x\sqrt{-g}\Big[R+6-\frac{1}{4}F^2+\dots\Big]$$

Dual to d=3 CFT with global U(1) symmetry

Assume for now:

• There is a single, static black hole Killing horizon with planar topology and periodic deformations

Dual to deformed CFT in thermal equilibrium, preserving T-reversal invariance e.g. no magnetic fields

# DC Conductivity and Stokes Flows

- Holographic lattice black holes - planar horizons

$$ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + g_{ij}dx^i dx^j$$

$$A = A_t dt$$

with $g_{\mu\nu}$ and $A_t$ functions of $(r, x^i)$, periodic in the $x^i$

- Behaviour at AdS boundary $r \to \infty$

$$ds^2 \to r^{-2}dr^2 + r^2[g^{\infty}_{tt}(x)dt^2 + g^{\infty}_{ij}(x)dx^i dx^j]$$

$$A \to A^{\infty}_t(x)dt$$

i.e. sources for boundary $T^{tt}$, $T^{ij}$ and $J^t$

- Killing horizon at $r = 0$: is spatially modulated (in general)

- Setup: Perturb the black hole by DC sources $E_i, \ \zeta_i$

$$\delta(ds^2) = \delta g_{\mu\nu} dx^\mu dx^\nu + 2t g_{tt} \zeta_i dt dx^i$$

$$\delta A = \delta a_\mu dx^\mu - tE_i dx^i + tA_t \zeta_i dx^i$$

- Behaviour at AdS boundary:

  The only sources are $E_i \quad \zeta_i \qquad\qquad \zeta \leftrightarrow -(\nabla T)/T$

- Behaviour at the horizon: perturbation is regular

- Naively: now solve full bulk perturbation. However…

• Result 1: Zero modes of the electric and heat currents at the horizon are equal the zero modes of those in the CFT

Illustrate with electric currents:

Bulk equations of motion $\nabla_\mu F^{\mu\nu} = 0$

Define the bulk electric current density as $J^i = \sqrt{-g} F^{ir}$

$$\boxed{\partial_i J^i = 0\,, \qquad \partial_r J^i = \partial_j \left(\sqrt{-g} F^{ji}\right)}$$

At AdS boundary, $J^i(x)|_\infty$, is local current of dual CFT

Observe zero mode $\bar{J}^i = \int d^2x J^i$ is independent of radius

$$\boxed{\bar{J}^i|_\infty = \bar{J}^i_0}$$

• Result 2: Obtain local currents $J^i(x)|_0$ , $Q^i(x)|_0$ on horizon as functions of $E_i$ , $\zeta_i$ and hence DC conductivity of CFT

Use Hamiltonian decomposition of equations of motion with respect to the radial coordinate:

$$\mathcal{H} = N\,H + N_\mu\,H^\mu + \Phi\,C\,,$$

Evaluate constraints at the "stretched horizon"

Find a decoupled sector for a subset of the perturbation which forms a closed set of equations.
Moreover, they give $J^i|_0$ and $Q^i|_0$

Define

$$\boxed{(v^i, p, w) \quad \leftrightarrow \quad (\delta g_{it}^{(0)}, \delta g_{rt}^{(0)}, \delta a_t^{(0)})}$$

$$Q_0^i = 4\pi T\sqrt{h}v^i$$

$$J_0^i = \sqrt{h}[a_t^{(0)}v^i + h^{ij}(E_j + \partial_j w)]$$

Find system of Stokes equations: [Donos,JPG]

$$\boxed{\begin{aligned} &\partial_i Q_0^i = 0 \\ &\partial_i J_0^i = 0 \\ &-2\nabla^i\nabla_{(i}v_{j)} = (4\pi T\,\zeta_j - \nabla_j\, p) + a_t^{(0)}(E_j + \nabla_j w) \end{aligned}}$$

Linear, time-independent, forced Navier-Stokes equations for a charged, incompressible fluid on the curved black hole horizon

# Comments I

- The formalism can be generalised from Einstein-Maxwell to any two derivative theory of gravity in holography

E.g. Scalar fields $\phi$ give extra viscous terms appear in the Stokes equations $\nabla_j \phi^{(0)} \nabla_i \phi^{(0)} v^i$ [Banks,Donos,JPG]

where $\phi^{(0)}(x)$ is the value of the scalar on the horizon

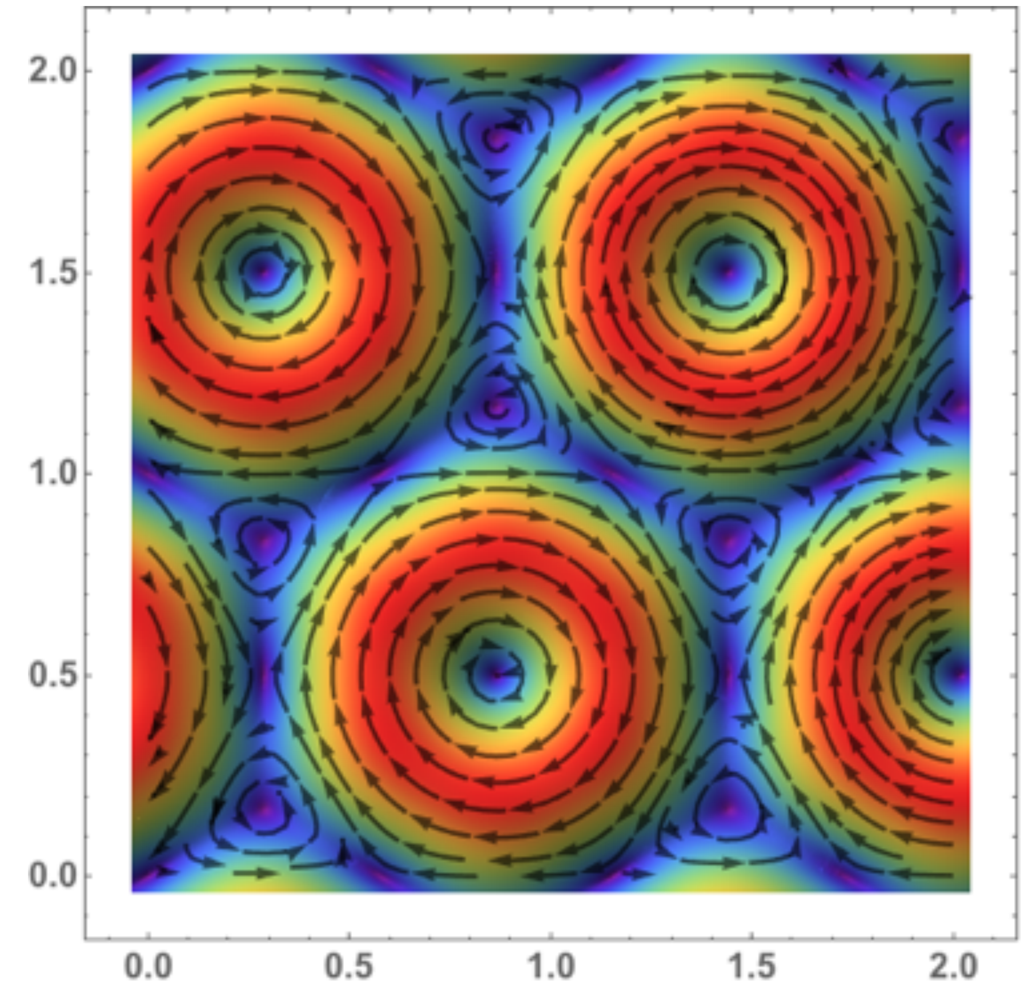
# Comments II

- Break T-reversal invariance [Donos,JPG,Griffin,Melgar]

Allow for equilibrium holographic lattices with magnetic fields but also local magnetisation currents and heat magnetisation currents

$$J_\infty^{(B)i} = \partial_j M^{(B)ij}$$

$$Q_\infty^{(B)i} = \partial_j M_T^{(B)ij}$$




Find: extra Lorentz and Coriolis terms in Stokes equations on the horizon

# Comments III

- Onsager relations [Donos,JPG,Griffin,Melgar]

$$\sigma^T(S) = \sigma(S^t) \qquad \alpha^T(S) = \bar{\alpha}(S^t) \qquad \bar{\kappa}^T(S) = \bar{\kappa}(S^t)$$

Prove by obtaining the Stokes equations from a novel variational principle, which is a functional of the fluid degrees of freedom and the time-reversed degrees of freedom.

# Comments IV

[Donos,JPG,Griffin,Melgar]

Generalise to higher derivative theories of gravity

E.g. Einstein-Gauss-Bonnet gravity in D dimensions, coupling $\tilde{\alpha}$

$$\boxed{\begin{aligned}&\nabla_i(\delta^i_j - 4\tilde{\alpha}G^i_j)v^j = 0\\ &-2\nabla^i\left(S^{kl}_{ij}\nabla_k v_l\right) = \left(\delta^i_j - 4\tilde{\alpha}G^i_j\right)\left(4\pi T\zeta_i - \nabla_i p\right)\end{aligned}}$$

where

$$S^{kl}_{ij} = [1 - \tilde{\alpha}2(D-4)(D-1)]\,\delta_i^{(k}\delta_j^{l)} + \tilde{\alpha}\left[2h_{ij}R^{kl} + 4\delta^{(k}_{(i}R^{l)}_{j)} + 4R_i{}^{(k}{}_j{}^{l)}\right]$$

The $\tilde{\alpha}$ corrected shear viscosity

[Brigante,Liu,Myers,Shenker,Yaida]

$h_{ij}$ Horizon metric

$G_{ij}$ and $R_{ijkl}$ Horizon metric Einstein and Riemann tensor

## Comments V

There are some special holographic lattices for which we can say more about the Stokes equations [Banks,Donos,JPG,Griffin,Melgar,…

- a) Q-lattices - can solve Stokes equations in closed form
- b) One dimensional lattices - can also solve in closed form
- c) Perturbative lattice (Drude physics)
  - can solve Stokes equations perturbatively and obtain general results on leading order effects
- d) Hydrodynamic limit of holographic lattice: $k_L/T \ll 1$

  Fluid-gravity also valid in this limit and the Stokes equations can be expressed in terms of UV data
- e) Interesting bounds on conductivity can be found

[Grozdanov,Lucas,Sachdev,Schalm]

# Diffusion modes

Holographic lattices

- Momentum not conserved

- Charge is conserved $\Rightarrow$ quasinormal modes associated with diffusion of charge

  Hydrodynamic modes $\omega, k_i \to 0$ and should exist for any $k_L$

- Satisfy Einstein relation?

$$D = \sigma\chi^{-1}$$

Quick hydrodynamic argument:

- Conservation law $\partial_t \rho + \nabla_i j^i = 0$
- Constitutive relation $j^i = \sigma^{ij} \nabla_j \mu = \sigma^{ij} \chi^{-1} \nabla_j \rho$
- Diffusion equation

$$\partial_t \rho = D^{ij} \nabla_i \nabla_j \rho \qquad D^{ij} = \sigma^{ij} \chi^{-1}$$

Can include heat currents and get another diffusive mode (coupled) [Hartnoll]

Issues:

- To get finite results need momentum dissipation
- Not in hydrodynamic limit in general
- "Conductivity" in Stokes equations is not the CFT conductivity
- Diffusion in inhomogeneous media is somewhat murky

Construct time source free quasinormal perturbation

$$P \equiv \{g_{\mu\nu}, A_\mu, \phi\}$$

- Time dependent ansatz $\delta P(t,r,x) = e^{-i\omega[t+S(r)]}\delta\hat{P}(r,x)$

  Ingoing boundary conditions at horizon, source free at $\infty$

- Einstein equations: radial equations plus constraints at horizon

$$\partial_i J_0^i = i\omega(\dots) \qquad \partial_i Q_0^i = i\omega(\dots)$$

$$-2\nabla^j\nabla_{(i}v_{j)}\dots = i\omega(\dots)$$

Set $i\omega = 0$ get Stokes equations with $E_i = \zeta_i = 0$

No longer a closed systems of equations when $\omega \neq 0$
Nevertheless we get dispersion relation for diffusion modes!

Solution:

- Identify the quasinormal mode at $\omega = k_i = 0$

  Start with background holographic lattice solution

$$P_{BG}(r,x)[\mu,T]$$

  Generate static perturbation via $\begin{array}{l} T \to T + \delta T \\ \mu \to \mu + \delta\mu \end{array}$

  Correct with diffeos and gauge transformation to respect b.c.'s

- Solve constraint equations in long wavelength expansion

  Data on horizon for DC conductivity appear

  Data on horizon for static susceptibilities appear

Result:

- Susceptibilities $c_\rho \equiv T(\partial s/\partial T)_\rho \qquad \chi \equiv (\partial\rho/\partial\mu)_T$

$$\xi \equiv (\partial s/\partial\mu)_T = (\partial\rho/\partial T)_\mu$$

- Define $\bar{\kappa}(k) = \bar{\kappa}^{ij} k_i k_j, \quad \alpha(k) = \alpha^{ij} k_i k_j, \quad \sigma(k) = \sigma^{ij} k_i k_j$

$$\kappa(k) \equiv \bar{\kappa}(k) - \frac{\alpha^2(k) T}{\sigma(k)}$$

Dispersion relation for coupled diffusion modes

$$\boxed{\begin{aligned} i\omega^+ \, i\omega^- &= \frac{\kappa(k)}{c_\rho} \frac{\sigma(k)}{\chi} \\ i\omega^+ + i\omega^- &= \frac{\kappa(k)}{c_\rho} + \frac{\sigma(k)}{\chi} + \frac{T\,[\chi\,\alpha(k) - \xi\,\sigma(k)]^2}{c_\rho \chi^2 \sigma(k)} \end{aligned}}$$

## Final Comments

• DC thermoelectric conductivity can be obtained by solving Stokes equations on black hole horizons

• Diffusion modes can also be obtained - also can be found in spontaneous case

• What else can be found at the horizon?