UC DAVIS
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# THERMAL EQUIVARIANCE AND ITS APPLICATIONS

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- Basic philosophy: [1510.02494]*
- Dissipative hydrodynamic actions: [1511.07809]
- Origins in Schwinger-Keldysh: [1610.01940]
- **Thermal Equivariance: [1610.01941]**

- * Classification of solutions to hydro axioms: [1412.1090] [1502.00636]

*Prelude: On that which came before....*

# CONTEXTUALIZATION

Schwinger 1961
Keldysh 1964
Feynman, Vernon 1963

Kubo 1957
Martin, Schwinger 1959

Review: Chou, Su, Hao, Yu 1985

*Schwinger-Keldysh & KMS*

Martin, Siggia, Rose 1973
Parisi, Sourlas 1982

Review: Zinn-Justin 2002

cf., Janssen 1976 &
de Dominicis, Peliti 1978

*Topological symmetry & Dissipation*

Witten 1988
Vafa, Witten 1994
Dijkgraaf, Moore 1996

Review: Cordes, Moore, Ramgoolam 1995
Guillemin, Sternberg 2013

cf., Blau, Thompson 1997
Zucchini 1998
Gozzi, Reuter 1990

*Equivariance &*
*Topological Sigma Models*

# CONTEXTUALIZATION

Banerjee, Bhattacharya, Bhattacharyya, Jain, Minwalla, Sharma 2012
Jensen, Kaminski, Kovtun, Myer, Ritz, Yarom 2012

**Sayantani Bhattacharyya** 2012, 2014

*Equilibrium, Entropy, and all that…*

Jarzynski 1996, 1997
Crooks 1999

Mallick, Moshe, Orland 2010
Gaspard 2012

*Non-equilibrium 2nd law*

Nickel, Son 2010

Dubovsky, Hui, Nicolis 2011

Kovtun, Moore, Romatschke 2014

Haehl, Loganayagam, MR 2014-15

*Effective actions for hydro*

*and that which is...*

# *RELATED WORK*

Crossley, Glorioso, Liu [1511.03646]
Glorioso, Liu [1612.07705]

Crossley, Glorioso, Liu [1701.07817]
Gao, Liu [1701.07445]

Jensen, Pinzani-Fokeeva, Yarom [1701.07436]

*Commentary & comparison of 1st two papers (CGL/GL) in 1701.07896 (HLR)*

# *Act I*

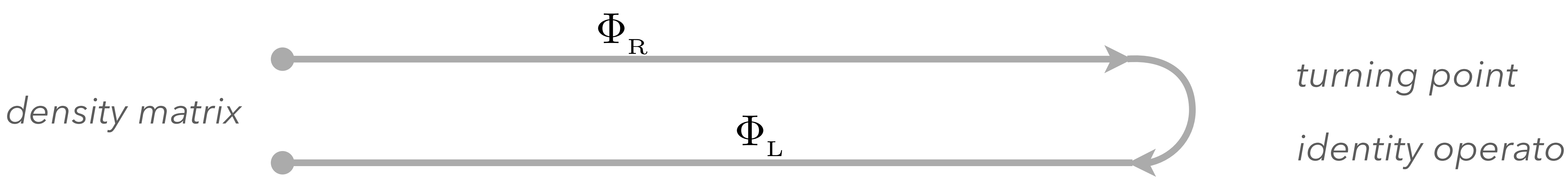
*in which we meet*

*Schwinger-Keldysh and Kubo-Martin-Schwinger*

*and find a useful way to represent them...*

# SCHWINGER-KELDYSH FORMALISM

- The Schwinger-Keldysh formalism computes singly out-of-time ordered correlation functions in a generic (mixed) state.




$$S_{SK} = S[\Phi_{\mathrm{R}}] - S[\Phi_{\mathrm{L}}]$$

$$\delta S_{SK} = \int d^d x \sqrt{-g}\,(\mathcal{J}_{\mathrm{R}}\,\mathbb{O}_{\mathrm{R}} - \mathcal{J}_{\mathrm{L}}\,\mathbb{O}_{\mathrm{L}})$$

Generating functional

$$\mathcal{Z}_{SK}[J_{\mathrm{R}}, J_{\mathrm{L}}] \equiv \mathrm{Tr}\Big\{U[J_{\mathrm{R}}]\ \hat{\rho}_{\text{initial}}\ (U[J_{\mathrm{L}}])^\dagger\Big\}$$

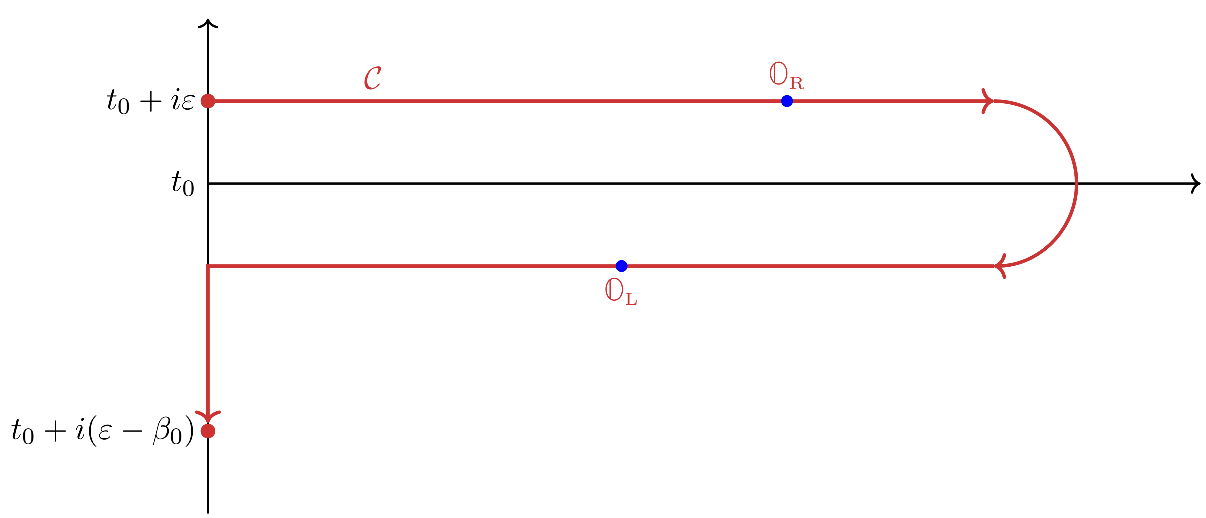
Time ordered correlations

$$\mathrm{Tr}\Big(\ \hat{\rho}_{\text{initial}}\ \bar{\mathcal{T}}\Big(U^\dagger\mathbb{O}_{\mathrm{L}}U^\dagger\mathbb{O}_{\mathrm{L}}\ldots\Big)\ \mathcal{T}\,(U\mathbb{O}_{\mathrm{R}}U\mathbb{O}_{\mathrm{R}}\ldots)\ \Big)$$

# THERMAL DENSITY MATRICES & KMS CONDITION

✦ Thermal density matrices $\hat{\rho}_T = e^{-\beta\,(\widehat{\mathbb{H}} - \mu_I\,\widehat{\mathbb{Q}}^I)}$ define stationary equilibrium configurations.

✦ Correlation functions have analyticity properties which allows for a Euclidean (Matsubara) formulation, cf., $\mathcal{Z}_T(\beta, \mu_I) = \mathrm{Tr}\,(\hat{\rho}_T)$




$$\mathcal{Z}_T[\mathcal{J}_{\mathrm{R}}, \mathcal{J}_{\mathrm{L}}] = \mathrm{Tr}\left(U[\mathcal{J}_{\mathrm{R}}]\,\hat{\rho}_T\,(U[\mathcal{J}_{\mathrm{L}}])^\dagger\right)$$

✦ KMS condition asserts that the correlation functions are analytic in the time strip $0 < \Im(t) < \beta$.

✦ This can be rephrased as a thermal Ward identity for correlation functions which involve operators shifted by a imaginary thermal period.

# TWO SUM RULES

- *Unitarity* of Schwinger-Keldysh path integral implies vanishing difference operator correlators:

$$\langle \mathcal{T}_{SK} \prod_k \left( \mathbb{O}_{\mathrm{R}}^{(k)} - \mathbb{O}_{\mathrm{L}}^{(k)} \right) \rangle \equiv \langle \mathcal{T}_{SK} \prod_k \mathbb{O}_{dif}^{(k)} \rangle = 0$$

*Weldon '05*

- The KMS condition translates into a second sum rule for thermal differences:

$$\langle \mathcal{T}_{SK} \prod_{k=1}^{n} \left( \mathbb{O}_{\mathrm{R}}^{(k)} - \tilde{\mathbb{O}}_{\mathrm{L}}^{(k)} \right) \rangle = \langle \mathcal{T}_{SK} \prod_{i=1}^{n} \mathbb{O}_{ret} \rangle = 0$$

- Keldysh (light-cone) basis $\qquad \mathbb{O}_{dif} \equiv \mathbb{O}_{\mathrm{R}} - \mathbb{O}_{\mathrm{L}} \,, \qquad \mathbb{O}_{av} \equiv \frac{1}{2} \left( \mathbb{O}_{\mathrm{R}} + \mathbb{O}_{\mathrm{L}} \right)$

- Adv-Ret basis $\qquad \mathbb{O}_{adv} \equiv \mathbb{O}_{\mathrm{R}} - \mathbb{O}_{\mathrm{L}} \,, \qquad i\Delta_{\boldsymbol{\beta}} \mathbb{O}_{ret} = \mathbb{O}_{\mathrm{R}} - e^{-i\delta_{\boldsymbol{\beta}}} \mathbb{O}_{\mathrm{L}}$

- Furthermore, a *largest time* and *thermal smallest time* equations hold.

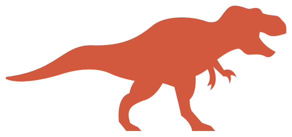
# THE SCHWINGER-KELDYSH SUPER-QUARTET

- Difference operator correlation functions vanish because they are trivial elements of a BRST cohomology.
- There exists a pair of Grassmann odd charges which act on the doubled operator algebra.
- The SK theory is covariantly expressed in terms of a quartet of fields, which usual doubled formalism being a gauge  fixed version (ghosts =0).

$$\begin{tikzcd}
& \mathbb{O}_{\mathrm{R}}, \mathbb{O}_{\mathrm{L}} \arrow[dl, "\mathcal{Q}_{SK}"] \arrow[dr, "\overline{\mathcal{Q}}_{SK}"] & \\
\mathbb{O}_{G} \arrow[dr, "\overline{\mathcal{Q}}_{SK}"] & & \mathbb{O}_{\overline{G}} \arrow[dl, "-\mathcal{Q}_{SK}"'] \\
& \mathbb{O}_{\mathrm{R}} - \mathbb{O}_{\mathrm{L}} &
\end{tikzcd}$$

$$\mathcal{Q}_{SK}^2 = \overline{\mathcal{Q}}_{SK}^2 = \left[\mathcal{Q}_{SK}, \overline{\mathcal{Q}}_{SK}\right]_\pm = 0$$

$$\left[\mathcal{Q}_{SK}, \mathbb{O}_{dif}\right]_\pm = 0 \ , \qquad \left[\overline{\mathcal{Q}}_{SK}, \mathbb{O}_{dif}\right]_\pm = 0$$



CGL argue that this should only be interpreted as a single supercharge $\delta$, but the pair above are CPT conjugates (cf., anti-BRST).

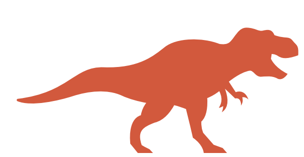
# *THE KMS SUPERCHARGES: I*

- The second sum rule suggests an analogous structure should pertain in the thermal sector, with new supercharges aligned to the thermal translations.

$$\begin{tikzcd}
& \mathbb{O}_{adv} \arrow[dl, "\mathcal{Q}_{KMS}"'] \arrow[dr, "\overline{\mathcal{Q}}_{KMS}"] & \\
\Delta_{\boldsymbol{\beta}} \mathbb{O}_{G} \arrow[dr, "\overline{\mathcal{Q}}_{KMS}"] & & -\Delta_{\boldsymbol{\beta}} \mathbb{O}_{\overline{G}} \arrow[dl, "-\mathcal{Q}_{KMS}"'] \\
& \Delta_{\boldsymbol{\beta}} \Delta_{\boldsymbol{\beta}} \mathbb{O}_{ret} &
\end{tikzcd}$$

$$\mathcal{Q}_{KMS}^2 = \overline{\mathcal{Q}}_{KMS}^2 = \left[\mathcal{Q}_{KMS}, \overline{\mathcal{Q}}_{KMS}\right]_{\pm} = 0$$

$$\left[\mathcal{Q}_{KMS}, \mathbb{O}_{ret}\right]_{\pm} = \left[\overline{\mathcal{Q}}_{KMS}, \mathbb{O}_{ret}\right]_{\pm} = 0$$



CGL posit that the KMS condition should be viewed as an involution leading to a second supercharge $\overline{\delta}$.

# THE SK-KMS ALGEBRA

- The SK and KMS operations (Grassmann odd) form a closed super algebra with further two Grassmann even operations
- One the even operations is a thermal translation: Lie drag along the Euclidean thermal circle.

$$
\begin{gathered}
\mathcal{Q}_{SK}^2 = \overline{\mathcal{Q}}_{SK}^2 = \mathcal{Q}_{KMS}^2 = \overline{\mathcal{Q}}_{KMS}^2 = 0 \;, \\
[\mathcal{Q}_{SK}, \mathcal{Q}_{KMS}]_\pm = \left[\overline{\mathcal{Q}}_{SK}, \overline{\mathcal{Q}}_{KMS}\right]_\pm = \left[\overline{\mathcal{Q}}_{SK}, \mathcal{Q}_{SK}\right]_\pm = \left[\mathcal{Q}_{KMS}, \overline{\mathcal{Q}}_{KMS}\right]_\pm = 0 \;, \\
\left[\mathcal{Q}_{SK}, \overline{\mathcal{Q}}_{KMS}\right]_\pm = \left[\overline{\mathcal{Q}}_{SK}, \mathcal{Q}_{KMS}\right]_\pm = \mathcal{L}_{KMS} \;, \\
\left[\mathcal{Q}_{KMS}, \mathcal{Q}_{KMS}^0\right]_\pm = \left[\overline{\mathcal{Q}}_{KMS}, \mathcal{Q}_{KMS}^0\right]_\pm = 0 \,, \\
\left[\mathcal{Q}_{SK}, \mathcal{Q}_{KMS}^0\right]_\pm = \mathcal{Q}_{KMS} \;, \qquad \left[\overline{\mathcal{Q}}_{SK}, \mathcal{Q}_{KMS}^0\right]_\pm = -\overline{\mathcal{Q}}_{KMS} \;.
\end{gathered}
$$

# *SUPERSPACE CHARGES*

- The structure is easily understood by passing onto a superspace construction, where the SK charges act as superderivations.

$$\mathring{\mathbb{O}} \equiv \mathbb{O}_{ret} + \theta\, \mathbb{O}_{\overline{G}} + \bar{\theta}\, \mathbb{O}_{G} + \bar{\theta}\theta\, \mathbb{O}_{adv}$$

$$\mathcal{Q}_{SK} \longrightarrow \partial_{\bar{\theta}}\,, \qquad \overline{\mathcal{Q}}_{SK} \longrightarrow \partial_{\theta}$$

$$\begin{tikzcd}
 & \mathring{\mathcal{I}}_0^{\rm KMS} \arrow[dl, "\partial_{\bar{\theta}}"'] \arrow[dr, "\partial_{\theta}"] & \\
\mathring{\mathcal{I}}^{\rm KMS} \arrow[dr, "\partial_{\theta}"'] & & -\mathring{\overline{\mathcal{I}}}^{\rm KMS} \arrow[dl, "-\partial_{\bar{\theta}}"] \\
 & \mathring{\mathcal{L}}^{\rm KMS} &
\end{tikzcd}$$

$$\begin{aligned}
\mathring{\mathcal{I}}_0^{\rm KMS} &= \mathcal{Q}^0_{KMS} + \bar{\theta}\, \mathcal{Q}_{KMS} - \theta\, \overline{\mathcal{Q}}_{KMS} + \bar{\theta}\theta\, \mathcal{L}_{KMS}\,, \\
\mathring{\mathcal{I}}^{\rm KMS} &= \mathcal{Q}_{KMS} + \theta\, \mathcal{L}_{KMS}\,, \\
\mathring{\overline{\mathcal{I}}}^{\rm KMS} &= \overline{\mathcal{Q}}_{KMS} + \bar{\theta}\, \mathcal{L}_{KMS}\,, \\
\mathring{\mathcal{L}}^{\rm KMS} &= \mathcal{L}_{KMS}\,.
\end{aligned}$$

# *Act II*

*in which we meet Weil and Cartan and learn of equivariance....*

# *EQUIVARIANCE*

- Equivariance = cohomology with gauge symmetry
- To understand cohomology on an orbifold $\mathcal{M}/\mathcal{G}$ we use the Borel construction to work with the contractible universal $\mathcal{G}$ bundle $\mathcal{E}_\mathcal{G}$.
- Classifying space $\mathcal{B}_\mathcal{G} = \mathcal{E}_\mathcal{G}/\mathcal{G}$ is smooth as group action is free on the universal bundle.
- eg., $\mathbf{S}^1 = \mathbb{R}/\mathbb{Z}$
- cohomology of $\mathcal{M}/\mathcal{G}$ = cohomology of $(\mathcal{E}_\mathcal{G} \times \mathcal{M})/\mathcal{G}$
- Physically think of $\mathcal{E}_\mathcal{G}$ as the space of the universal $\mathcal{G}$ gauge connections and $\mathcal{B}_\mathcal{G}$ as the space of gauge orbits. This picture is efficient to write superspace Lagrangians.
- Cohomology of interest will be the space of invariant horizontal forms.

*Matthai, Quillen ’86*
*Kalkman ‘93*

# *THE WEIL MODEL*

- Gauge structure can be captured by a Grassmann odd gauge potential (fermions = differential forms) and its field strength

$$\mathbb{d}^{\mathrm{E}}_{\mathrm{W}} G^i + \frac{1}{2} f^i_{jk} G^j G^k = \phi^i \,, \qquad \mathbb{d}^{\mathrm{E}}_{\mathrm{W}} \phi^i + f^i_{jk} G^j \phi^k = 0 \,.$$

Cartan equations for gauge structure

$$\delta^i_j + \overline{\mathcal{I}}^{\mathrm{E}}_j G^i = 0 \,, \qquad \overline{\mathcal{I}}^{\mathrm{E}}_j \phi^i = 0 \,.$$

interior contractions pick out gauge directions

$$\mathcal{L}^{\mathrm{E}}_j \equiv \left[ \mathbb{d}^{\mathrm{E}}_{\mathrm{W}} , \overline{\mathcal{I}}^{\mathrm{E}}_j \right]_{\pm}$$

Lie derivations follow from above

Weil superalgebra

$$\begin{aligned}
\left[ \overline{\mathcal{I}}^{\mathrm{E}}_i , \overline{\mathcal{I}}^{\mathrm{E}}_j \right]_{\pm} &= 0 \,, & \left[ \mathbb{d}^{\mathrm{E}}_{\mathrm{W}} , \overline{\mathcal{I}}^{\mathrm{E}}_j \right]_{\pm} &= \mathcal{L}^{\mathrm{E}}_j \,, \\
\left[ \mathcal{L}^{\mathrm{E}}_i , \overline{\mathcal{I}}^{\mathrm{E}}_j \right]_{\pm} &= -f^k_{ij} \, \overline{\mathcal{I}}^{\mathrm{E}}_k \,, & \left[ \mathbb{d}^{\mathrm{E}}_{\mathrm{W}} , \mathcal{L}^{\mathrm{E}}_j \right]_{\pm} &= 0 \,, \\
\left[ \mathcal{L}^{\mathrm{E}}_i , \mathcal{L}^{\mathrm{E}}_j \right]_{\pm} &= -f^k_{ij} \, \mathcal{L}^{\mathrm{E}}_k \,, & \left[ \mathbb{d}^{\mathrm{E}}_{\mathrm{W}} , \mathbb{d}^{\mathrm{E}}_{\mathrm{W}} \right]_{\pm} &= 0 \,.
\end{aligned}$$

- *invariant horizontal forms are polynomial functions of field strengths.*

# *CARTAN MODEL*

- One can similarly account for the group action on the manifold by working with coordinate vectors and Grassmann fields (for 1-forms).
- Since cohomology is in gauge invariant data, helpful to pass to gauge covariant language (Kalkman automorphism)

$$\begin{aligned}
\mathbb{d}_{\mathrm{C}} X^\mu &\equiv \mathbb{d}_{\mathrm{W}} X^\mu + G^k \xi_k^\mu = \psi_{\mathrm{C}}^\mu \,, \\
\mathbb{d}_{\mathrm{C}} \psi_{\mathrm{C}}^\mu &\equiv \mathbb{d}_{\mathrm{W}} \psi_{\mathrm{C}}^\mu + G^k (\partial_\nu \xi_k^\mu) \psi_{\mathrm{C}}^\nu = \phi^k \xi_k^\mu \\
\mathbb{d}_{\mathrm{C}} \phi^k &\equiv \mathbb{d}_{\mathrm{W}} \phi^k + f_{ij}^k G^i \phi^j = 0 \,.
\end{aligned}$$

action of Cartan charges on target space and field strengths

The two charges act isomorphically on horizontal, invariant forms.

$$\mathbb{d}_{\mathrm{C}} = \mathbb{d}_{\mathrm{W}} + G^i \mathcal{L}_i + \left( \frac{1}{2} f_{ij}^k G^i G^j + \phi^k \right) \overline{\mathcal{I}}_k$$

The Cartan charge however squares to a gauge transformation

$$\mathbb{d}_{\mathrm{C}}^2 = \phi^k \mathcal{L}_k - [G, \phi]^k \, \overline{\mathcal{I}}_k$$

# *EXTENDED EQUIVARIANCE I*

- One can extend the algebraic constructions to situations with more than one differential. We will focus on the case with 2 generators of the cohomology and swiftly pass to superspace: $\mathbb{d}_{\mathrm{W}} = \partial_{\bar{\theta}}(\ldots)| \,, \qquad \bar{\mathbb{d}}_{\mathrm{W}} = \partial_{\theta}(\ldots)| \,.$

- The Weil model closes on 6 generators: 2 derivations, 3 interior contraction, and one Lie derivation

$$
\begin{gathered}
\mathbb{d}_{\mathrm{W}}^2 = \bar{\mathbb{d}}_{\mathrm{W}}^2 = \left[\mathbb{d}_{\mathrm{W}}, \bar{\mathbb{d}}_{\mathrm{W}}\right]_{\pm} = 0 \\
\left[\mathbb{d}_{\mathrm{W}}, \overline{\mathcal{I}}_j\right]_{\pm} = \left[\bar{\mathbb{d}}_{\mathrm{W}}, \mathcal{I}_j\right]_{\pm} = \mathcal{L}_j \,, \qquad \left[\mathbb{d}_{\mathrm{W}}, \mathcal{I}_j\right]_{\pm} = \left[\bar{\mathbb{d}}_{\mathrm{W}}, \overline{\mathcal{I}}_j\right]_{\pm} = 0 \\
\left[\mathbb{d}_{\mathrm{W}}, \mathcal{I}_j^0\right]_{\pm} = \mathcal{I}_j \,, \qquad \left[\bar{\mathbb{d}}_{\mathrm{W}}, \mathcal{I}_j^0\right]_{\pm} = -\overline{\mathcal{I}}_j \\
\left[\mathbb{d}_{\mathrm{W}}, \mathcal{L}_j\right]_{\pm} = \left[\bar{\mathbb{d}}_{\mathrm{W}}, \mathcal{L}_j\right]_{\pm} = 0 \\
\left[\overline{\mathcal{I}}_i, \mathcal{I}_j\right]_{\pm} = f_{ij}^k \, \mathcal{I}_k^0 \\
\left[\mathcal{L}_i, \overline{\mathcal{I}}_j\right]_{\pm} = -f_{ij}^k \, \overline{\mathcal{I}}_k \,, \qquad \left[\mathcal{L}_i, \mathcal{I}_j\right]_{\pm} = -f_{ij}^k \, \mathcal{I}_k \,, \qquad \left[\mathcal{L}_i, \mathcal{I}_j^0\right]_{\pm} = -f_{ij}^k \, \mathcal{I}_k^0 \,.
\end{gathered}
$$

*Blau, Thompson '91*
*Cordes, Moore, Ramgoolam '94*

- *this should be reminiscent of structures in Act 1….*

*Vafa, Witten '94*
*Dijkgraaf, Moore '96*

# EXTENDED EQUIVARIANCE II

- Package the universal data into a set of gauge superfield 1-form which we assume lives on a worldvolume with coordinates $\sigma^a$.

$$\mathring{\mathcal{A}} = \mathring{\mathcal{A}}_I\, dz^I = \mathring{\mathcal{A}}_a\, d\sigma^a + \mathring{\mathcal{A}}_\theta\, d\theta + \mathring{\mathcal{A}}_{\bar\theta}\, d\bar\theta\,.$$

$$\mathring{\mathcal{D}}_I = \partial_I + [\mathring{\mathcal{A}}_I,\ \cdot\ ]\,, \qquad \mathring{\mathfrak{F}}_{IJ} \equiv (1 - \frac{1}{2}\,\delta_{IJ})\Big(\partial_I\,\mathring{\mathcal{A}}_J - (-)^{IJ}\,\partial_J\,\mathring{\mathcal{A}}_I + [\mathring{\mathcal{A}}_I, \mathring{\mathcal{A}}_J]\Big)$$

| ghost charge | Faddeev-Popov ghost triplet | | Vafa-Witten ghost of ghost quintet | | Vector quartet | | |
|---|---|---|---|---|---|---|---|
| 2 | | | $\phi$ | | | | |
| 1 | $G$ | | | $\eta$ | | $\lambda_a$ | |
| 0 | | $B$ | $\phi^0$ | | $\mathcal{A}_a$ | | $\mathcal{F}_a$ |
| -1 | $\bar{G}$ | | | $\bar\eta$ | | $\bar\lambda_a$ | |
| -2 | | | $\bar\phi$ | | | | |

- Cartan charges are gauge-covariant super derivations and obey:

$$\mathring{\mathcal{D}}_{\bar\theta}^2 = \mathring{\mathcal{L}}_{\mathring{\mathfrak{F}}_{\bar\theta\bar\theta}}\,, \qquad \mathring{\mathcal{D}}_{\theta}^2 = \mathring{\mathcal{L}}_{\mathring{\mathfrak{F}}_{\theta\theta}}\,, \qquad \Big[\mathring{\mathcal{D}}_{\bar\theta}, \mathring{\mathcal{D}}_\theta\Big]_\pm = \mathring{\mathcal{L}}_{\mathring{\mathfrak{F}}_{\theta\bar\theta}}\,.$$

# *Act III*

*where we  attempt a synthesis of SK-KMS with a touch of equivariance…*

# SK-KMS THERMAL EQUIVARIANCE

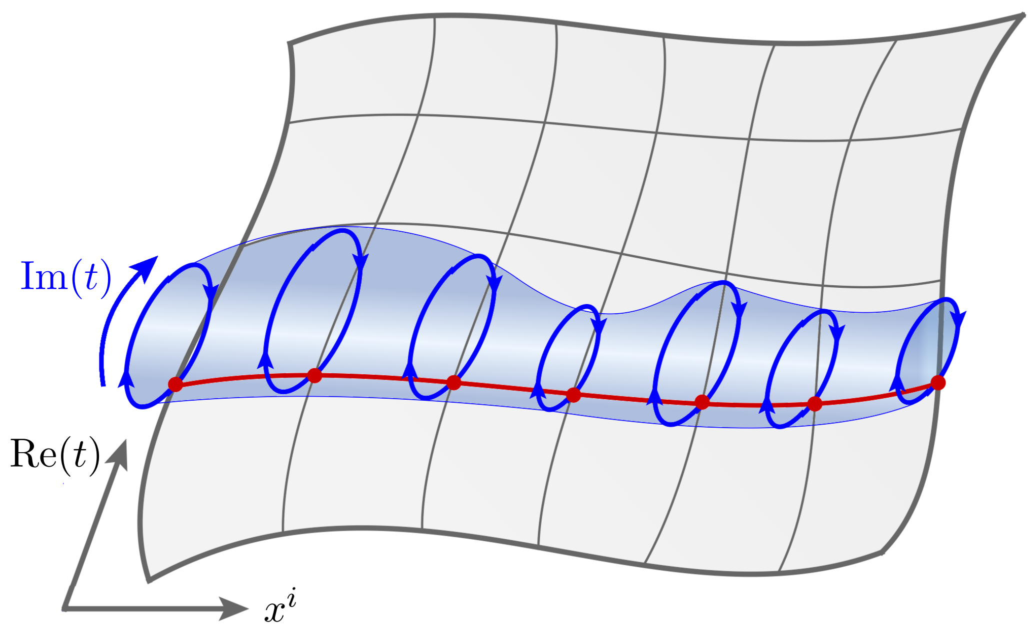
- SK charges are akin to Weil differentials, while the KMS charges fill out the interior contractions.
- The Lie derivation takes operators around the thermal circle.

$$
\begin{array}{rcl}
\underline{\mathcal{N}_{\mathsf{T}} = 2 \text{ algebra}} & | & \underline{\text{SK-KMS symmetries}} \\
\{\mathbb{d}_{\text{W}}, \overline{\mathbb{d}}_{\text{W}}\} & \leftrightarrow & \{\mathcal{Q}_{SK}, \overline{\mathcal{Q}}_{SK}\}\,, \\
\{\mathcal{I}_k, \overline{\mathcal{I}}_k\} & \leftrightarrow & \{\mathcal{Q}_{KMS}, \overline{\mathcal{Q}}_{KMS}\}\,, \\
\{\mathcal{L}_k, \mathcal{I}^0_k\} & \leftrightarrow & \{\mathcal{L}_{KMS}, \mathcal{Q}^0_{KMS}\}\,.
\end{array}
$$

- The algebraic structure for arbitrary temperature is complicated by non-locality of thermal translations.
- Some form of deformation of the group of circle diffeomorphisms…

# SK-KMS THERMAL EQUIVARIANCE

- Life is simpler at high temperatures when thermal circle is small.




- Literally implement thermal translations as diffeomorphisms along the thermal circle and demand equivariance with this symmetry. $\mathring{\mathcal{L}}^{\mathrm{KMS}}\mathring{\mathbb{O}} = \Delta_{\boldsymbol{\beta}}\mathring{\mathbb{O}}$

$$\begin{array}{ccc} \underline{\mathcal{N}_{\mathsf{T}} = 2 \text{ algebra}} & | & \underline{\text{SK-KMS symmetries}} \\ \left[\mathring{\mathfrak{F}}_1, \mathring{\mathfrak{F}}_2\right]^k = f^k_{ij}\,\mathring{\mathfrak{F}}_1\mathring{\mathfrak{F}}_2 & \leftrightarrow & (\mathring{\mathfrak{F}}_1, \mathring{\mathfrak{F}}_2)_{\boldsymbol{\beta}} = \mathring{\mathfrak{F}}_1\,\Delta_{\boldsymbol{\beta}}\,\mathring{\mathfrak{F}}_2 - \mathring{\mathfrak{F}}_2\,\Delta_{\boldsymbol{\beta}}\,\mathring{\mathfrak{F}}_1 \end{array}$$

- This leads to the $U(1)_{\mathsf{T}}$ KMS symmetry discovered in during our attempt to classify hydro transport.

# *THERMAL CARTAN AND WEIL MODELS*

✦ The gauge covariant Cartan charges (supercovariant derivations) can be mapped to the basic building blocks as follows:

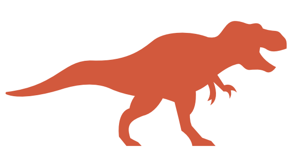
$$\begin{aligned} \mathcal{Q} &\equiv \mathcal{Q}_{_{SK}} + \phi_{_\mathsf{T}}\, \overline{\mathcal{Q}}_{_{KMS}} + \phi^0_{_\mathsf{T}}\, \overline{\mathcal{Q}}_{_{KMS}} + \eta_{_\mathsf{T}}\, \mathcal{Q}^0_{_{KMS}}\,, \\ \overline{\mathcal{Q}} &\equiv \overline{\mathcal{Q}}_{_{SK}} + \overline{\phi}_{_\mathsf{T}}\, \mathcal{Q}_{_{KMS}}\,. \end{aligned}$$

✦ The superalgebra structure can then be captured by the anti-commutation relation among the Cartan charges as

$$\mathcal{Q}^2 = (\mathring{\mathfrak{F}}_{\bar\theta\bar\theta}|_{\bar\theta=\theta=0})\, \mathcal{L}_{_{KMS}}\,, \qquad \overline{\mathcal{Q}}^2 = (\mathring{\mathfrak{F}}_{\theta\theta}|_{\bar\theta=\theta=0})\, \mathcal{L}_{_{KMS}}\,, \qquad \left[\mathcal{Q}, \overline{\mathcal{Q}}\right]_\pm = (\mathring{\mathfrak{F}}_{\theta\bar\theta}|_{\bar\theta=\theta=0})\, \mathcal{L}_{_{KMS}}$$

➤ Assume: dynamically consistent in dissipative systems to set all but the zero ghost number element of the Vafa-Witten quintet to zero: $\langle \mathring{\mathcal{F}}_{\theta\bar\theta}|\rangle = -i$

$$\mathcal{Q}^2 = 0\,, \qquad \overline{\mathcal{Q}}^2 = 0\,, \qquad \left[\mathcal{Q}, \overline{\mathcal{Q}}\right]_\pm = -i\, \mathcal{L}_{_{KMS}} \mapsto i\pounds_{\boldsymbol{\beta}}$$



The final algebra is also the one CGL/GL work with in the high temperature limit and appears to be well known in the stat mech literature (Mallick, Moshe, Orland 2010).

$$\delta^2 = \bar\delta^2 = 0, \qquad \{\delta, \bar\delta\} = 2\, \tanh\left(\frac{i}{2}\,\beta\partial_t\right) \approx i\,\beta\partial_t$$

# *Act IV*

*in which the Brownian particle is thermally equivariantized...*

# TOY MODEL: LANGEVIN DYNAMICS

✦ Point particle in external potential subject to external forcing and noise

$$m\frac{d^2x}{dt^2} + \frac{\partial U}{\partial x} + \nu\;\Delta_{\boldsymbol{\beta}} x = \mathbb{N}$$

✦ One can write down a SK effective action for this dissipative dynamics

$$x = -i\,\Delta_{\boldsymbol{\beta}}^{-1}\left(x_{\rm R} - e^{-i\,\delta_{\boldsymbol{\beta}}}\,x_{\rm L}\right)\,, \qquad \tilde{x} = x_{\rm R} - x_{\rm L}$$

*Martin, Siggia, Rose 1973*

$$\mathcal{L}_{SK} = \left[\tilde{x}\;\frac{\partial U}{\partial x} + \bar{\psi}\frac{\partial^2 U}{\partial x^2}\psi\right] - m\left[\tilde{x}\;\frac{d^2x}{dt^2} + \bar{\psi}\;\frac{d^2\psi}{dt^2}\right] - \nu\left[\tilde{x}\;\Delta_{\boldsymbol{\beta}} x - \bar{\psi}\;\Delta_{\boldsymbol{\beta}}\psi\right] + i\,\nu\,\tilde{x}^2\,.$$

✦ The dissipative part of the action is controlled by ghosts and is related to the fluctuation terms difference fields - fluctuation/dissipation relation.

✦ Convergence of the path integral fixes the sign of dissipative terms.

✦ Simplest realization of the extended equivariant cohomology algebra.

# *BROWNIAN BRANES*

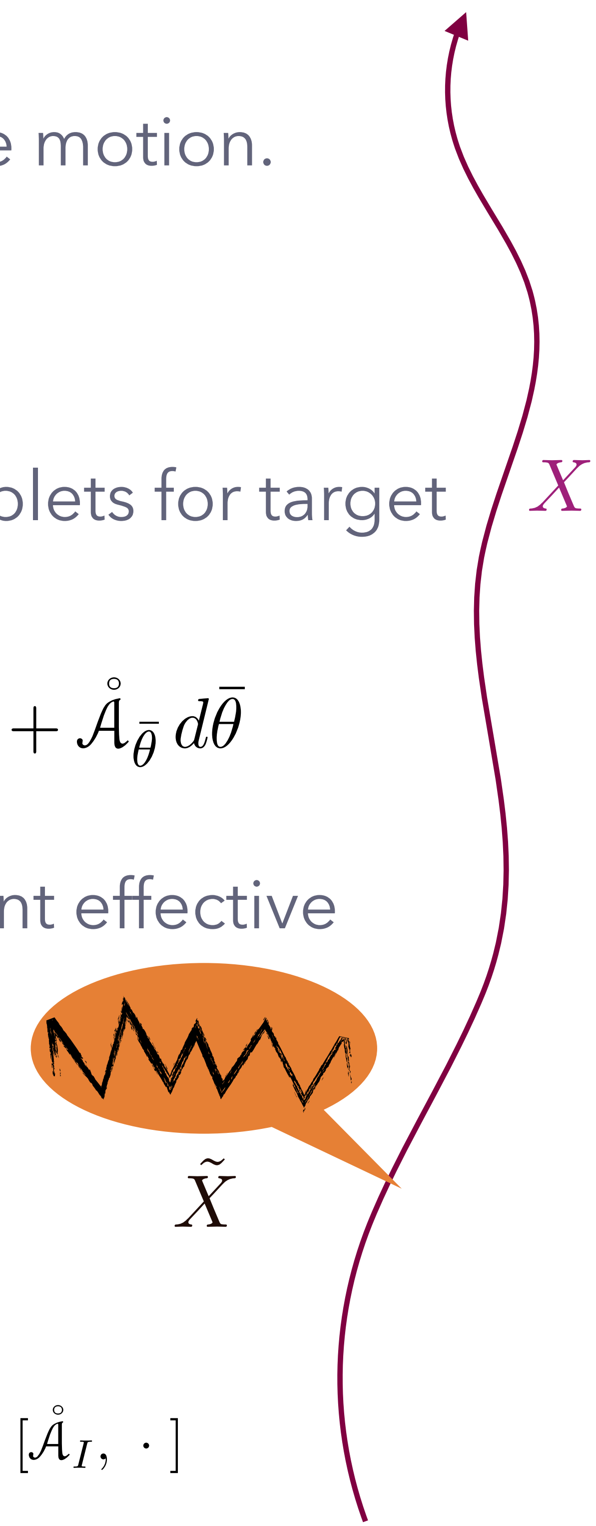
- Brownian particle immersed in a fluid undergoes dissipative motion.
- Langevin effective action: worldvolume B0-brane theory.
- Data for the worldvolume theory: thermal equivariant multiplets for target space coordinate map and thermal gauge field data.

$$\mathring{X} = \{X, X_\psi, X_{\bar{\psi}}, \tilde{X}\} \qquad \mathring{\mathcal{A}} \equiv \mathring{\mathcal{A}}_t\, dt + \mathring{\mathcal{A}}_\theta\, d\theta + \mathring{\mathcal{A}}_{\bar{\theta}}\, d\bar{\theta}$$

- MSR action follows as the basic thermal *U(1)*$_T$ gauge invariant effective action of the worldline theory

$$S_{\text{B0}} = \int dt\, d\theta\, d\bar{\theta} \left\{ \frac{m}{2} \left(\mathring{\mathcal{D}}_t \mathring{X}\right)^2 - U(\mathring{X}) - i\, \nu\, \mathring{\mathcal{D}}_\theta \mathring{X} \mathring{\mathcal{D}}_{\bar{\theta}} \mathring{X} \right\}$$

$$(\mathring{\Lambda}, \mathring{X})_{\boldsymbol{\beta}} = \mathring{\Lambda}\, \pounds_{\boldsymbol{\beta}} \mathring{X} = \mathring{\Lambda}\, \Delta_{\boldsymbol{\beta}} \mathring{X} = \mathring{\Lambda}\, \boldsymbol{\beta} \frac{d}{dt} \mathring{X} \qquad \mathring{\mathcal{D}}_I = \partial_I + [\mathring{\mathcal{A}}_I,\ \cdot\ ]$$



# FLUCTUATION DISSIPATION AS CPT BREAKING

- Stochasticity and dissipation arises because of spontaneous CPT symmetry breaking.

- BRST supersymmetry + spontaneous CPT breaking leads to Jarzynski relation which is a generalized fluctuation dissipation relation

*Jarzynski 1997*
*Crooks 1999*

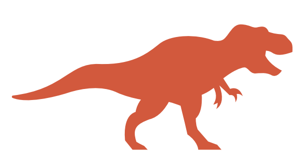
$$S_{\text{B0}} \mapsto S_{\text{B0}} - i\,\langle \mathring{\mathcal{F}}_{\theta\bar{\theta}}|\rangle\, \boldsymbol{\beta}\,(\Delta G + W) \quad \Longrightarrow \quad \langle e^{-\beta W}\rangle = e^{-\beta\,\Delta G}$$

*Mallick, Moshe, Orland 2010*

- The CPT symmetry in our construction is implemented as R-parity in superspace and its breaking encoded in the vev for the ghost number zero field strength: $\langle \mathring{\mathcal{F}}_{\theta\bar{\theta}}|\rangle = -i$

*Gaspard 2012*

- Useful moral: *dissipation = ghost condensation.*



The combined CPT + $U(1)_T$ *transformation ends up being the transformation used by GL to prove entropy positivity.*

# *Act V*

*in which thermal equivariance allows one to write down dissipative fluid dynamics in terms of an effective action, a topological sigma model….*

# *FLUID DYNAMICS AS A SIGMA MODEL*

- Hydrodynamics: low energy dynamics of conserved currents in near equilibrium situations.
- The hydrodynamic modes are Goldstone modes for spontaneously broken difference diffeomorphisms and difference gauge transformation.

*Nickel, Son 2010*

- The order parameter for broken difference diffeomorphisms is a vector field, which we identify with the hydrodynamic velocity rescaled by the local temperature (the pions of hydrodynamics):

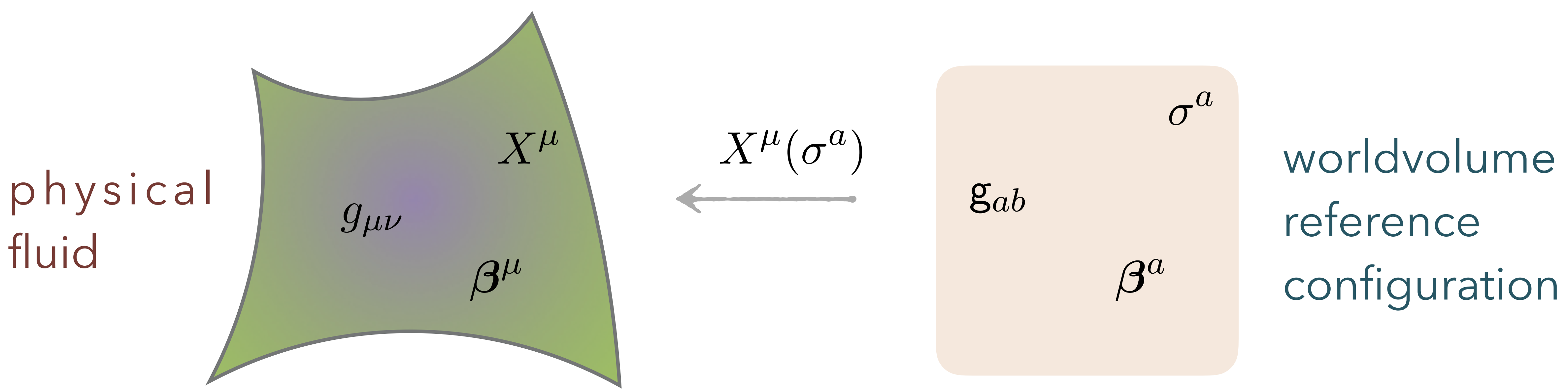
$$\boldsymbol{\beta}^\mu = \frac{u^\mu}{T}, \quad \Lambda_{\boldsymbol{\beta}} = \frac{\mu}{T} - \boldsymbol{\beta}^\alpha A_\alpha$$

- A Landau-Ginzburg theory of this vector field captures a part of hydrodynamic transport (Class L), but getting all of hydrodynamic transport requires more ingredients (cf., eightfold classification).

*Haehl, Loganayagam, MR 2015*

# LANDAU-GINZBURG SIGMA MODELS

- *Class L:* effective action is just a sigma model parameterized by a scalar functional (free energy density) $\mathcal{L}[\boldsymbol{\beta}^a, g_{ab}(X)]$ .
- *Adiabatic fluids:* Invariance under diffeomorphisms and flavour transformations forces non-dissipative dynamics.
- *Dynamics:* conservation follows from variational of the pullback maps with reference thermal vector being fixed.



# THE EIGHTFOLD LAGRANGIAN

- More generally the full set of adiabatic transport derives from an Lagrangian density

$$\mathcal{L}_{\text{wv}} = \frac{1}{2}\,\mathbf{T}^{ab}\,\tilde{\mathrm{g}}_{ab} - \mathbf{N}^{a}_{\mathcal{L}}\,\tilde{\mathcal{A}}_a$$

- New variables $\tilde{\mathrm{g}}_{ab}, \tilde{\mathcal{A}}_a$ : former is the SK partner of the worldvolume metric.

- The one-form is an abelian gauge field which couples to the entropy current.

*Haehl, Loganayagam, MR 2015*

- The linear couplings to the partners is highly suggestive of structures encountered in analysis of linear dissipative systems and topological sigma models.

*Martin, Siggia, Rose 1973*

*Kovtun, Moore, Romatschke 2014*

- Take the symmetry seriously and attempt to work out a full theory including dissipation.

# *TOPOLOGICAL SIGMA MODELS FOR HYDRODYNAMICS*

- Hydrodynamic modes are equivariant maps from the worldvolume to the target space (physical manifold).
- The symmetry being gauged is thermal translations.
- Variables: superfields with top and bottom components being SK difference and average fields respectively

$$\mathcal{Y} \to \mathring{\mathcal{Y}} = \mathcal{Y} + \theta\,\mathcal{Y}_{\bar{\psi}} + \bar{\theta}\,\mathcal{Y}_{\psi} + \bar{\theta}\,\theta\,\tilde{\mathcal{Y}} \equiv \frac{\mathcal{Y}_{\text{L}} + \mathcal{Y}_{\text{R}}}{2} + \theta\,\mathcal{Y}_{\bar{\psi}} + \bar{\theta}\,\mathcal{Y}_{\psi} + \bar{\theta}\,\theta\,(\mathcal{Y}_{\text{R}} - \mathcal{Y}_{\text{L}})$$

- Thermal translations act via Lie drag along reference thermal vector $\mathring{\boldsymbol{\beta}}^I(z)$ .

$$(\mathring{\Lambda}, \mathring{\mathcal{Y}})_{\boldsymbol{\beta}} = \mathring{\Lambda}\, \pounds_{\boldsymbol{\beta}} \mathring{\mathcal{Y}}$$

- KMS gauge superfield implements thermal equivariance.

# TOPOLOGICAL SIGMA MODELS FOR HYDRODYNAMICS

- Symmetries we impose are:
  - Superdiffeomorphisms in target space and world volume
  - CPT symmetry of SK path integrals ( $\mathcal{Z}^*_{SK}[\mathcal{J}_{\mathrm{L}}, \mathcal{J}_{\mathrm{R}}] = \mathcal{Z}_{SK}[\mathcal{J}_{\mathrm{R}}, \mathcal{J}_{\mathrm{L}}]$ )
  - worldvolume ghost number conservation
  - KMS gauge invariance

- Dynamical fields are the pull-back maps which induce a worldvolume super-metric $\mathring{\mathrm{g}}_{IJ}(z) = g_{\mu\nu}(\mathring{X}(z))\, \mathring{\mathcal{D}}_I \mathring{X}^\mu\, \mathring{\mathcal{D}}_J \mathring{X}^\nu$

- Its top component is the SK difference metric which couples to the physical stress tensor.

- Physical fluid dynamics obtained by deforming the topological theory.

$$\mathring{\mathrm{g}}_{IJ}(z) \;\to\; \mathring{\mathrm{g}}_{IJ}(z) + \bar{\theta}\,\theta\; \mathsf{h}_{IJ}(\sigma)$$

# DISSIPATIVE HYDRODYNAMIC ACTIONS

- Working in superspace the symmetries suffice to constrain the terms that can appear in the worldvolume sigma model

$$S_{\rm wv} \equiv \int d^d\sigma\, \mathcal{L}_{\rm wv}\,, \qquad \mathcal{L}_{\rm wv} = \int d\theta\, d\bar{\theta}\, \frac{\sqrt{-\mathring{\mathfrak{g}}}}{1+\boldsymbol{\beta}^e \mathring{\mathcal{A}}_e} \left( \mathring{\mathcal{L}} - \frac{i}{4}\, \mathring{\boldsymbol{\eta}}^{(ab)(cd)}\, \mathring{\mathcal{D}}_\theta \mathring{\mathfrak{g}}_{ab}\, \mathring{\mathcal{D}}_{\bar{\theta}} \mathring{\mathfrak{g}}_{cd} \right)$$

- In ordinary space we get back the adiabatic lagrangian + dissipation

$$\mathcal{L}_{\rm wv} = \frac{\sqrt{-\mathfrak{g}}}{1+\boldsymbol{\beta}^e \mathcal{A}_e} \left\{ \frac{1}{2} \left[ \mathbf{T}^{ab}_{\mathcal{L}} - \frac{i}{2} \boldsymbol{\eta}^{(ab)(cd)}\, (\mathcal{F}_{\theta\bar{\theta}}, \mathfrak{g}_{cd})_{\boldsymbol{\beta}} \right] \tilde{\mathfrak{g}}_{ab} - \mathbf{N}^a_{\mathcal{L}} \tilde{\mathcal{A}}_a \right.$$
$$\left. + \frac{i}{8} \left( \boldsymbol{\eta}^{(ab)(cd)} + \boldsymbol{\eta}^{(cd)(ab)} \right) \tilde{\mathfrak{g}}_{ab}\, \tilde{\mathfrak{g}}_{cd} + \dots \right\},$$

*Class LT Lagrangian*

*Noise fluctuations*

*Kovtun, Moore, Romatschke 2014*
*Crossley, Glorioso, Liu 2015*

- Again dissipative dynamics spontaneously breaks CPT, KMS field strength picks up a vev and ghost condenses.

# HOLOGRAPHIC FLUIDS

- Known second order transport of holographic fluids follows from:

$$\mathcal{L}_{\mathrm{wv}} = c_{\mathrm{eff}} \int d\theta\, d\bar{\theta}\, \frac{\sqrt{-\mathring{\mathsf{g}}}}{1 + \boldsymbol{\beta}^e \mathring{\mathcal{A}}_e} \Bigg\{ \left(\frac{4\pi\, \mathring{\mathsf{T}}}{d}\right)^d \left(1 - \frac{i\, d}{8\pi}\, \mathring{\mathsf{P}}^{c\langle a}\mathring{\mathsf{P}}^{b\rangle d}\ \mathring{\mathcal{D}}_\theta \mathring{\mathsf{g}}_{ab}\ \mathring{\mathcal{D}}_{\bar{\theta}} \mathring{\mathsf{g}}_{cd}\right)$$
$$- \left(\frac{4\pi\, \mathring{\mathsf{T}}}{d}\right)^{d-2} \left[\frac{{}^{\mathcal{W}}\mathring{\mathsf{R}}}{d-2} + \frac{1}{d}\, \mathrm{Harmonic}\left(\frac{2}{d} - 1\right) \mathring{\sigma}^2 + \frac{1}{2}\mathring{\omega}^2\right] \Bigg\}$$

- How does the bulk gravity theory realize this effective action?
- Recent attempts get the ideal fluid part correct, but no clear story beyond…

*Nickel, Son 2010 (ideal)*

*Crossley, Glorioso, Liu, Wang 2015 (incomplete at second order)*
*deBoer, Heller, Pinzani-Fokeeva 2015 (ideal)*

# *FLUCTUATION-DISSIPATION & JARZYNSKI*

- Presence of a gauge symmetry which couples to entropy current appears to be manifestly contradicting second law.
- Claim: entropy flows into the physical sector from the ghost sector. Appears to work in superspace cleanly…
- The MMO argument goes through in the hydrodynamic effective action leading to a derivation of the Jarzynski relation which them implies the 2nd law using convexity of the exponential function.

$$\langle e^{-\frac{W}{\mathsf{T}}} \rangle = e^{-\frac{1}{\mathsf{T}}\left(G_f - G_i\right)} \qquad\qquad \langle W \rangle \geq G_f - G_i$$

- Note only stochastic fluctuations accounted for thus far. Requires understanding of full KMS structure for quantum effects.

## *Epilogue: Of that which is yet to be...*

# LOOKING AHEAD…

- Near-equilibrium dynamics appears to be under control (should however write down the eightfold topological sigma model). What about non-equilibrium?
- Open quantum systems & renormalization *Avinash, Jana, Loganayagam, Rudra 2017*
- How does thermal equivariance extend to include non-stochastic fluctuations? Deformation quantization? *Basart, Flato, Lichnerowicz, Sternheimer 1984*
- Microscopic unitary which enforces fluctuation-dissipation etc., is upheld thanks to the ghost couplings. Lessons for gravity?
- What is the analogous story for higher out-of-time-order correlators?
- Are the similar statements for modular evolutions (equivalent in some contexts), and if so what does it imply for geometry = entanglement?

# A ROADMAP FOR THE FUTURE….

*Microscopic Schwinger-Keldysh construction*

★ *doubling of degrees of freedom*

★ *entanglement structure in initial state*

Wilsonian IR physics

ER=EPR

*Black hole dynamics*

★ *emergence of horizons?*

★ *reality of the interior?*

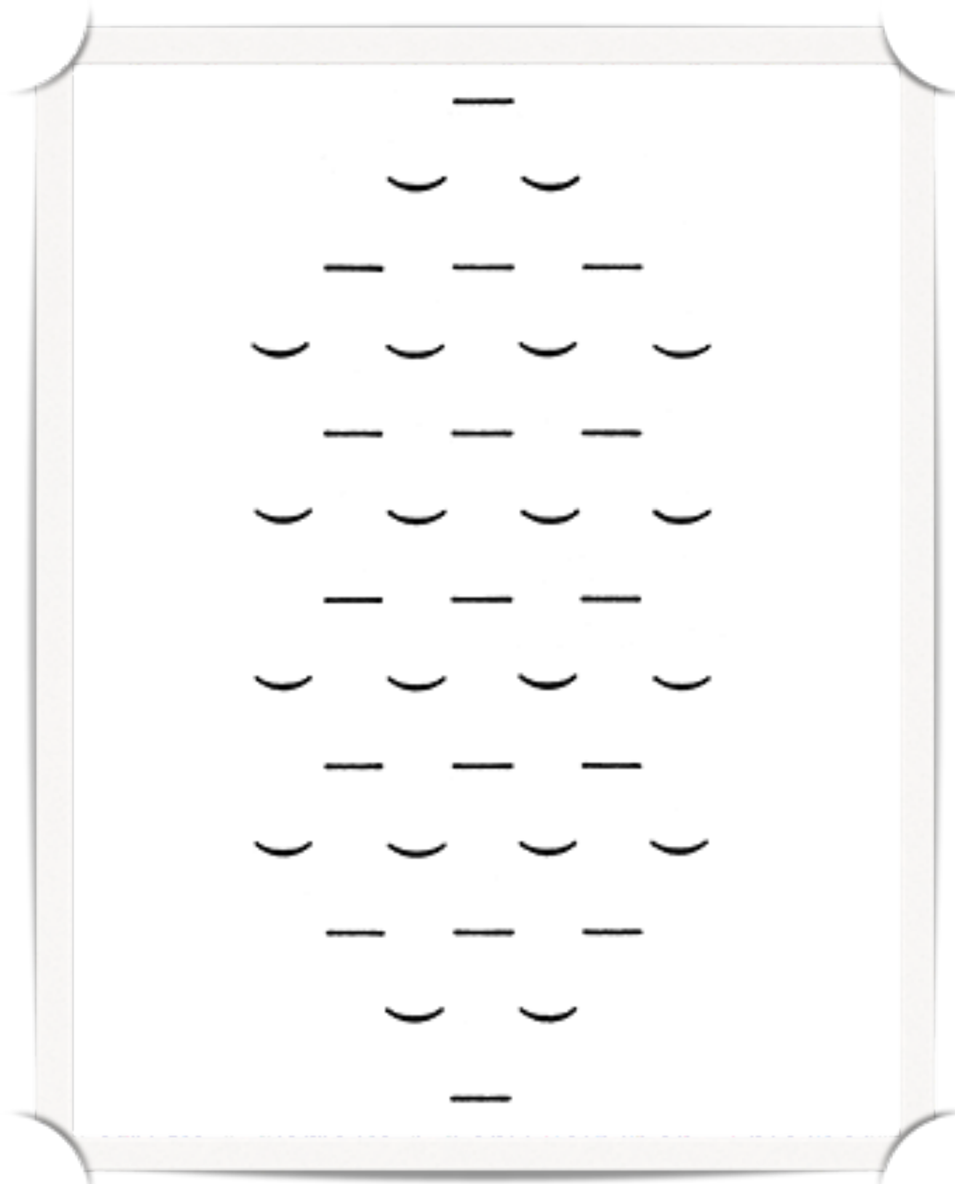
*Macrophysics: cf., hydrodynamics*

★ *no doubling!*

★ *emergent IR collective fields*

★ *entropy & second law of thermodynamics*

Fluid/Gravity



*Fisches Nachtgesang: Christian Morgenstern*