SUPERSYMMETRY & GEOMETRY OF STRINGY CORRECTIONS

Plan:

- Motivation [works with J.T. Liu; T. Pugh & R. Savelli]
  - ▷ R^4 corrections
    - ◇ dualities, compactifications ($\mathcal{N} = 2$)
  - ▷ exotic kynematics and $\mathcal{N} = 1$ 4D physics
    - ◇ Higher-derivative couplings with varying dilaton-axion
    - ◇ more to come (?)
- Supersymmetry and generalised Lichnerowicz formula [with A. Coimbra, H. Triendl & D. Waldram; A. Coimbra]
  - ◇ Heterotic strings
  - ◇ M-theory

R^4 corrections in string theory

⋆ "CP-odd" part:

- (Heterotic strings: GS terms)
- M5 (NS5) anomalies $\sim C_3(B_2) \wedge [\frac{1}{4}p_1^2(TX) - p_2(TX)]$

⋆ CP-even part

- tree-level: $e^{-2\phi}(t_8 t_8 + \frac{1}{4}\epsilon_8\epsilon_8)R^4$
- one loop: $(t_8 t_8 \mp \frac{1}{4}\epsilon_8\epsilon_8)R^4$ (IIA/IIB)
  $(t_8\epsilon_8 + \epsilon_8 t_8)R^4 \sim [\frac{1}{4}p_1^2(TX) - p_2(TX)]$
- D-instanton contributions for IIB: $f(\rho,\bar\rho) = \sum_{mn\in\mathbb{Z}} \frac{\mathrm{Im}\rho^{3/2}}{|m+n\rho|^3}$ for $\rho = C_0 + ie^{-\phi_{10}}$

⋆ susy completions

- checked at linearized level
- $\mathcal{N}=1$ superinvariants:

$$J_0(\Omega) = \left(t_8 t_8 + \frac{1}{8}\epsilon_{10}\epsilon_{10}\right) R^4$$

$$J_1(\Omega) = t_8 t_8 R^4 - \frac{1}{4}\epsilon_{10} t_8 B R^4$$

Summary of type II $(\alpha')^3$ one-loop couplings (10D):

	No B	With B
e-o **+** **o-e**	$\frac{1}{8}(t_8\epsilon_{10}+\epsilon_{10}t_8)BR^4$ $= B\wedge X_8(\Omega^{\rm LC})$ $= \frac{1}{192(2\pi)^4}B\wedge\left(\text{tr}\,R^4-\frac{1}{4}(\text{tr}\,R^2)^2\right)$	$B\wedge X_8(\Omega^{\rm LC})$ + exact terms ?
e-e	$t_8t_8R^4$	?
o-o	$\frac{1}{8}\epsilon_{10}\epsilon_{10}R^4$	??

◇ $t_8t_8R^4 = t_{\mu_1\cdots\mu_8}t_{\nu_1\cdots\nu_8}R^{\mu_1\mu_2}{}_{\nu_1\nu_2}R^{\mu_3\mu_4}{}_{\nu_3\nu_4}R^{\mu_5\mu_6}{}_{\nu_5\nu_6}R^{\mu_7\mu_8}{}_{\nu_7\nu_8}$

  * $t_8M^4 = 24\left(\text{tr}\,M^4-\frac{1}{4}(\text{tr}\,M^2)^2\right)$

◇ $\epsilon_{10}\epsilon_{10}R^4 = \epsilon_{\alpha\beta\mu_1\cdots\mu_8}\epsilon^{\alpha\beta\nu_1\cdots\nu_8}R^{\mu_1\mu_2}{}_{\nu_1\nu_2}R^{\mu_3\mu_4}{}_{\nu_3\nu_4}R^{\mu_5\mu_6}{}_{\nu_5\nu_6}R^{\mu_7\mu_8}{}_{\nu_7\nu_8}$

◇ At linearised level (5 & 4-pt functions at one-loop) : $\Omega^{\rm LC} \longrightarrow \Omega_\pm$

◇ Curvature: $R(\Omega_\pm)_{\mu\nu}{}^{\alpha\beta} = R_{\mu\nu}{}^{\alpha\beta} \pm \nabla_{[\mu}H_{\nu]}{}^{\alpha\beta} + \frac{1}{2}H_{[\mu}{}^{\alpha\gamma}H_{\nu]\gamma}{}^{\beta}$

  * H closed $\Rightarrow$ $R(\Omega_+)_{\mu\nu\alpha\beta} = R(\Omega_-)_{\alpha\beta\mu\nu}$

$\mathcal{N}=2$: M-theory/IIA on Calabi-Yau threefolds

- 4D quantum corrected effective action:

$$S = \frac{1}{2\kappa_4^2} \int d^4x\sqrt{g^\sigma}\Big[\Big((1+\tfrac{\chi_T}{v_6})\,e^{-2\phi_4}-\chi_1\Big)\,\mathcal{R}_{(4)} + \Big((1-\tfrac{\chi_T}{v_6})\,e^{-2\phi_4}-\chi_1\Big)\,G_{vv}(\partial v)^2 + \Big((1+\tfrac{\chi_T}{v_6})\,e^{-2\phi_4}+\chi_1\Big)\,G_{hh}(\partial h)^2\Big]$$

  - ▷ $v_6 = \mathcal{V}_3(2\pi l_s)^{-6}$
  - ▷ G_{vv} - the metric of the $h_{(1,1)}-1$ vector-multiplets
  - ▷ G_{hh} - the metric of the $h_{(1,2)}$ non-universal hypermultiplets
  - ▷ $\chi_T = 2\zeta(3)\chi/(2\pi)^3$
  - ▷ $\chi_1 = 4\zeta(2)\chi/(2\pi)^3$

- Weyl rescaling:
  - ▷ Quantum corrections to vector and hyper moduli space metrics
  - ▷ Corrections to the Kähler potential $\quad (\mathcal{V}_3 \;\to\; \mathcal{V}_3 \;-\; \frac{\zeta(3)}{32\pi^3\, g_s^{3/2}}\,\chi(X_3))$

B-field

⋆ Appearance of (at linearized level)

$$\hat{R}_{\mu\nu}{}^{\lambda\sigma}(\omega + \tfrac{1}{2}\mathcal{H}) = R_{\mu\nu}{}^{\lambda\sigma} + \tfrac{1}{2}\nabla_{[\mu}H_{\nu]}{}^{\lambda\sigma}$$

⋆ Inclusion of higher orders in B_2 required by

- supersymmetry
- T-duality (similarly for RR couplings to D-branes $C \wedge \sqrt{\hat{A}(X)}\mathrm{ch}(x)$)
- generalized geometry ??? hope for systematic geometric calculation?
- Heterotic/Type II duality ((refined) map between tree-level and one-loop terms)

⋆ In fact, 10d one-loop term ${\alpha'}^3\, R^3 H^2$ (in the string frame):

$$\int d^{10}x\sqrt{G}\,\delta^{r_1\cdots r_9}_{s_1\cdots s_9}\, R^{r_1 r_2}_{s_1 s_2} R^{r_3 r_4}_{s_3 s_4} R^{r_5 r_6}_{s_5 s_6}\left(H_{r_7 r_8 s_9}H_{s_7 s_8 r_9} - \tfrac{1}{9}H_{r_7 r_8 r_9}H_{s_7 s_8 s_9}\right)$$

$$\to \quad \chi \int d^4x\sqrt{g^\sigma}\, H_{r_1 r_2 r_3}H^{r_1 r_2 r_3}$$

is at the origin of ....

Quantum corrections to $\mathcal{N} = 2$ moduli spaces:

- vector moduli (IIA): $G_{vv} \to (1 - \chi \frac{4}{(2\pi)^3} \frac{\zeta(3)}{v_6}) G_{vv}$ ($e^{-2\phi_4} = v_6\, e^{-2\phi_{10}}$)
- (non-"universal") hyper moduli (IIA): $G_{h\bar{h}} \to (1 + e^{2\phi_4} \frac{1}{6\pi} \chi) G_{h\bar{h}}$

2 "loop counting" parameters:

- for the corrections to the metric of vector multiplets(σ-model) :
  $e^{-2\tilde{\phi}_4} \simeq e^{-2\phi_4} \left(1 + \mu_T \frac{\chi_T}{v_6} + \cdots\right)$ $(\chi_T = 2\zeta(3)\chi/(2\pi)^3)$
- for the corrections to the metric of hypermultiplets :
  $\tilde{v}_6 \simeq v_6 \left(1 - \frac{3\mu_1}{2} \chi_1\, e^{2\phi_4} + \mathcal{O}(e^{4\phi_4})\right)$ ($\mu_1^2 = 4$ and $\chi_1 = 4\zeta(2)\chi/(2\pi)^3$)

classical "universal" hypermultiplet (ϕ_4, B_2, C_0 : $C_3 \to C_0 \Omega_3 + \bar{C}_0 \wedge \bar{\Omega}_3 + \cdots$)

- classically - $SU(2,1)/U(2)$ coset (3 isometries)
- Loop corrections - self dual Enstein metric defined by a single function:
  $e^{-2\tilde{\phi}_4} - 4\zeta(2)\chi/(2\pi)^3$

Summary of type II $(\alpha')^3$ one-loop couplings (10D):

	No B	With B
e-o **+** **o-e**	$\frac{1}{8}(t_8\epsilon_{10}+\epsilon_{10}t_8)BR^4$ $= B\wedge X_8(\Omega^{\rm LC})$ $= \frac{1}{192(2\pi)^4}B\wedge\left(\text{tr}\,R^4-\frac{1}{4}(\text{tr}\,R^2)^2\right)$	$\frac{1}{8}(t_8\epsilon_{10}+\epsilon_{10}t_8)BR^4(\Omega_+)$ $=\frac{1}{8}t_8\epsilon_{10}B(R^4(\Omega_+)+R^4(\Omega_-))$ $=\frac{1}{2}B\wedge[X_8(\Omega_+)+X_8(\Omega_-)]$ $=\frac{1}{192\cdot(2\pi)^4}B\wedge\left(\text{tr}\,R^4-\frac{1}{4}(\text{tr}\,R^2)^2+\text{exact terms}\right)$
e-e	$t_8t_8R^4$	$t_8t_8R^4(\Omega_+)=t_8t_8R^4(\Omega_-)$
o-o	$\frac{1}{8}\epsilon_{10}\epsilon_{10}R^4$	$\frac{1}{8}\epsilon_{10}\epsilon_{10}\left(R(\Omega_+)^4+\frac{8}{3}H^2R(\Omega_+)^3+\cdots\right)$ $=\frac{1}{8}\epsilon_{10}\epsilon_{10}\left(R(\Omega_-)^4+\frac{8}{3}H^2R(\Omega_-)^3+\cdots\right)$

- ◇ new kinematic structures in o-o sector
- ◇ lifting to 11d: $H \;\mapsto\; G_4$ (with lifting ambiguities)
- ◇ get RR couplings via reduction

Lift from D=10 to D=11

... is heavy and the ambiguities are not fully resolved.

Lift from D=6 to D=7 has the main features and can be done explicitly.

CP-odd part:

$$\frac{1}{4} B_2 \wedge \overline{X}_4 \quad \longrightarrow \quad -\frac{1}{32\pi^2} \quad C_3 \wedge \left(\text{tr}\, R^2 - \frac{1}{12} d\left(\mathcal{G}^{abc} \wedge (\nabla\mathcal{G})^{abc}\right)\right).$$

◇ $\mathcal{G}_1^{abc} = 4 G_{\mu\rho\lambda} d\hat{x}^\mu \hat{e}^{a\nu} \hat{e}^{b\rho} \hat{e}^{c\lambda}$

CP-even part:

$$\begin{aligned} e^{-1}\delta\mathcal{L}^{\text{lift}} \quad = \quad & R^2_{\mu\nu\lambda\sigma} - 2R^2_{\mu\nu} + \tfrac{1}{2}R^2 - \tfrac{1}{6}R_{\mu\nu}G^{2\,\mu\nu} + \tfrac{1}{48}RG^2 + \tfrac{1}{6}\nabla_\mu G_{\nu\alpha\beta\gamma}\nabla^\nu G^{\mu\alpha\beta\gamma} \\ & + \tfrac{1}{48}G_{\mu\nu\lambda\rho}G^{\mu\rho\alpha}{}_\beta G^{\nu\sigma\beta}{}_\gamma G^{\lambda\gamma}{}_{\alpha\sigma} + \tfrac{1}{288\cdot 12}(G^2)^2 - \tfrac{1}{216}(G_{\mu\nu})^2 + (\text{eom})^2 \end{aligned}$$

Reducing back to 10D/6D

◇ $\mathcal{G}_1^{abc} \quad \longrightarrow \quad (e^{\phi/2}\mathcal{F}^{abc}; \mathcal{H}^{ab})$

allows to recover RR completion (1-loop only!)

Puzzles (at tree-level)

- One-loop results would suggest

$$J_0(\Omega) = \left(t_8 t_8 + \tfrac{1}{8}\epsilon_{10}\epsilon_{10}\right) R^4 \quad \longrightarrow \quad \left(t_8 t_8 + \tfrac{1}{8}\epsilon_{10}\epsilon_{10}\right) R^4(\Omega_+) + \tfrac{1}{3}\epsilon_{10}\epsilon_{10} H^2 R^3(\Omega_+) + \ldots$$
$$= J_0(\Omega_+) + \Delta J_0(\Omega_+, H)$$
$$J_1(\Omega) = t_8 t_8 R^4 - \tfrac{1}{4}\epsilon_{10} t_8 B R^4 \quad \longrightarrow \quad t_8 t_8 R^4(\Omega_+) - \tfrac{1}{8}\epsilon_{10} t_8 B \left(R^4(\Omega_+) + R(\Omega_-)\right) = J_1(\Omega_+)$$

$\Delta J_0(\Omega_+, H)$ needs susy completion. $\mathcal{N} = 2, 4$ tests:

$\Rightarrow$ ${c_2}^I (t_I \mathrm{tr}\, R^2 + u_I R \wedge R)$ with ${c_2}^I = \int_X \omega^I \wedge \mathrm{tr}\, R^2, \quad \omega^I \in H^{(1,1)}(X)$

◇ $\mathcal{F}_1 W^2|_{\text{F-term}}$ with W - $\mathcal{N} = 2$ chiral Weyl superfield and $\mathcal{F}_1$ - function of chiral vector superfields

$\Rightarrow$ important cancellations

▷ Extra corrections (??): $J_0(\Omega) \longrightarrow J_0(\Omega_+) + \Delta J_0 + 2\delta \quad J_1(\Omega) \longrightarrow J_1(\Omega_+) + \delta$

▷ *Different* superinvariant appear at tree-level and one-loop (!?)

- In IIB use $SL(2)$ and try $e^{-\phi} \ \rightarrow \ \tau = C_0 + i e^{-\phi}$

▷ Tree 4pt : $\Delta S_{IIB}|_{4-\text{pt}} \sim \frac{1}{l_s^2} \int \left\{ t_8 t_8 [R^4 + 24 R^2 |DP|^2] + \hat{\mathcal{O}}_1 \left[(|DP|^2)^2 \right] \right\} *_{10} 1$

Exotic kynematcis, dilaton in R^4, $SL(2)$ and F-theory

- IIB strings with varying dialton-axion $\tau = C_0 + ie^{-\phi}$
  - ◇ $S_{IIB} \sim \frac{1}{l_s^8} \int (R - P\bar{P}) *_{10} 1$
    - ▷ $P = \frac{i}{2\mathrm{Im}\tau} \nabla\tau$ ($U(1)_R$ covariant, charge 2)
    - ▷ *formally* obtained from $R^{(12)}$ at fixed volume ν
    - ▷ $\mathbb{T}^2$ metric: $\frac{\nu}{\mathrm{Im}\tau} \begin{pmatrix} 1 & \mathrm{Re}\tau \\ \mathrm{Re}\tau & |\tau|^2 \end{pmatrix}$
    - ▷ F-theory space - *elliptically fibered CY* ◁
    - ▷ D7/O7 - codim 2 defects (with *deficit angle*) - degenerations of el. fiber
    - ▷ 10D slice integral: $S_0^{12} \sim \frac{1}{l_s^8} \int R^{(12)} *_{10} 1$
- Decompactification limit of M-theory (on X_e)
  - ◇ $S^{11} \sim \frac{1}{l_M^9} \int R *_{11} 1 \quad \Rightarrow \quad S^9 \sim \frac{\nu}{l_M^7} \int (R - P\bar{P}) *_9 1 \quad \Rightarrow \quad S_{IIB}$
    - ▷ IIB limit $\nu \to 0$:

$\sim \alpha'^3$ in IIB

- kinematics:
  - ▷ $(t_8 t_8 + \frac{1}{8}\epsilon_{10}\epsilon_{10})R^4$
  - ▷ no CP-odd (GS-like) couplings
  - ▷ tree-level, 1 loop and non-perturbative contributions
- M-theory (11D) perspective
  - ▷ For M-th/$\mathbb{T}^2$, $\quad \nu \to 0 \quad \Rightarrow \quad S_3^9 \to 0$
  - ▷ need to account for KK modes on $\mathbb{T}^2$
- For constant τ: $\Delta S^{11} \sim \frac{1}{l_M^3}\int t_8 t_8 R^4 *_{11} 1 \quad \Rightarrow \quad \Delta S_{IIB} \sim \frac{1}{l_s^2}\int f_0(\tau,\bar\tau)\, t_8 t_8 R^4 *_{10} 1$
  - ▷ $$f_0(\tau,\bar\tau) = \sum_{(m,n)\neq(0,0)} \frac{\tau_2^{3/2}}{|m+n\tau|^3} \to \frac{2\zeta(3)}{g_3^{3/2}} + \frac{2\pi}{3} g_s^{1/2} + \mathcal{O}(e^{-1/g_s})$$
- Varying dilaton-axion:
  - ▷ $$\Delta S_{IIB}|_{4-\mathrm{pt}} \sim \frac{1}{l_s^2}\int \left\{ t_8 t_8 [R^4 + 24 R^2 |DP|^2] + \hat{\mathcal{O}}_1 \left[(|DP|^2)^2\right] \right\} *_{10} 1$$
  - ▷ Complete agreement with 0-mode reduction of

$$S_3^{12}(t_8) \quad \sim \quad \frac{1}{l_s^2}\int \hat{t}_8 \hat{t}_8 {R^{12}}^4 *_{10} 1$$

$\sim \alpha'^3$ in F-theory...

- Odd-odd sector:
  - ▷ 0-mode reduction of

$$S_3^{12}(\epsilon_8) \quad \sim \quad \frac{1}{l_s^2} \int \hat{\epsilon}_8 \hat{\epsilon}_8 R^{12^4} *_{10} 1$$

  - ▷ $S_3^{12}(\epsilon_8)$ restricted to 4pt *vanishes*

- Missing τ dynamics - all g_s corrections

- (Conjectured) complete coupling:

$$S_3^{12} \quad = \quad \frac{1}{(4\pi)^9 \cdot 3 \cdot l_s^4} \int f_0(\tau, \bar{\tau}) \left[\hat{t}_8 \hat{t}_8 + \frac{1}{96} \hat{\epsilon}_{12} \hat{\epsilon}_{12} \right] (R^{(12)})^4 *_{10} 1$$

  - ▷ $SL(2, \mathbb{Z})$ and SUSY compatibility
  - ▷ perturbatively *tree* + 1-loop terms
  - ▷ No "cusp forms" - non-perturbative part captured by $f_0(\tau, \bar{\tau})$
  - $*$ g_s-exact $\sim \mathcal{O}(\alpha'^3)$ Type IIB action without flux
  - $*$ R and τ couplings *beyond* 4pt

4D $\mathcal{N}=1$ compactifications of F-theory

- Smooth four-fold $B_3 \to CY_4$, fibered over B_3 with zero-section
- 2 + 8 derivative reductions ($+\mathcal{O}(\alpha'^4)$):

$$S^4_{0+3} = \frac{1}{2\pi\alpha'}\int \left(\mathcal{V}_b - \frac{1}{64\pi^3}\int_{B_3} f_0(\tau,\bar\tau)\, c_3(X_4)|_{B_3}\right) R_{(4\mathrm{d})} *_4 1$$

  - ◇ Correction

$$\sim \int R_{(4\mathrm{d})} *_4 1 \int_{B_3} f_0 *_8 (J\wedge c_3(X_4)) *_6 1 + \dots ,$$

  - ◇ Use $*_8(J\wedge *_6 1) = 1 + \mathcal{O}(\alpha')$

▷ Verify that the correction is *finite*

▷ *constant* τ ($B_3 = CY_3$) known $\mathcal{N}=2$ results

▷ τ varies over B_3 - the correction is *non-topological*

▷ $\Rightarrow$ Kähler potential via Weyl rescaling (plenty of ifs and buts!)

Weak string coupling limit

Sen limit: a region of the complex structure moduli space of the CY fourfold X_4, where none of the monodromies acting on τ involves the string coupling τ_2^{-1}:

- ▷ τ_2^{-1} kept small in a globally well-defined way
- ▷ Type IIB on orientifolded CY threefold X_3 - branched double cover of B_3
- ▷ O7-plane - branching locus: in cohomolgy $D_{\mathrm{O7}} \equiv c_1(B_3)$
- ∗ Correction (*topological*) to the classical volume $\mathcal{V}_3$ of the CY threefold:

$$\tilde{\mathcal{V}}_3 = \mathcal{V}_3 \; - \; \frac{\zeta(3)}{32\pi^3\, g_s^{3/2}} \left(\chi(X_3) + 2 \int_{X_3} D_{\mathrm{O7}}^3 \right) \quad + \quad \mathcal{O}(g_s^{-1/2})$$

  - ◇ Note integration over X_3 (not B_3)
  - ◇ Only Weyl rescaling contribution to Kähler potential is computed here

- ▷ New terms from tree-level closed string scattering in this CY orientifold background
- ▷ *Absent* in toroidal models!

Supersymmetry and Generalised geometry

gen. Lichnerowicz theorem (gLBT) $\Rightarrow$ effective actions (*Local data*)

- (gen.) Lichnerowicz theorem: $\left(D^A D_A - D^2\right)\epsilon = \left[\frac{1}{4}S + \gamma^{abcd} I_{abcd}\right]\epsilon$ (S tensorial!)

◇ Heterotic effective action: $S = R + 4\nabla^2\phi - 4(\partial\phi)^2 - \frac{1}{12}H^2 - \frac{\alpha'}{4}\text{tr}\,\hat{\mathcal{F}}^2$

◇ $I_{abcd} = \frac{1}{6}\nabla_{[a}H_{bcd]} - \frac{\alpha'}{8}\text{tr}\,\hat{\mathcal{F}}_{[ab}\hat{\mathcal{F}}_{cd]} = 0$

◇ $\left.\begin{aligned} \delta\psi_a &= D_a\epsilon = \nabla_a\epsilon - \frac{1}{8}H_{abc}\gamma^{bc}\epsilon \\ \delta\zeta_\alpha &= D_\alpha\epsilon = -\frac{1}{8}\sqrt{2\alpha'}\hat{\mathcal{F}}_{ab\alpha}\gamma^{ab}\epsilon \end{aligned}\right\}$ $\leftarrow$ covariant derivative ($A = \{a, \alpha\}$)

◇ $\delta\lambda = D\epsilon = \left(\gamma^a\nabla_a - \frac{1}{24}H_{abc}\gamma^{abc} - \gamma^a\partial_a\phi\right)\epsilon$ $\leftarrow$ Dirac operator

Gravitational terms (obstruction to gen. tangent bundle E) ?

◇ take $G \to G_{\text{gauge}} \times O(d)$....

◇ reduce the structure group of E to $O(d) \times G \times O(d) \subset O(d + \dim(\mathfrak{g})) \times O(d)$

◇ Identify $O(d) \in G$ with $O(d)$ in $\tilde{C}_+$

▷ Works only for $\hat{\mathcal{A}} = \Omega_+ = \omega^{\text{LC}} + \frac{1}{2}\mathcal{H}$!!! (cf susy for Ω_-!!!)

▷ For type II $G \to O(d) \times O(d)$ does **NOT** work

▷ "Tensoriality" without generalised geometry:

$$\not{D}\not{D}\epsilon - D_M D^M \epsilon - \frac{\alpha'}{64}\left(\text{tr } \not{F}\not{F}\epsilon - \text{tr } \not{R}^+ \not{R}^+ \epsilon\right) + 2\nabla^M \phi D_M \epsilon = -\frac{1}{4}\mathcal{L}_b \epsilon + \mathcal{O}(\alpha'^2)$$

(mod. heterotic BI: $\mathrm{d}H = -\frac{\alpha'}{4}\left(\text{tr } F \wedge F - \text{tr } R^+ \wedge R^+\right)$)

▷ Multiply by $e^{-2\phi}\epsilon^\dagger$ and integrate by parts ($\epsilon^\dagger \epsilon = 1$):

$$\frac{1}{4}\mathcal{L}_b = (\not{D}\epsilon)^\dagger \not{D}\epsilon - (D_M \epsilon)^\dagger D^M \epsilon + \frac{\alpha'}{64}\left(\text{tr } \epsilon^\dagger \not{F}\not{F}\epsilon - \text{tr } \epsilon^\dagger \not{R}^+ \not{R}^+ \epsilon\right) + \mathcal{O}(\alpha'^2)$$

▷ Works for *static* backgrounds without spacetime filling flux (!!)

▷ The (bosonic) action

$$S_b = \int_{M_{10}} e^{-2\phi}\mathcal{L}_b = BPS^2$$

▷ "intergrability" (Susy + BI $\Rightarrow$ solutions?) as an ... alternative to G. Ricci ?

$$\Gamma^M D^-_{[N} D^-_{M]}\epsilon - \frac{1}{2}D^-_N(\mathcal{O}\epsilon) + \frac{1}{2}\mathcal{O}D^-_N\epsilon = -\frac{1}{4}\mathcal{E}_{NM}\Gamma^M\epsilon + \frac{1}{8}\mathcal{B}_{NM}\Gamma^M\epsilon + \frac{1}{48}\mathrm{d}H_{NMPQ}\Gamma^{MPQ}\epsilon$$

$\mathcal{O} = \not{\partial}\phi - \frac{1}{12}\not{H}$ and $\mathcal{E}^0_{NM}, \mathcal{B}^0_{NM}$ - EOMs for metric and B-field

- ▷ Flip of the sign in $\mathcal{O}(\alpha')$ effective action wrt D_a : $\Omega_- \longrightarrow \Omega_+$!!!
  - ◇ $$R_{mnpq}(\Omega^-) - R_{pqmn}(\Omega^+) = -12\mathrm{d}H_{mnpq}$$
  - • leading to corrections all orders in α':
    - ▷ "gaugino" $\psi_{ab} \in \Gamma(\Lambda^2 C_+ \otimes S(C_-))$ for "gauge group" $O(d)_+$
      - ◇ $$\delta\psi_{O(d)ab} = \tfrac{1}{8}\sqrt{\alpha'}R(\Omega^+)_{\bar{a}\bar{b}ab}\gamma^{\bar{a}\bar{b}}\epsilon \ldots. = D_{ab}\epsilon \ \ (?)$$
    - ▷ ψ_{ab} - *composite* "gravitino curvature"
      - ◇ $\delta\psi_{ab} = D_{ab}\epsilon + \frac{1}{8}\sqrt{\alpha'}\left(\frac{1}{8}\alpha'[\mathrm{tr}\, F \wedge F - \mathrm{tr}\, R(\Omega^+) \wedge R(\Omega^+)]_{ab\bar{a}\bar{b}}\right)\gamma^{\bar{a}\bar{b}}\epsilon \ \rightarrow \ \hat{D}_{ab}\epsilon$
    - ▷ $D_{ab} \ \rightarrow \ \hat{D}_{ab}$ in gLBT $\Rightarrow$ $\mathcal{O}(\alpha'^2)$ modifications of susy for
      $\gamma^{\bar{a}}\hat{D}_{\bar{a}}\gamma^{\bar{b}}\hat{D}_{\bar{b}}\epsilon - \hat{D}^a\hat{D}_a\epsilon + \hat{D}^\alpha\hat{D}_\alpha\epsilon + \hat{D}^{ab}\hat{D}_{ab}\epsilon = -\frac{1}{4}S^-\epsilon + \gamma^{abcd}I_{abcd}\epsilon$
    - ▷ hierarchy of higher α' corrections (consistent with GCG)
    - ▷ $\mathcal{O}(\alpha'^3)$ agreement with literature
    - ▷ new $\mathcal{O}(\alpha'^4)$ corrections
    - ▷ iterative all order formulae ?

- Higher orders in α':
  - **Tensorial action of** $\hat{D}^{\dagger a}\hat{D}_a\epsilon - \hat{D}^{\dagger ab}\hat{D}_{ab}\epsilon + \hat{D}^{\dagger\alpha}\hat{D}_\alpha$
  - **No (new) corrections to Bianchi Identity** *
  - RHS vanishes on shell (can be checked)
- The corrected operators (at order $\sim(\alpha')^3$):

$$\delta\psi_{ab} = \hat{D}_{ab}\epsilon = \frac{1}{4}\sqrt{(\alpha')}\not{R}_{ab}\epsilon + \frac{1}{8}(\alpha')^{3/2}\not{T}_{ab}\epsilon + \frac{1}{8}(\alpha')^{5/2}\nabla_{[a}\nabla^c\left(\not{T}_{b]c}\epsilon\right)$$

$$\delta\psi_a = \hat{D}_a\epsilon = \nabla_a\epsilon + \frac{1}{8}(\alpha')^2\nabla^b\left(\not{T}_{ab}\epsilon\right) + \frac{1}{64}(\alpha')^3\nabla^b\left(X^3(R)_{ab}\epsilon\right) + \frac{1}{64}(\alpha')^3(Y(R,T)_a + Y(T,R)_a)\epsilon$$

$$\delta\chi_\alpha = \hat{D}_\alpha\epsilon = -\frac{1}{4}\sqrt{(\alpha')}\not{F}_\alpha\epsilon + \frac{1}{64}(\alpha')^3X^3(F)_\alpha\epsilon$$

where: $T = -\frac{1}{4}(F\wedge F - R\wedge R)$ and $\tilde{S}^a{}_b \equiv R^{ca}R_{cb} - F^{ca}F_{cb}$

Definitoins:

$$X^3(F)_\alpha = \frac{1}{3!}F^e{}_{a\alpha}T_{ebcd}\gamma^{abcd} + \frac{1}{2}F^{ef}{}_\alpha T_{efab}\gamma^{ab} - \frac{1}{2}F^e{}_{a\alpha}\tilde{S}_{eb}\gamma^{ab}$$

$$Y(R,T)_a\epsilon = \not{R}_{ba}\nabla_c(\not{T}^{cb}\epsilon) + \not{R}_{ba}\nabla_c(\not{T}^{cb})\epsilon$$

Note:

- $\diamond$ $R = R^{+} \qquad \& \qquad \nabla_a \to D_a$
- $\diamond$ $(\nabla^-_{[a}\nabla^-_{b]} + \frac{1}{2}H_a{}^c{}_b\nabla^-_c)\epsilon = \frac{1}{8}R^-_{abcd}\gamma^{cd}\epsilon = \frac{1}{4}\not{R}_{ab}\epsilon + \frac{1}{8}(\alpha')\not{T}_{ab}\epsilon$

Adjoint operators:

$\hat{D}_a : S \to V \otimes S \qquad \& \qquad \hat{D}^{\dagger a} : V \otimes S \to S$

$\hat{D}_{ab} : S^- \to \Lambda^2 C_+ \otimes S^- \qquad \& \qquad \hat{D}^{\dagger ab} : \Lambda^2 C_+ \otimes S^- \to S^-$

defined as $\int \bar{\psi}^a \hat{D}_a \epsilon = -\int \overline{\hat{D}^{\dagger a}\psi_a}\, \epsilon$

$\triangleright$ $\sim \mathcal{O}((\alpha')^3)$ (and $\sim \mathcal{O}((\alpha')^4)$) action:

$$\hat{D}^{\dagger a}\hat{D}_a\epsilon - \hat{D}^{\dagger ab}\hat{D}_{ab}\epsilon + \hat{D}^{\dagger \alpha}\hat{D}_\alpha\epsilon = \overbrace{D^a D_a\epsilon - D^{ab}D_{ab}\epsilon + D^\alpha D_\alpha\epsilon}^{\text{action } \sim \mathcal{O}(\alpha')}$$

$$+ 2(\alpha')^2\left[\text{e.o.m.}|_{\mathcal{O}((\alpha')^1)}\right]_{ab}\not{T}^{ab}\epsilon - (\alpha')^3(\tfrac{1}{4}T\lrcorner T + \tfrac{1}{64}\tilde{S}\lrcorner\tilde{S})\epsilon + (\alpha')^3\left[\text{e.o.m.}|_{\mathcal{O}((\alpha')^0)}\right]_{ab}\not{T}^{cd}R_{cd}{}^{ab}\epsilon$$

(schematically)

M-theory

- Fields: $\{g_{mn}, \mathcal{A}_{mnp}, \psi_m\}$
  - ◇ $S_{\mathsf{B}} = \frac{1}{2\kappa^2}\int\left(\sqrt{-g}\,R - \frac{1}{2}\mathcal{F}\wedge *\mathcal{F} - \frac{1}{6}\mathcal{A}\wedge\mathcal{F}\wedge\mathcal{F}\right)$
  - ◇ $S_{\mathsf{F}} = \frac{1}{\kappa^2}\int\sqrt{-g}\Big(\bar\psi_m\gamma^{mnp}\nabla_n\psi_p + \mathcal{F}_{p_1\ldots p_4}\left(\frac{1}{96}\bar\psi_m\gamma^{mp_1\ldots p_4 n}\psi_n + \frac{1}{8}\bar\psi^{p_1}\gamma^{p_2p_3}\psi^{p_4}\right)\Big)$
    - ▷ susy $\delta\psi_m = \nabla_m\varepsilon + \frac{1}{288}\left(\gamma_m{}^{n_1\ldots n_4} - 8\delta_m{}^{n_1}\gamma^{n_2n_3n_4}\right)\mathcal{F}_{n1\ldots n_4}\varepsilon = D_m\varepsilon$
    - ▷ eom $\gamma^{mnp}\nabla_n\psi_p + \frac{1}{96}\left(\gamma^{mnp_1\ldots p_4}\mathcal{F}_{p_1\ldots p_4} + 12\mathcal{F}^{mn}{}_{p_1p_2}\gamma^{p_1p_2}\right)\psi_n = 0 = L^{mn}\psi_n$

- exact sequence: $S \xrightarrow{D} V\otimes S \xrightarrow{L} V\otimes S \xrightarrow{D^\dagger} S$
  - ▷ $L\circ D = 0$ follows from supersymmetry
  - ▷ reality of $\int\bar\psi_a L^{ab}\psi_b \Rightarrow L = -L^\dagger \Rightarrow D^\dagger\circ L = 0$.

- Lichnerowicz-type relation $\tilde D^a D_a\varepsilon = \gamma^{ab}D_aD_b\varepsilon \propto$ (trace of Einstein $+$ 8-form)
  - ▷ $L^{ab}\psi_b = \gamma^{abc}D_b\psi_c$ and $\tilde D^c = \frac{1}{9}\gamma_a L^{ac} = \gamma^{bc}D_b$
  - ▷ coefs in D_a and $\tilde D^a$ are *uniquely* fixed by tensoriality of rhs!

- LT $\Rightarrow$ $S_{\mathsf{B}} = \int\sqrt{-g}(\mathcal{R} - \frac{1}{3}\frac{1}{48}\mathcal{F}_{b_1\ldots b_4}\mathcal{F}^{b_1\ldots b_4}) - \frac{1}{3}\mathcal{A}\wedge(\mathrm{d}*\mathcal{F} + \frac{1}{2}\mathcal{F}\wedge\mathcal{F})$

Higher order

$$D : S \to T^* \otimes S$$

$$(D\varepsilon)_a = \nabla_a \varepsilon + \alpha \left(\nabla^b X_{abcd}\right) \gamma^{cd} \varepsilon + \beta X_{abcd} \gamma^{cd} \nabla^b \varepsilon$$

$$\tilde{D} : T^* \otimes S \to S$$

$$(\tilde{D}\psi) = \gamma^{ab} \left(\nabla_a \psi_b + \tilde{\alpha} \left(\nabla^c X_{acef}\right) \gamma^{ef} \psi_b + \tilde{\beta} X_{acef} \gamma^{ef} \nabla^c \psi_b\right)$$

where $X_{abcd} \in [0,2,0,0,0]$ and α, β etc. parametrise higher-order corrections

$$\begin{aligned} (\tilde{D}D\varepsilon) &= \gamma^{ab} \nabla_a \nabla_b \varepsilon + (\alpha - \tilde{\alpha} - \beta) \left(\nabla^a X_{abcd}\right) \gamma^{cd} \nabla^b \varepsilon \\ &\quad - \alpha \left(\nabla^a \nabla^b X_{abcd}\right) \gamma^{cd} \varepsilon - \left(\tilde{\beta} + \beta\right) X_{abcd} \gamma^{cd} \nabla^a \nabla^b \varepsilon + \ldots \\ (\text{if } \alpha - \tilde{\alpha} - \beta = 0) &= -\tfrac{1}{4} \mathcal{R} \varepsilon + \tfrac{1}{2} \left(2\alpha - \tilde{\beta} - \beta\right) R^{abe}{}_c X_{abed} \gamma^{cd} \varepsilon \\ &\quad + \tfrac{1}{4} \left(\tilde{\beta} + \beta\right) R^{abcd} X_{abcd} \varepsilon - \tfrac{1}{8} \left(\tilde{\beta} + \beta\right) R^{ab}{}_{cd} X_{abef} \gamma^{cdef} \varepsilon + \ldots \end{aligned} \tag{1}$$

Ambiguities:

- ▷ $\alpha = \tilde{\alpha} = \beta = 0$ consistent: keeping susy classical and only correcting S_{F}
- ▷ $\tilde{\alpha} = 0, \alpha = \beta = \tilde{\beta}$: $\tilde{D}^a \nabla_a \varepsilon$ and $\gamma^{ab} \nabla_a D_b \varepsilon$ are separately tensorial (the fermionic action is in terms of ”supercovariant” objects)

Everything that can modify susy:

Projection of R^3	Rep of $so(10,1)$	Multiplicity	multiplicity of embeddings in $\delta\psi$	of which result in $[\nabla,\nabla]R^3$	projected into form of rank
X^i	[0,2,0,0,0]	8	1	1	2
W^i	[2,0,0,0,0]	3	3	1	2
S^i	[0,0,0,0,0]	2	1	1	2
Y^i	[0,1,0,0,2]	2	1	1	6
V^i	[1,0,0,0,2]	2	4	2	$4, 6$
T^i	[0,0,0,1,0]	3	5	3	$2, 4, 6$
Z^i	[0,1,0,1,0]	3	1	1	4
U^i	[1,0,1,0,0]	3	4	2	$2, 4$
L^i	[2,1,0,0,0]	3	1	0	-
M^i	[2,0,0,1,0]	6	1	0	-

The last two lead to symmetrised ∇ and so are immediately ruled out. The other terms all admit at least one combination which corresponds to $R^3[\nabla,\nabla]\varepsilon$ in the Lichnerowicz, which thus give rise to R^4 terms.

These R^4 will appear as p-forms contracted with gamma-matrices acting on the spinor. The different terms contribute to different forms as follows:

p-form :	0	1	2	3	4	5
R^4 multiplicity:	7	0	1	2	17	0
$X^i \otimes R$	•	-	•	-	•	-
$W^i \otimes R$	-	-	-	-	-	-
$S^i \otimes R$	-	-	-	-	-	-
$Y^i \otimes R$	-	-	-	•	•	-
$V^i \otimes R$	-	-	-	-	•	-
$T^i \otimes R$	-	-	-	-	•	-
$Z^i \otimes R$	-	-	•	-	•	-
$U^i \otimes R$	-	-	•	-	•	-

!!! Nothing is possible at R^2 and R^3 order !!!

- ▷ Tensoriality of $\tilde{D}D$ + cancellation of 2,4,6-forms yields 2 independent invariants
  - ◇ $x\Big(\left(t_8 t_8 - \frac{1}{8}\epsilon\epsilon\right) R^4 + \frac{1}{2}\epsilon t_8 C R^4\Big)$
  - ◇ $y \left(t_8 t_8 + \frac{1}{8}\epsilon\epsilon\right) R^4$
- ▷ what fixes $y = 0$?
  - ◇ fermion terms and “actual” supersymmetry
  - ◇ inclusion of $\mathcal{F}$
  - ◇ next order $\sim R^7$ (lift from 2-loop string terms)

Will quantum corrections be a key to the (generalised) geometry of M-theory?

(How much) can generalised geometry capture the systematics of string expansion?

Plenty of open questions....