修士学位論文

Gravitational effects on Dirac particles

(ディラック粒子に対する重力の効果)

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Abstract

We discuss the gravitational effects on Dirac particles which propagate in Kerr geometry. In particular, we provide a simple framework for studying the gravitational effects on a Dirac particle with infinitesimal mass. We perform our calculations in the slowly rotating, weak field limit. The two-component Weyl equations with the corrections arising from the infinitesimal mass and the gravitational field are obtained from the covariant Dirac equation. Our approach is also applied to neutrino oscillations in the presence of the gravitational field.

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Chapter 1

Introduction

The discoveries of a quantum concept and a new space-time concept are great achievements in physics in this century. The former concept was systematized as a theory, together with quantum mechanics, while the latter was first introduced by Einstein in the form of special relativity in 1905, and the space-time theory involving gravity, general relativity, was completed in 1916. On the other hand, one of the great physical problems in this century is to unify the theories of the forces of nature. The various physical phenomena we know can be explained by these theories. The electromagnetic interaction and the weak interaction have been unified with the Weinberg-Salam theory [1], [2]. Moreover, attempts to incorporate the strong interaction into a wider theory seem to be successful with the so-called Grand Unified Theories (GUTs) [3]. These interactions except the gravitational interaction are described in terms of the quantum theory. Although the gravitational interaction was the first force to be investigated classically, it was the most difficult one to be quantized. Many physicists have pursued the quantization of the gravitational field with intense vigor over the past half of a century, and a considerable number of results are gained. However, a satisfactory quantum theory of gravity is not yet completed. Hence gravity appears to stand apart from the other three forces.

Next, we shall see what circumstances the quantum theory of gravity becomes important under. Since the gravitational force is weaker than the other three forces by a factor of about 10^{40}, the gravitational interaction can be ignored in ordinary particle accelerator experiments. However, at an energy scale of about 10^{19} GeV, the gravitational interaction becomes dominant. To see this, let us recall that the Newtonian gravitational force is given by $F = Gm_1m_2/r^2$, where the gravitational constant G is approximately

$(1.2 \times 10^{19}\,\mathrm{GeV})^{-2}$ in the system of natural units ($\hbar = c = 1$). Note that the mass of a particle can be regarded as the energy divided by c^2. Therefore, if the particle has the so-called Planck energy, $10^{19}\,\mathrm{GeV}$, then the gravitational force will be comparable to the other forces. The Compton wavelength corresponding to this energy scale, $10^{-35}\,\mathrm{m}$, is called the Planck length. These are, of course, far beyond the range of our instruments. However, at the singularity at the center of a black hole, or at the instant of the Big Bang, the quantum corrections will become important. Do we always have to consider these extreme situations to the problems in which gravity cannot be separated from the quantum theory? In other words, do the physical phenomena in which gravitational effects and quantum effects appear simultaneously occur only under these extreme situations? The answer is "No". There also exist these physical phenomena under ordinary situations, where there seems to be no need to consider the quantization of the gravitational field. One of the representative examples is the so-called Colella-Overhauser-Werner (COW) experiment [4] using a neutron interferometer (for a review see [5]). This kind of experiment has become possible by the grace of recent progress in technology, and at the same time the other experiments which have been described as thought experiments have also become realizable.

The COW experiment was the first experiment that measures the gravitational effects on a wave function. The gravitational effects on this elegant experiment were often compared with the Aharonov-Bohm (AB) effect [6]. Aharonov and Bohm suggested that even if electromagnetic fields vanish, there exist the effects of the electromagnetic potentials on quantum interference. Hence this effect is called AB effect. On the other hand, the COW experiment showed the effect of the gravitational potential of Earth on quantum interference. This effect and the detectability were first suggested by Overhauser and Colella [7], and the effect was verified by Colella, Overhauser and Werner [4] using a neutron interferometer. Although their analysis, which was based on inserting the Newtonian gravitational potential into the Schrödinger equation, was so simple, this experiment was conceptually very important in the history of the quantum theory. After the COW experiment had been done, another effect arising from the rotation of Earth was discussed by Page [8]. The experimental verification of this effect was provided by Werner, Staudenmann and Colella [9]. This theoretical expression was also derived by various authors using various methods [10], [11], [12]. Furthermore, the general relativistic effects including these effects were derived by Kuroiwa, Kasai and Futamase [13] starting with the covariant Klein-Gordon equation, which describes a spinless particle. However, a neutron has spin-1/2 and is described by the Dirac equation. Under this consideration, Wajima, Kasai and Futamase [14] have derived the general relativistic effects including the spin effects by using the covariant Dirac equation.

On the other hand, gravitational effects on another physical phenomenon, neutrino oscillations, have been much discussed recently [15], [16], [17], [18], [19], [20], [21]. The COW experiment and this have common ground that the gravitational effects appear in the quantum interference. However, there are some differences between the two. In the former case, the gravitational effects on a single mass eigenstate are investigated, and the spatial spread of the wave function plays a significant role. On the other hand, in the latter case, the existence of the different mass eigenstates and the linear superposition are important. Moreover, it is another important difference whether the related particle is non-relativistic or ultra-relativistic.

It seems that the controversy about the gravitationally induced neutrino oscillation phases arises. Ahluwalia and Burgard [15] state that the phases amount to approximately 20 % of the kinematic counterparts in the vicinity of a neutron star. Nevertheless, the definition of the neutrino energy and the derivation of the phases were not clear in the original paper [15]. The other groups [17], [18], [20], [21] have obtained similar results for a radially propagating neutrino; the results seem to be different from that in Ref. [15]. However, the authors of Ref. [17] assume that the different mass eigenstates are produced at different times. This assumption seems to be questionable because the relative phase between the two mass eigenstates initially becomes arbitrary. These papers except Ref. [20] are based on the previous work [22], in which the classical action is taken as a quantum phase. Therefore, the effects arising from the spin of the particle are not considered in these papers. On the other hand, the authors of Ref. [20] use the covariant Dirac equation, but they also calculate the classical action along the particle trajectory in the end.

In this situation, we shall provide a simple framework different from the previous work for studying the gravitational effects on a Dirac particle with infinitesimal mass such as a neutrino. (The experimental confirmation which shows that neutrinos have nonzero mass is not yet obtained. However, the recent experimental report [23] seems to suggest neutrinos to be massive.) In particular, we consider the propagation of the particle in the Kerr geometry, by which the external field of a rotating star can be described. We do not merely calculate the classical action along the particle trajectory, but start from the covariant Dirac equation. We shall perform our calculations in the slowly rotating, weak field limit, and derive the two-component Weyl equations with the corrections arising from the small mass and the gravitational field. Furthermore, we shall discuss the neutrino oscillations in the presence of the gravitational field.

The organization of this thesis is as follows. In Chapter 2, we summarize the covariant formalism of general fields and derive the covariant Dirac equation from the viewpoint of the strong Principle of Equivalence. In Chapter 3, we consider the gravitational field arising from a rotating object, and specify the metric and the coordinates in terms of the (3+1) formalism. In the last part of this chapter, we discuss the covariant Dirac equation in this field, and derive an equation of the Schrödinger-type. In Chapter 4, following the discussion of Ref. [14], we derive the Schrödinger equation with general relativistic corrections for a non-relativistic particle, and summarize the general relativistic effects on a quantum interferometer. In Chapter 5, we derive the Weyl equations with general relativistic corrections for a ultra-relativistic particle, and discuss the gravitationally induced neutrino oscillation phases. Finally, we shall give a summary and conclusion in Chapter 6.

We use the following notation; Latin indices i, j, k, and so on generally run over three spatial coordinate labels 1, 2, 3 or x, y, z, Latin indices a, b, c, and so on over the four space-time inertial coordinate labels 0, 1, 2, 3 or t, x, y, z, and Greek indices α, β, γ, and so on over the four coordinate labels in a general coordinate system. Furthermore, we use the metric signature $(+,-,-,-)$.

Chapter 2

Covariant formalism

In this chapter, we review the covariant formalism of general fields including spinors. We shall derive the covariant Dirac equation in the last part of this chapter. (See also Ref. [24], [25], [26].)

In the case of tensor equations, we replace all Lorentz tensors $T^{a\cdots}_{b\cdots}$ with objects $T^{\alpha\cdots}_{\beta\cdots}$ which behave like tensors under general coordinate transformations to derive the general-relativistic equations from the special-relativistic ones. Moreover, we replace all derivatives ∂_a with covariant derivatives ∇_α, and replace η_{ab} with $g_{\alpha\beta}$. The equations are then generally covariant. Thus we can make equations describing scalar fields or tensor fields generally covariant forms. This method actually works only for objects which behave like tensors under Lorentz transformations, and not for spinor fields describing spin-1/2 particles. How then can we incorporate spinors into general relativity? The clue to the question lies in a fact that spinors are well defined in the Minkowski space-time.

We now consider locally inertial coordinate systems at every space-time point, and define general fields involving spinors in these coordinate systems. By relating the locally inertial coordinate systems to general non-inertial coordinate systems, we shall extend the theories of the fields into the curved space-time. From the viewpoint of the strong Principle of Equivalence, which states that all the laws of nature in a locally inertial coordinate system take the same form as in unaccelerated Cartesian coordinate systems in the absence of gravitation, this approach is reasonable.

2.1 Covariant differentiation

To begin with, let us introduce the so-called tetrad, or vierbein formalism. We erect a set of locally inertial coordinates ${\xi_X}^a$ at every space-time point X. The metric in any general non-inertial coordinate system is then

$$g_{\alpha\beta}(x) = {e^{(a)}}_\alpha(x)\, {e^{(b)}}_\beta(x)\, \eta_{ab}, \tag{2.1}$$

where

$$ {e^{(a)}}_\alpha(X) \equiv \left(\frac{\partial {\xi_X}^a(x)}{\partial x^\alpha} \right)_{x=X}. \tag{2.2}$$

If we change the general non-inertial coordinates from x^α to x'^α, then ${e^{(a)}}_\alpha$ changes according to

$$ {e^{(a)}}_\alpha \to {e'^{(a)}}_\alpha = \frac{\partial x^\beta}{\partial x'^\alpha}\, {e^{(a)}}_\beta. \tag{2.3}$$

Thus, the tetrad ${e^{(a)}}_\alpha$ forms four covariant vector fields.

Given any contravariant vector field $A^\alpha(x)$, we can use the tetrad to refer its components to the locally inertial coordinate system ${\xi_X}^a$ at x:

$$\tilde{A}^a \equiv {e^{(a)}}_\alpha A^\alpha, \tag{2.4}$$

which behaves like scalars under general coordinate transformations. We can do the same with general tensor fields:

$$\tilde{A}_a \equiv {e_{(a)}}^\alpha A_\alpha, \tag{2.5}$$
$$\tilde{B}^a{}_b \equiv {e^{(a)}}_\alpha\, {e_{(b)}}^\beta\, B^\alpha{}_\beta, \quad \text{etc}, \tag{2.6}$$

where

$$ {e_{(a)}}^\alpha \equiv \eta_{ab}\, g^{\alpha\beta}\, {e^{(b)}}_\beta. \tag{2.7}$$

Furthermore, it is easy to show that the tetrad satisfies the relations

$$ {e_{(a)}}^\alpha\, {e^{(a)}}_\beta = \delta^\alpha{}_\beta, \tag{2.8}$$
$$ {e_{(a)}}^\alpha\, {e^{(b)}}_\alpha = \delta_a{}^b. \tag{2.9}$$

We have shown how to derive objects which behave like scalars under general coordinate transformations. Since the Principle of Equivalence requires that special relativity should apply in locally inertial frames, the scalar field components $\tilde{A}^a$, $\tilde{B}^a{}_b$, and so on, which are defined in an arbitrarily chosen locally inertial coordinate system, must behave like vectors or tensors with respect to Lorentz transformations $\Lambda^a{}_b(x)$ at x:

$$\begin{aligned} \tilde{A}^a(x) &\rightarrow \Lambda^a{}_b(x)\, \tilde{A}^b(x), && (2.10)\\ \tilde{B}^a{}_b(x) &\rightarrow \Lambda^a{}_c(x)\, \Lambda_b{}^d(x)\, \tilde{B}^c{}_d(x), \quad \text{etc}, && (2.11) \end{aligned}$$

where

$$\eta_{ab}\Lambda^a{}_c(x)\Lambda^b{}_d(x) = \eta_{cd}. \tag{2.12}$$

Similarly, the tetrad $e^{(a)}{}_\alpha$ changes according to

$$e^{(a)}{}_\alpha(x) \rightarrow \Lambda^a{}_b(x)\, e^{(b)}{}_\alpha(x). \tag{2.13}$$

In general, an arbitrary field $\tilde{\psi}_m$ defined in a locally inertial coordinate system will change in the following way:

$$\tilde{\psi}_m(x) \rightarrow \sum_n \left[U\left(\Lambda(x)\right)\right]_{mn} \tilde{\psi}_n(x), \tag{2.14}$$

where $U\left(\Lambda(x)\right)$ is a matrix representation of the Lorentz group. For example, if $\tilde{\psi}$ is a covariant vector $\tilde{A}_a$, the $U\left(\Lambda(x)\right)$ is simply

$$\left[U\left(\Lambda(x)\right)\right]_a{}^b = \Lambda_a{}^b(x), \tag{2.15}$$

whereas for a contravariant tensor $\tilde{T}^{ab}$,

$$\left[U\left(\Lambda(x)\right)\right]^{ab}{}_{cd} = \Lambda^a{}_c(x)\, \Lambda^b{}_d(x). \tag{2.16}$$

An ordinary derivative is, of course, a coordinate vector in the sense that it transforms as a vector under a general coordinate transformation $x \rightarrow x'$:

$$\frac{\partial}{\partial x^\alpha} \rightarrow \frac{\partial}{\partial x'^\alpha} = \frac{\partial x^\beta}{\partial x'^\alpha}\frac{\partial}{\partial x^\beta}. \tag{2.17}$$

We can also use the tetrad to form a coordinate scalar derivative:

$$e_{(a)}{}^\alpha \frac{\partial}{\partial x^\alpha}\,. \tag{2.18}$$

This coordinate scalar derivative corresponds to the ordinary derivative defined in locally inertial coordinate systems. Although this is actually a coordinate scalar, it does not have simple transformation properties under position-dependent Lorentz transformations when it acts on general fields. Given a general field $\tilde{\psi}$, we have the following transformation rule under the Lorentz transformation rule (2.14):

$$\begin{aligned} e_{(a)}{}^{\alpha}(x)\,\frac{\partial}{\partial x^{\alpha}}\,\tilde{\psi}(x) \;\;&\rightarrow\;\; \Lambda_a{}^b(x)\,e_{(b)}{}^{\alpha}(x)\,\frac{\partial}{\partial x^{\alpha}}\left\{U\left(\Lambda(x)\right)\tilde{\psi}(x)\right\} \\ &= \;\Lambda_a{}^b(x)\,e_{(b)}{}^{\alpha}(x)\left[U\left(\Lambda(x)\right)\,\frac{\partial}{\partial x^{\alpha}}\tilde{\psi}(x)+\left\{\frac{\partial}{\partial x^{\alpha}}U\left(\Lambda(x)\right)\right\}\tilde{\psi}(x)\right]. \end{aligned} \tag{2.19}$$

In order to extend the theory in flat space-time into curved space-time, what we have to do is to make an operator $\tilde{D}_a$ which under a position-dependent Lorentz transformation $\Lambda^a{}_b(x)$, satisfies the transformation rule

$$\tilde{D}_a\tilde{\psi}(x) \rightarrow \Lambda_a{}^b(x)U\left(\Lambda(x)\right)\;\tilde{D}_b\tilde{\psi}(x). \tag{2.20}$$

Thus $\tilde{D}_a\tilde{\psi}$ behaves like a tensor with one extra covariant rank under position-dependent Lorentz transformations. By replacing $\partial_a\tilde{\psi}$ in field equations in flat space-time with $\tilde{D}_a\tilde{\psi}$, we can obtain the field equations which are independent of the choice of locally inertial coordinate systems.

Considering Eq. (2.19), we can construct a coordinate-scalar Lorentz-vector derivative $\tilde{D}_a$ of the form

$$\tilde{D}_a \equiv e_{(a)}{}^{\alpha}\left[\frac{\partial}{\partial x^{\alpha}}-\Gamma_{\alpha}\right], \tag{2.21}$$

where Γ_α is a matrix satisfying the Lorentz transformation rule

$$\Gamma_{\alpha}(x) \;\rightarrow\; U\left(\Lambda(x)\right)\;\Gamma_{\alpha}(x)\,U^{-1}\left(\Lambda(x)\right)+\left[\frac{\partial}{\partial x^{\alpha}}U\left(\Lambda(x)\right)\right]U^{-1}\left(\Lambda(x)\right). \tag{2.22}$$

In fact, we can obtain the transformation rule (2.20) under this definition.

The coordinate-scalar Lorentz-vector derivative $\tilde{D}_a$ is a covariant derivative with respect to the position-dependent Lorentz transformations. The introduction of this derivative allows us to extend the discussion of general fields involving spinors into curved space-time. Therefore, we can obtain the general field equations in curved space-time.

2.2 Connection

In this section, we shall investigate the structure of the connection Γ_α introduced in the last section.

For the purpose of this, it will be sufficient to consider infinitesimal Lorentz transformations close to the identity:

$$\Lambda^a{}_b(x) = \delta^a{}_b + \omega^a{}_b(x), \qquad |\omega^a{}_b| \ll 1. \tag{2.23}$$

Since the Lorentz transformation $\Lambda^a{}_b(x)$ is restricted by the condition (2.12), to first order in ω, we have

$$\omega_{ab}(x) = -\omega_{ba}(x), \tag{2.24}$$

where the indices on ω are lowered or raised with η. Under this transformation, the matrix representation $D\left(\Lambda(x)\right)$ must be written as

$$U\left(1 + \omega(x)\right) = \mathbf{1} + \frac{1}{2}\omega^{ab}(x)\,\sigma_{ab}, \tag{2.25}$$

where σ_{ab} are a set of constant matrices, which can always be chosen as an antisymmetric tensor:

$$\sigma_{ab} = -\sigma_{ba}. \tag{2.26}$$

For example, if we consider a covariant vector $\tilde{A}_a$, then we have

$$\left[\sigma_{ab}\right]_c{}^d = \eta_{ac}\delta_b{}^d - \eta_{bc}\delta_a{}^d, \tag{2.27}$$

and for a contravariant tensor $\tilde{T}^{ab}$, we have

$$\left[\sigma_{ab}\right]^{cd}{}_{ef} = \delta_a{}^c\,\eta_{be}\delta^d{}_f - \delta_b{}^c\,\eta_{ae}\delta^d{}_f + \delta_a{}^d\,\eta_{bf}\delta^c{}_e - \delta_b{}^d\,\eta_{af}\delta^c{}_e. \tag{2.28}$$

Furthermore, the matrices σ_{ab} satisfy the commutation relations:

$$\left[\sigma_{ab}, \sigma_{cd}\right] = \eta_{cb}\sigma_{ad} - \eta_{ca}\sigma_{bd} + \eta_{db}\sigma_{ca} - \eta_{da}\sigma_{cb} \tag{2.29}$$

with square brackets denoting the usual commutator

$$[A, B] \equiv AB - BA. \tag{2.30}$$

The details are summarized in Appendix A.

Using the transformation rule (2.22), we see that under the infinitesimal Lorentz transformation (2.23), the connection Γ_α transforms according to

$$\Gamma_\alpha(x) \;\rightarrow\; \Gamma_\alpha(x) + \frac{1}{2}\,\omega^{ab}(x)\left[\sigma_{ab}, \Gamma_\alpha(x)\right] + \frac{1}{2}\,\sigma_{ab}\frac{\partial}{\partial x^\alpha}\omega^{ab}(x). \tag{2.31}$$

We now assume that the connection Γ_α has the form:

$$\Gamma_\alpha(x) = \frac{1}{2}\, C^{ab}{}_\alpha(x)\, \sigma_{ab}, \tag{2.32}$$

where the Latin indices on C are, of course, lowered or raised with η, whereas the Greek index with g. In this case, $C^{ab}{}_\alpha(x)$ is antisymmetric in a and b. Using the transformation rule (2.31) and the commutation relations (2.29), we obtain the following transformation rule of $C^{ab}{}_\alpha(x)$:

$$C^{ab}{}_\alpha(x) \;\rightarrow\; C^{ab}{}_\alpha(x) + \omega^a{}_c(x) C^{cb}{}_\alpha(x) + \omega^b{}_c(x) C^{ac}{}_\alpha(x) + \frac{\partial}{\partial x^\alpha}\omega^{ab}(x). \tag{2.33}$$

Until now, we have investigated the transformation property of the connection Γ_α. Next, we turn our attention to the relation between the connection Γ_α and the tetrad $e_{(a)\alpha}$.

The coordinate-scalar Lorentz-vector derivative $\tilde{D}_a$ is a covariant derivative with respect to local Lorentz transformations. This derivative $\tilde{D}_a$ is different from the covariant derivative ∇_α used in tensor analysis. Nevertheless, there must be some relation between the two. In order to investigate the relation, we consider the case in which the derivative $\tilde{D}_a$ acts on the tetrad $e_{(a)\alpha}$. Using Eq. (2.27), we have

$$\begin{aligned} \tilde{D}_a\, e_{(b)\alpha} &= e_{(a)}{}^\mu \left[\frac{\partial}{\partial x^\mu} e_{(b)\alpha} - \frac{1}{2}\, C^{ef}{}_\mu \left[\sigma_{ef}\right]_b{}^c\, e_{(c)\alpha} \right] \\ &= e_{(a)}{}^\mu \left[\frac{\partial}{\partial x^\mu} e_{(b)\alpha} - \eta_{bc}\, C^{cd}{}_\mu\, e_{(d)\alpha} \right]. \end{aligned} \tag{2.34}$$

However, we have not considered the generally covariant index α. The tetrad $e_{(a)\alpha}$ behaves like not only a Lorentz-vector, but also a coordinate-vector. Hence we have to consider the property as a coordinate-vector as well. We know the covariant derivative of a coordinate covariant vector. We now introduce a total covariant derivative $\tilde{\mathcal{D}}_a$ defined as

$$\tilde{\mathcal{D}}_a\, e_{(b)\alpha} = e_{(a)}{}^\mu \left[\frac{\partial}{\partial x^\mu} e_{(b)\alpha} - \eta_{bc}\, C^{cd}{}_\mu\, e_{(d)\alpha} - \Gamma^\nu{}_{\alpha\mu}\, e_{(b)\nu} \right], \tag{2.35}$$

where $\Gamma^\nu{}_{\alpha\mu}$ is the Christoffel symbol. We can do the same with the tetrad $e^{(a)}{}_\alpha$:

$$\tilde{\mathcal{D}}_a\, e^{(b)}{}_\alpha = e_{(a)}{}^\mu \left[\frac{\partial}{\partial x^\mu} e^{(b)}{}_\alpha - \eta_{cd}\, C^{bc}{}_\mu\, e^{(d)}{}_\alpha - \Gamma^\nu{}_{\alpha\mu}\, e^{(b)}{}_\nu \right]. \tag{2.36}$$

By the way, the covariant derivative of the metric $g_{\alpha\beta}$ vanishes:

$$\nabla_\mu\, g_{\alpha\beta} \equiv 0. \tag{2.37}$$

Hence, from Eq. (2.1) we find

$$\begin{aligned} 0 &= \nabla_\mu \, g_{\alpha\beta} \\ &= e^{(a)}_{\ \ \mu} \, \tilde{\mathcal{D}}_a \, g_{\alpha\beta} \\ &= e^{(a)}_{\ \ \mu} \left[\left(\tilde{\mathcal{D}}_a e^{(b)}_{\ \ \alpha} \right) e_{(b)\beta} + e^{(b)}_{\ \ \alpha} \left(\tilde{\mathcal{D}}_a e_{(b)\beta} \right) \right]. \end{aligned} \tag{2.38}$$

The simplest solution of this equation is

$$\tilde{\mathcal{D}}_a \, e^{(b)}_{\ \ \alpha} = \tilde{\mathcal{D}}_a \, e_{(b)\alpha} = 0. \tag{2.39}$$

From now on, we regard this solution as a fundamental condition. Using this condition, we can find the relation between the connection Γ_α and the tetrad $e_{(a)\alpha}$. From Eqs. (2.35) and (2.36), we obtain

$$C^{ab}_{\ \ \ \alpha}(x) = -\eta^{ac}\eta^{bd} \, e_{(c)}^{\ \ \lambda} \, \nabla_\alpha \, e_{(d)\lambda}. \tag{2.40}$$

Therefore, the connection Γ_α is given by

$$\Gamma_\alpha(x) = -\frac{1}{2} \, \sigma^{ab} g_{\mu\nu} \, e_{(a)}^{\ \ \mu} \, \nabla_\alpha \, e_{(b)}^{\ \ \nu}. \tag{2.41}$$

Note that under the infinitesimal Lorentz transformations (2.23), the tetrad $e_{(a)\alpha}$ transforms according to

$$e_{(a)\alpha}(x) \ \rightarrow \ e_{(a)\alpha}(x) + \omega_a^{\ \ b}(x) \, e_{(b)\alpha}(x), \tag{2.42}$$

and, hence, we have

$$\begin{aligned} e_{(a)}^{\ \ \lambda}(x) \, \nabla_\alpha \, e_{(b)\lambda}(x) \ \ \rightarrow \ \ & e_{(a)}^{\ \ \lambda}(x) \, \nabla_\alpha \, e_{(b)\lambda}(x) + \omega_a^{\ \ c}(x) \, e_{(c)}^{\ \ \lambda}(x) \, \nabla_\alpha \, e_{(b)\lambda}(x) \\ & + \omega_b^{\ \ c}(x) \, e_{(a)}^{\ \ \lambda}(x) \, \nabla_\alpha \, e_{(c)\lambda}(x) - \frac{\partial}{\partial x^\alpha}\omega_{ab}(x). \end{aligned} \tag{2.43}$$

Therefore, we see that $C^{ab}_{\ \ \ \alpha}(x)$ given by Eq. (2.40) satisfies the transformation rule (2.33).

2.3 Dirac equation in curved space-time

In this section, we shall use the covariant derivative $\tilde{D}_a$ discussed in the previous sections to derive the Dirac equation in curved space-time.

Let us consider the Dirac equation in locally inertial coordinates systems. This equation takes the well-known form with respect to locally inertial coordinates $\xi_X^{\ \ a}$:

$$\left[i\hbar\gamma^{(a)} \frac{\partial}{\partial \xi_X^{\ \ a}} - mc \right] \tilde{\Psi}\,(\xi) = 0, \tag{2.44}$$

where the Dirac matrices $\gamma^{(a)}$ satisfy the relations

$$\gamma^{(a)}\gamma^{(b)} + \gamma^{(b)}\gamma^{(a)} = 2\,\eta^{ab}, \tag{2.45}$$

and $\tilde{\Psi}$ is the Dirac spinor defined in locally inertial coordinate systems.

In order to extend Eq. (2.44) into curved space-time, we replace the derivative $\partial/\partial\xi_X{}^a$ with covariant derivative $\tilde{D}_a$:

$$\left[i\hbar\gamma^{(a)}e_{(a)}^{\ \alpha}\left(\frac{\partial}{\partial x^\alpha} - \Gamma_\alpha\right) - mc\right]\Psi\left(x\right) = 0, \tag{2.46}$$

It is convenient to introduce a generally covariant derivative D_α defined as

$$D_\alpha \equiv \frac{\partial}{\partial x^\alpha} - \Gamma_\alpha \tag{2.47}$$

and generally covariant Dirac matrices γ^α defined as

$$\gamma^\alpha \equiv \gamma^{(a)}e_{(a)}^{\ \alpha}. \tag{2.48}$$

Using these definitions, we can derive the Dirac equation in the generally covariant form

$$\left[i\hbar\gamma^\alpha D_\alpha - mc\right]\Psi\left(x\right) = 0. \tag{2.49}$$

Furthermore, we find that the covariant Dirac matrices γ^α satisfy the relations

$$\gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha = 2\,g^{\alpha\beta}. \tag{2.50}$$

In the case of spinors, we have

$$\sigma^{ab} = \frac{1}{4}\left[\gamma^{(a)}, \gamma^{(b)}\right]. \tag{2.51}$$

(See Appendix A.) Therefore, from Eq. (2.41) we obtain the so-called spin connection

$$\Gamma_\alpha = -\frac{1}{8}\left[\gamma^{(a)}, \gamma^{(b)}\right] g_{\mu\nu}\, e_{(a)}^{\ \mu}\, \nabla_\alpha\, e_{(b)}^{\ \nu}. \tag{2.52}$$

Finally, let us confirm that the covariant Dirac equation derived above gives the natural extension of the Klein-Gordon equation.

We now multiply Eq. (2.49) by the operator $\left[i\hbar\gamma^\beta D_\beta + mc\right]$ on the left. Then we have

$$\left[\hbar^2\gamma^\alpha D_\alpha\gamma^\beta D_\beta + m^2c^2\right]\Psi = 0. \tag{2.53}$$

Using a generally covariant total derivative $\mathcal{D}_\alpha$:

$$\mathcal{D}_\alpha \equiv e^{(a)}{}_\alpha \tilde{\mathcal{D}}_a, \tag{2.54}$$

we can write

$$\begin{aligned}
&\gamma^\alpha D_\alpha \gamma^\beta D_\beta \Psi \\
&\quad = \gamma^\alpha \left[\left(\mathcal{D}_\alpha \gamma^\beta \right) D_\beta \Psi + \gamma^\beta \left(\mathcal{D}_\alpha D_\beta \Psi \right) \right] \\
&\quad = \gamma^\alpha \gamma^\beta \mathcal{D}_\alpha \mathcal{D}_\beta \Psi \\
&\quad = \left(g^{\alpha\beta} + \frac{1}{2} \left[\gamma^\alpha, \gamma^\beta \right] \right) \mathcal{D}_\alpha \mathcal{D}_\beta \Psi \\
&\quad = \left(g^{\alpha\beta} \mathcal{D}_\alpha \mathcal{D}_\beta + \frac{1}{4} \left[\gamma^\alpha, \gamma^\beta \right] \left[\mathcal{D}_\alpha, \mathcal{D}_\beta \right] \right) \Psi,
\end{aligned} \tag{2.55}$$

where we have used the following relation:

$$\mathcal{D}_\alpha \gamma^\beta = \gamma^{(a)} \mathcal{D}_\alpha \, e_{(a)}{}^\beta = 0. \tag{2.56}$$

Moreover, we can derive

$$\frac{1}{4} \left[\gamma^\alpha, \gamma^\beta \right] \left[\mathcal{D}_\alpha, \mathcal{D}_\beta \right] = -\frac{1}{4} R, \tag{2.57}$$

where R is the Ricci scalar. (See Appendix B for the details of the calculations.) Therefore, we obtain the following equation in curved space-time:

$$\left[\hbar^2 g^{\alpha\beta} \mathcal{D}_\alpha \mathcal{D}_\beta + m^2 c^2 - \frac{1}{4} \hbar^2 R \right] \Psi = 0. \tag{2.58}$$

This shows the non-minimally coupled generalization of the Klein-Gordon equation. Furthermore, it is shown that this equation for a massless particle is invariant under conformal transformations. (See Appendix C.)

In the case of a scalar field Φ, however, we have

$$\left[\hbar^2 g^{\alpha\beta} \nabla_\alpha \nabla_\beta + m^2 c^2 \right] \Phi = 0, \tag{2.59}$$

because σ^{ab} vanish. Nevertheless, if we demand that the equation for a massless particle becomes conformally invariant, then we derive the following Klein-Gordon equation:

$$\left[\hbar^2 g^{\alpha\beta} \nabla_\alpha \nabla_\beta - \frac{1}{6} \hbar^2 R \right] \Phi = 0. \tag{2.60}$$

(See Appendix C for the details.)

2.4 Another approach

We have obtained the covariant Dirac equation in the last section. In this section, however, we provide another approach. Using this approach, we can derive the covariant Dirac equation again.

In this approach, natural generalization is applied to spinors as well as tensors. The spinors are connected with space-time through the relations

$$\gamma^\alpha \gamma^\beta + \gamma^\beta \gamma^\alpha = 2\, g^{\alpha\beta}. \tag{2.61}$$

These relations are the natural generalization of the relations

$$\gamma^{(a)} \gamma^{(b)} + \gamma^{(b)} \gamma^{(a)} = 2\, \eta^{ab}. \tag{2.62}$$

Moreover, we have to generalize the covariant differentiation in tensor analysis. For the sake of the generalization, it is necessary to introduce four 4×4 matrices Γ_α, which are called a spin connection. Using this quantity, we can define the total generally covariant derivative of the covariant Dirac matrices γ_α as

$$\mathcal{D}_\alpha \gamma_\beta = \frac{\partial}{\partial x^\alpha} \gamma_\beta - \Gamma^\mu{}_{\beta\alpha} \gamma_\mu - \Gamma_\alpha \gamma_\beta + \gamma_\beta \Gamma_\alpha. \tag{2.63}$$

The spin connection Γ_α is uniquely determined up to an additive multiple of the unit matrix by

$$\mathcal{D}_\alpha \gamma_\beta = 0. \tag{2.64}$$

This condition corresponds to the identity

$$\nabla_\alpha g_{\mu\nu} \equiv 0. \tag{2.65}$$

We now introduce the constant Dirac matrices $\gamma^{(a)}$ defined as

$$\gamma^{(a)} = e^{(a)}{}_\alpha \gamma^\alpha. \tag{2.66}$$

The Dirac matrices $\gamma^{(a)}$, of course, satisfy the relations (2.62). Using this quantity, from Eqs. (2.63) and (2.64) we find

$$\gamma_{(b)} \nabla_\alpha e^{(b)}{}_\beta - \Gamma_\alpha\, \gamma_{(b)}\, e^{(b)}{}_\beta + e^{(b)}{}_\beta\, \gamma_{(b)}\, \Gamma_\alpha = 0. \tag{2.67}$$

Contracting this with $e_{(a)}^{\ \ \beta}$, we derive

$$\left[\Gamma_\alpha, \gamma_{(a)}\right] = \gamma_{(b)}\, e_{(a)}^{\ \ \beta}\, \nabla_\alpha\, e^{(b)}_{\ \ \beta}. \tag{2.68}$$

From this, we can derive

$$\Gamma_\alpha = -\frac{1}{8}\, \left[\gamma^{(a)}, \gamma^{(b)}\right] g_{\mu\nu}\, e_{(a)}^{\ \ \mu}\, \nabla_\alpha\, e_{(b)}^{\ \ \nu} + a_\alpha I, \tag{2.69}$$

where a_α is arbitrary and I is the unit matrix. (See Appendix D for the derivation of Eq. (2.69).) In order to derive Eq. (2.69), we have utilized the fact that the following 16 matrices are linearly independent:

$$\Gamma^A = \left\{I\ ,\ \gamma_{(a)}\ ,\ \tilde{\sigma}_{ab}\ ,\ \gamma_{(5)}\gamma_{(a)}\ ,\ \gamma_{(5)}\right\}, \tag{2.70}$$

where $\tilde{\sigma}_{ab}$ are given by

$$\tilde{\sigma}_{ab} = 2i\sigma_{ab} = \frac{i}{2}\, \left[\gamma_{(a)}, \gamma_{(b)}\right], \tag{2.71}$$

and $\gamma_{(5)}$ is defined as

$$\gamma_{(5)} = \gamma^{(5)} = -i\, \gamma_{(0)}\gamma_{(1)}\gamma_{(2)}\gamma_{(3)}. \tag{2.72}$$

(See Appendix E for the proof of the linear independence.)

Chapter 3

Covariant Dirac equation in Kerr geometry

We now consider the gravitational field arising from a rotating object. We assume that the external field of this object is described by the Kerr metric. In particular, we restrict ourselves to the slowly rotating, weak field limit, and specify the metric and the coordinates in terms of the (3+1) formalism. Furthermore, we shall derive an equation of the Schrödinger-type from the covariant Dirac equation (2.49) in the last part of this chapter.

3.1 Space-time

To start, let us consider the following line element called the Boyer-Lindquist form:

$$\begin{aligned} ds^2 &= \frac{1}{\rho^2}\left(\Delta - a'^2\sin^2\theta\right)c^2dt^2 + \frac{4ma'}{\rho^2}r'\sin^2\theta cdtd\phi \\ &\quad - \frac{\rho^2}{\Delta}dr'^2 - \rho^2 d\theta^2 - \frac{1}{\rho^2}\left[\left(r'^2 + a'^2\right)^2 - a'^2\Delta\sin^2\theta\right]\sin^2\theta d\phi^2, \end{aligned} \tag{3.1}$$

$$\Delta \equiv r'^2 - 2mr' + a'^2, \tag{3.2}$$

$$\rho^2 \equiv r'^2 + a'^2\cos^2\theta, \tag{3.3}$$

where using the mass of the rotating object, M, m is defined as follows:

$$m \equiv \frac{GM}{c^2}, \tag{3.4}$$

and a' is the Kerr parameter expressed in terms of the mass M and angular momentum J:

$$a' \equiv \frac{J}{Mc}. \tag{3.5}$$

Assuming that the rotating object is a sphere of radius R with uniform density, we have

$$a' \equiv \frac{J}{Mc} = \frac{2}{5c} R^2 \omega_S, \tag{3.6}$$

where ω_S denotes the angular velocity of this object. (If the rotating object deviates from a sphere, or has an inhomogeneous density distribution, then the numerical factor 2/5 might be changed by a factor of order unity.)

The slow rotation approximation up to first order in a', gives

$$\begin{aligned} ds^2 &= \left(1 - \frac{2m}{r'}\right) c^2 dt^2 + \frac{4ma'}{r'} \sin^2\theta c dt d\phi \\ &\quad - \left(1 - \frac{2m}{r'}\right)^{-1} dr'^2 - r'^2 d\theta^2 - r'^2 \sin^2\theta d\phi. \end{aligned} \tag{3.7}$$

Next, we perform the two continuous coordinate transformations

$$r' = r\left(1 + \frac{m}{2r}\right), \tag{3.8}$$

$$\begin{cases} x' = r\sin\theta\cos\phi, \\ y' = r\sin\theta\sin\phi, \\ z' = r\cos\theta, \end{cases} \tag{3.9}$$

so that we obtain

$$\begin{aligned} ds^2 &= \frac{\left(1 - \frac{m}{2r}\right)^2}{\left(1 + \frac{m}{2r}\right)^2} c^2 dt^2 + \frac{4ma}{r^3\left(1 + \frac{m}{2r}\right)^2} \left(x'dy' - y'dx'\right) dt \\ &\quad - \left(1 + \frac{m}{2r}\right)^4 \left(dx'^2 + dy'^2 + dz'^2\right), \end{aligned} \tag{3.10}$$

where a is defined as $a = ca'$. Moreover, the weak field limit up to $O(1/c^2)$, gives

$$\begin{aligned} ds^2 &= \left(c^2 + 2\,\phi + 2\,\frac{\phi^2}{c^2}\right) dt^2 + \frac{4GMa}{c^2 r^3} \left(x'dy' - y'dx'\right) dt \\ &\quad - \left(1 - 2\,\frac{\phi}{c^2}\right) \left(dx'^2 + dy'^2 + dz'^2\right), \end{aligned} \tag{3.11}$$

where ϕ is the Newtonian gravitational potential, $\phi = -GM/r$.

Furthermore, we consider the case that the observer is revolving around this object in a plane perpendicular to the rotation axis. Assuming that the angular velocity of the observer, ω_O, is constant, the relation between the observer's rest frame (t, x, y, z) and the asymptotically static coordinate frame (t, x', y', z') is

$$\begin{cases} x' &= x\cos\omega_O t - y\sin\omega_O t, \\ y' &= x\sin\omega_O t + y\cos\omega_O t, \\ z' &= z. \end{cases} \tag{3.12}$$

Performing this coordinate transformation, we obtain the following line element:

$$\begin{aligned} ds^2 &= \left[c^2 + 2\phi - \omega_O^2\left(x^2+y^2\right) + 2\frac{\phi^2}{c^2} + \frac{8GMR^2}{5c^2r^3}\omega_O\omega_S\left(x^2+y^2\right) + 2\frac{\phi}{c^2}\omega_O^2\left(x^2+y^2\right)\right]dt^2 \\ &\quad - \left[\omega_O - 2\frac{\phi}{c^2}\omega_O - \frac{4GMR^2}{5c^2r^3}\omega_S\right](xdy - ydx)\,dt - \left(1 - 2\frac{\phi}{c^2}\right)\left(dx^2+dy^2+dz^2\right), \end{aligned} \tag{3.13}$$

where we have used Eq. (3.6). Using this metric, we can calculate the Christoffel symbol. The results are shown in Appendix F.

3.2 (3+1) decomposition

The covariant Dirac equation (2.49) has beautiful space-time symmetry. However, it is sometimes convenient to break the symmetrical form of this equation. In particular, this will be useful for investigating the time evolution of a certain physical quantity.

For the purpose of this, we use the (3+1) formalism. In the (3+1) formalism, the metric $g_{\alpha\beta}$ is split as follows:

$$g_{00} = N^2 - \gamma_{ij}N^iN^j, \tag{3.14}$$

$$g_{0i} = -\gamma_{ij}N^j \equiv -N_i, \tag{3.15}$$

$$g_{ij} = -\gamma_{ij}, \tag{3.16}$$

where N is the lapse function, N^i is the shift vector, and γ_{ij} is the spatial metric on the 3D hypersurface. We define γ^{ij} as the inverse matrix of γ_{ij}. Then $g^{\alpha\beta}$ is also split as

$$g^{00} = \frac{1}{N^2}, \tag{3.17}$$

$$g^{0i} = -\frac{N^i}{N^2}, \tag{3.18}$$

$$g^{ij} = \frac{N^i N^j}{N^2} - \gamma^{ij}. \tag{3.19}$$

Using the metric (3.13) derived in the last section, we can write the lapse function, the shift vector and the spatial metric in the following way:

$$N = c\left(1 + \frac{\phi}{c^2} + \frac{\phi^2}{2c^4}\right), \tag{3.20}$$

$$N^x = -\left(\omega_O - \frac{4GMR^2}{5c^2r^3}\omega_S\right) y, \tag{3.21}$$

$$N^y = \left(\omega_O - \frac{4GMR^2}{5c^2r^3}\omega_S\right) x, \tag{3.22}$$

$$N^z = 0, \tag{3.23}$$

$$\gamma_{ij} = \left(1 - 2\frac{\phi}{c^2}\right)\delta_{ij}. \tag{3.24}$$

3.3 Equation of Schrödinger-type

In this section, we shall derive an equation of the Schrödinger-type from the covariant Dirac equation (2.49). For the purpose of this, we choose the tetrad

$$e_{(0)}{}^{\mu} = c\left(\frac{1}{N}, -\frac{N^i}{N}\right), \tag{3.25}$$

$$e_{(k)}{}^{\mu} = \left(0, e_{(k)}{}^{i}\right), \tag{3.26}$$

where the spatial triad $e_{(k)}{}^{i}$ is defined as

$$\gamma_{ij}\, e_{(k)}{}^{i}\, e_{(l)}{}^{j} = \delta_{kl}. \tag{3.27}$$

From Eqs. (3.20)–(3.24), we derive

$$e_{(0)}{}^{0} = 1 - \frac{\phi}{c^2}, \tag{3.28}$$

$$e_{(0)}{}^{1} = \left(\omega_O - \frac{\phi}{c^2}\omega_O - \frac{4GMR^2}{5c^2r^3}\omega_S\right) y, \tag{3.29}$$

$$
\begin{aligned}
e_{(0)}^{\ \ 2} &= -\left(\omega_O - \frac{\phi}{c^2}\omega_O - \frac{4GMR^2}{5c^2r^3}\omega_S\right)x, && (3.30)\\
e_{(0)}^{\ \ 3} &= 0, && (3.31)\\
e_{(j)}^{\ \ i} &= \left(1+\frac{\phi}{c^2}\right)\delta_j^{\ i}. && (3.32)
\end{aligned}
$$

From this, we can also calculate the components of the spin connection. (See Appendix G.)

Using our choice of the tetrad, the covariant Dirac matrices γ^α are written as

$$
\begin{aligned}
\gamma^0 &= \gamma^{(a)}e_{(a)}^{\ \ 0} = \gamma^{(0)}\frac{c}{N}, && (3.33)\\
\gamma^i &= \gamma^{(a)}e_{(a)}^{\ \ i} = -\gamma^{(0)}\frac{c}{N}N^i + \gamma^{(j)}e_{(j)}^{\ \ i}. && (3.34)
\end{aligned}
$$

Hence the covariant Dirac equation (2.49) becomes

$$
i\hbar\gamma^{(0)}\frac{c}{N}\frac{\partial}{\partial t}\Psi = \left[\left(-\gamma^{(0)}\frac{c}{N}N^i + \gamma^{(j)}e_{(j)}^{\ \ i}\right)\left(-i\hbar\frac{\partial}{\partial x^i} + i\hbar\Gamma_i\right) + i\hbar\gamma^{(0)}\frac{c}{N}\Gamma_0 + mc\right]\Psi. \quad (3.35)
$$

Multiplying this by $\gamma^{(0)}cN$, we derive the equation of the Schrödinger-type:

$$
\begin{aligned}
i\hbar\frac{\partial}{\partial t}\Psi &= H\Psi\\
&= \left[\left(\gamma^{(0)}\gamma^{(j)}cNe_{(j)}^{\ \ i} - N^i\right)(\overline{p}_i + i\hbar\Gamma_i) + i\hbar\Gamma_0 + \gamma^{(0)}mc^2N\right]\Psi, && (3.36)
\end{aligned}
$$

where $\overline{p}_i$ is the momentum operator in flat space-time, and we have used $\left(\gamma^{(0)}\right)^2 = 1/c^2$. If we adopt the standard representation as the constant Dirac matrices, then in flat space-time, we have the well-known form

$$
i\hbar\frac{\partial}{\partial t}\Psi = \left(c\boldsymbol{\alpha}\cdot\overline{\boldsymbol{p}} + mc^2\beta\right)\Psi. \quad (3.37)
$$

On the other hand, if we use the Weyl representation, then for a massless particle we derive the Weyl equations

$$
i\hbar\frac{\partial}{\partial t}\psi = \pm c\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}\psi, \quad (3.38)
$$

where ψ denotes the two-component spinor. (See also Appendix G.)

Chapter 4

Non-relativistic limit

In this chapter, we restrict ourselves to the non-relativistic limit, that is, the case that the rest energy of the particle is much larger than the kinetic energy. In this case, it will be sufficient to expand the Hamiltonian with respect to $1/c$, because the ratio of the kinetic energy to the rest energy, p/mc, is much smaller than unity.

We shall obtain the non-relativistic Hamiltonian by performing the Foldy-Wouthuysen-Tani(FWT) [27], [28] transformation. Furthermore, we shall consider the gravitational effects on quantum interferometry experiments, and investigate the gravitationally induced phase difference in the quantum interferometer. Note that the quantum interferometry experiments in the laboratory are done on Earth. Thus we can choose

$$\boldsymbol{\omega}_{S} = \boldsymbol{\omega}_{O} = \boldsymbol{\omega}. \tag{4.1}$$

4.1 Non-relativistic Hamiltonian

Before performing the FWT transformation, we redefine the spinor and the Hamiltonian in the following way:

$$\Psi' = \gamma^{1/4}\Psi, \quad H' = \gamma^{1/4} H \gamma^{-1/4}, \tag{4.2}$$

where γ is the determinant of the spatial metric:

$$\gamma = \det\left(\gamma_{ij}\right). \tag{4.3}$$

Since the invariant scalar product is

$$(\psi, \varphi) \equiv \int \overline{\psi}\varphi\sqrt{\gamma}d^3x, \tag{4.4}$$

the definition of the scalar product becomes the same as that in flat space-time under this redefinition:

$$\langle \psi', \varphi' \rangle \equiv \int \overline{\psi'} \varphi' d^3x. \tag{4.5}$$

It is sometimes convenient to adopt this definition of the scalar product.

We shall obtain the non-relativistic Hamiltonian by applying the FWT transformation to the Hamiltonian H'. Performing this transformation, we can obtain the "even" operator up to the order of our interest:

$$\begin{aligned} \tilde{H}' &= U H' U^{\dagger} \\ &= \begin{pmatrix} \tilde{H}_{+} & 0 \\ 0 & \tilde{H}_{-} \end{pmatrix} + O\left(\frac{1}{c^4}\right). \end{aligned} \tag{4.6}$$

The spinor is also divided into each of the two-component spinors:

$$\tilde{\Psi} = \begin{pmatrix} \Phi \\ \chi \end{pmatrix}, \tag{4.7}$$

where Φ and χ is called the "large" and "small" component, respectively.

Next, we define the reduced Hamiltonian as follows:

$$H_{+} \equiv \tilde{H}_{+} - mc^2. \tag{4.8}$$

Using this, we obtain the Schrödinger equation with general relativistic corrections for the "large" component in the form

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \Phi &= H_{+} \Phi \\ &= \left[\frac{\overline{\boldsymbol{p}}^2}{2m} + m\phi - \boldsymbol{\omega} \cdot (\boldsymbol{L} + \boldsymbol{S}) \right. \\ &\quad + \frac{1}{c^2} \left(\frac{4GMR^2}{5r^3} \boldsymbol{\omega} \cdot (\boldsymbol{L} + \boldsymbol{S}) - \frac{\overline{\boldsymbol{p}^4}}{8m^3} + \frac{1}{2} m\phi^2 + \frac{3}{2m} \overline{\boldsymbol{p}} \cdot \phi \overline{\boldsymbol{p}} \right) \\ &\quad \left. + \frac{1}{c^2} \left(\frac{3GM}{2mr^3} \boldsymbol{L} \cdot \boldsymbol{S} + \frac{6GMR^2}{5r^5} \boldsymbol{S} \cdot [\boldsymbol{r} \times (\boldsymbol{r} \times \boldsymbol{\omega})] \right) \right] \Phi, \end{aligned} \tag{4.9}$$

where $\boldsymbol{S} = \hbar \boldsymbol{\sigma}/2$ is the spin of the particle with the Pauli spin matrices $\boldsymbol{\sigma}$. The details of the calculations are summarized in Appendix H. Furthermore, following the canonical quantization procedure, we can obtain this non-relativistic Hamiltonian with $\boldsymbol{S} = 0$, again. (See Appendix I.)

4.2 Gravitational effects on quantum interferometer

In this section, we consider the gravitational effects on quantum interferometer experiments, and investigate the gravitationally induced phase difference in the quantum interferometer. In particular, we shall derive the phase difference from the non-relativistic Hamiltonian derived in the last section.

4.2.1 Quantum interferometer experiments

Before we discuss the gravitationally induced phase difference, we briefly review the principle of the quantum interferometer.

The neutron interferometer is a typical one. The neutron interferometer is a extraordinary piece of experimental equipment which allows us to check the basic ideas of quantum mechanics in the laboratory. One of the most important attributes of the neutron interferometer is its conceptual simplicity. We here present a simplified model of the interferometer.

The simplest type of the interferometer is constructed from a single crystal. The schematic drawing of the interferometer is shown in Fig. 4.1. The incident neutron beam is split into two coherent sub-beams at point A. This split occurs as a result of Bragg scattering off the atomic planes perpendicular to the face of the crystal. At points B and C, the sub-beams are redirected. Finally, they interfere at point D.

4.2.2 Gravitationally induced phase difference

The non-relativistic Hamiltonian in Eq. (4.9) can be written as

$$H_+ = H_0 + \sum_k \Delta H_k, \tag{4.10}$$

where H_0 corresponds to the Hamiltonian for a non-relativistic particle which propagates in flat space-time:

$$H_0 = \frac{\overline{\boldsymbol{p}}^2}{2m}. \tag{4.11}$$

If we use Φ_0 satisfying the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t}\Phi_0 = H_0 \Phi_0, \tag{4.12}$$

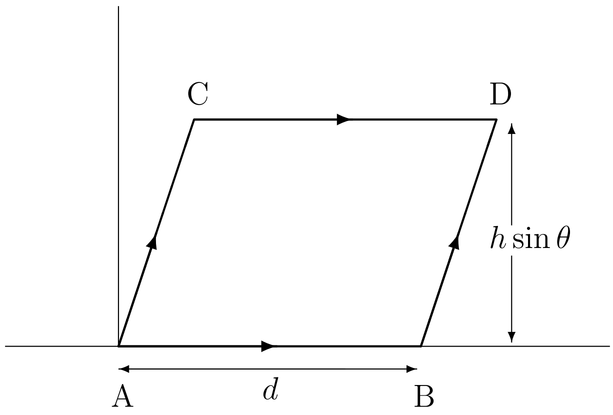

Figure 4.1: The neutron interferometer.

then the Schrödinger equation (4.9) is formally solved in the following way:

$$\Phi = \Phi_0 \exp\left(i \sum_k \beta_k\right), \tag{4.13}$$

$$\beta_k = -\frac{1}{\hbar}\int^t \Delta H_k dt. \tag{4.14}$$

Therefore, for the two neutron beams which follow the path ABD and the path ACD, respectively, the phase difference at point D is

$$\Delta\beta_k = \beta_{k\,(\text{path ACD})} - \beta_{k\,(\text{path ABD})} = -\frac{1}{\hbar}\oint \Delta H_k dt. \tag{4.15}$$

(See Fig. 4.1.) Let us evaluate the phase difference arising from each correction term in order.

First, the gravitational potential term $\Delta H_1 = m\phi$ gives the phase shift

$$\Delta\beta_1 = \frac{m^2 g A\lambda}{2\pi\hbar^2}\sin\theta, \tag{4.16}$$

where g is the acceleration of gravity, A is the area inside the interferometry loop; A is given by $A = dh\sin\theta$ (See Fig. 4.1), λ is the de Broglie wavelength, and θ is the rotation angle of the interferometer relative to the horizontal plane. The effect arising from this phase shift was first predicted by Overhauser and Colella [7], and observed with a neutron interferometer [4].

Next, we consider the term $\Delta H_2 = -\boldsymbol{\omega} \cdot \boldsymbol{L}$, which represents the coupling between the rotation of Earth and the angular momentum. The phase shift arising from this term is

$$\Delta\beta_2 = \frac{2m}{\hbar}\boldsymbol{\omega} \cdot \boldsymbol{A}, \tag{4.17}$$

where

$$\boldsymbol{A} = \frac{1}{2}\oint \boldsymbol{r} \times d\boldsymbol{r} \tag{4.18}$$

is the area vector of the interferometry loop. This phase shift is caused by the inertial force, and hence it dose not depend on gravity. The phase shift (4.17) was first derived by Page [8] from the analogy with the Sagnac effect in optical interferometry, and later by other authors using various methods [10], [11], [12]. The experimental verification was provided in Ref. [9]. The Sagnac effect was observed also in atomic interferometry [29].

The third contribution arises from a general relativistic effect called Lense-Thirring effect:

$$\begin{aligned} \Delta\beta_3 &= -\frac{4GMR^2m}{5c^2\hbar}\boldsymbol{\omega} \cdot \oint \frac{\boldsymbol{r} \times d\boldsymbol{r}}{r^3} \\ &= \frac{2m}{5\hbar}\frac{r_g}{R}\left[\boldsymbol{\omega} - 3\left(\frac{\boldsymbol{R}}{R} \cdot \boldsymbol{\omega}\right)\frac{\boldsymbol{R}}{R}\right] \cdot \boldsymbol{A}, \end{aligned} \tag{4.19}$$

where $\boldsymbol{R}$ is the position vector of the interferometer from the center of Earth, and $r_g \equiv 2GM/c^2$ is the Schwarzschild radius of Earth. This is very similar to the Biot-Savart law in the classical electromagnetism. The phase difference (4.19) was derived in Ref. [13]. We here find that the phase difference (4.19) depends on the orientation. In particular, we have

$$\Delta\beta_3 = \frac{1}{5}\frac{r_g}{R}\Delta\beta_2 \quad \text{on the equator} \, (\boldsymbol{R} \perp \boldsymbol{\omega}) \, , \tag{4.20}$$

whereas

$$\Delta\beta_3 = -\frac{2}{5}\frac{r_g}{R}\Delta\beta_2 \quad \text{on the North Pole} \, (\boldsymbol{R} \| \boldsymbol{\omega}) \, . \tag{4.21}$$

Therefore, if we carry out the experiments in different places on Earth, then we can separate this effect from the Newtonian effect in principle. Until now, the Lense-Thirring effect has not yet been observed in any interferometer experiments. This is of course due to the smallness. (The phase shift arising from the Lense-Thirring effect is $r_g/R \sim 10^{-9}$ times smaller than that due to the Sagnac effect.)

The fourth correction term $\Delta H_4 = \overline{\boldsymbol{p}}^4/(8m^3c^2)$ is a purely special relativistic correction to the kinematic energy. Since this term is independent of the path, the phase difference in the interferometer experiments is not produced.

The fifth correction term $\Delta H_5 = m\phi^2/(2c^2)$ can be regarded as the red shift correction to the potential energy. The phase difference is

$$\Delta\beta_5 = -\frac{1}{2}\frac{r_g}{R}\Delta\beta_1. \tag{4.22}$$

The sixth contribution $\Delta H_6 = 3\overline{\boldsymbol{p}} \cdot \phi\overline{\boldsymbol{p}}/(2mc^2)$ is the redshift correction to the kinetic energy. The phase difference is

$$\Delta\beta_6 = \frac{3}{2}\left(\frac{\lambda_C}{\lambda}\right)^2 \Delta\beta_1, \tag{4.23}$$

where λ_C is the Compton wavelength.

The last two corrections have same rotation angle dependence as $\Delta\beta_1$. Therefore, as far as the experiments are done only in different rotation angles, these effects are not separable from the Newtonian effect.

Finally, we consider the spin corrections. If the spin of the particle is constant along the paths in the interferometer, then the term $-\boldsymbol{\omega} \cdot \boldsymbol{S}$ does not produce the phase difference. On the other hand, the remaining correction terms have typically the relative orders of magnitude to the orbital angular momentum, λ/l, where λ is the de Broglie wavelength and l is a typical size of the interferometer loop. For the neutron interferometers of the first generation, the typical values are $\lambda \sim 10^{-8}$cm and $l \sim 10$cm. Hence, for such interferometers, the effects of the spin corrections are generally 10^{-9} times smaller than those of the orbital angular momentum terms.

Chapter 5

Ultra-relativistic limit

We now turn our attention to the ultra-relativistic limit, in which the rest energy of the particle is much smaller than the kinetic energy. (We now consider the observer's frame.) In this limit, the energy the particle itself has is expanded with respect to c because the ratio of the rest energy to the kinetic energy, mc/p, is much smaller than unity, whereas objects arising from the gravitational field are expanded with respect to $1/c$. Hence we cannot consider the expansion only with respect to c. We here consider the slowly rotating, weak field approximation up to first orders in ϕ/c^2 and ω, respectively. On the other hand, we expand the energy the particle itself has up to $O\left(m^2c^4/pc\right)$.

By performing a unitary transformation similar to the FWT transformation, we shall obtain the ultra-relativistic Hamiltonian. Furthermore, we shall consider the gravitational effects on neutrino oscillations, and investigate the gravitationally induced neutrino oscillation phases. For this analysis, we ignore the observer's angular velocity $\boldsymbol{\omega}_O$:

$$\boldsymbol{\omega}_S \equiv \boldsymbol{\omega}, \quad \boldsymbol{\omega}_O = \mathbf{0}. \tag{5.1}$$

This assumption is valid for a neutrino propagating from a distant star.

5.1 Ultra-relativistic Hamiltonian

To begin with, we redefine the spinor and the Hamiltonian as in Sec. 4.1:

$$\Psi' = \gamma^{1/4}\Psi, \quad H' = \gamma^{1/4}H\gamma^{-1/4}. \tag{5.2}$$

Next, we perform a unitary transformation to derive the ultra-relativistic Hamiltonian which is the "even" operator up to the order of our interest. From this, we have

$$\begin{aligned}\tilde{H}' &= UH'U^{\dagger} \\ &= \begin{pmatrix} H_R & 0 \\ 0 & H_L \end{pmatrix} + \left[O\left(\frac{m^3c^6}{p^2c^2}\right) \text{ or } O\left(\frac{\phi^2}{c^4}, \omega^2\right)\right].\end{aligned} \tag{5.3}$$

The spinor is also divided into each of the two-component spinors:

$$\tilde{\Psi} = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}, \tag{5.4}$$

where the subscript R and L denote the right-handed and the left-handed component, respectively.

We restrict our attention to the left-handed component. Then the equation for the left-handed component is given by

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\psi_L &= H_L\psi_L \\ &= -\Bigg[c\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}} + \frac{1}{c^2}\left(c\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}\,\phi + \phi\, c\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}\right) \\ &\quad - \frac{1}{c^2}\left(\frac{4GMR^2}{5r^3}\boldsymbol{\omega}\cdot(\boldsymbol{L}+\boldsymbol{S}) + \frac{6GMR^2}{5r^5}\boldsymbol{S}\cdot[\boldsymbol{r}\times(\boldsymbol{r}\times\boldsymbol{\omega})]\right) \\ &\quad + \frac{1}{2}m^2c^3\frac{\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}}{\overline{p}^2} + \frac{1}{c^2}\frac{1}{8}m^2c^3\left\{\frac{\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}}{\overline{p}^2}\,\phi + \phi\,\frac{\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}}{\overline{p}^2} - \left(\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}\,\phi\,\frac{1}{\overline{p}^2} + \frac{1}{\overline{p}^2}\,\phi\,\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}\right)\right\} \\ &\quad + \frac{1}{c^2}\frac{1}{8}m^2c^2\Bigg\{\left(\frac{4GMR^2}{5r^3}\boldsymbol{\omega}\cdot(\boldsymbol{L}+\boldsymbol{S}) + \frac{6GMR^2}{5r^5}\boldsymbol{S}\cdot[\boldsymbol{r}\times(\boldsymbol{r}\times\boldsymbol{\omega})]\right)\frac{1}{\overline{p}^2} \\ &\qquad - 2\frac{\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}}{\overline{p}^2}\left(\frac{4GMR^2}{5r^3}\boldsymbol{\omega}\cdot(\boldsymbol{L}+\boldsymbol{S}) + \frac{6GMR^2}{5r^5}\boldsymbol{S}\cdot[\boldsymbol{r}\times(\boldsymbol{r}\times\boldsymbol{\omega})]\right)\frac{\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}}{\overline{p}^2} \\ &\qquad + \frac{1}{\overline{p}^2}\left(\frac{4GMR^2}{5r^3}\boldsymbol{\omega}\cdot(\boldsymbol{L}+\boldsymbol{S}) + \frac{6GMR^2}{5r^5}\boldsymbol{S}\cdot[\boldsymbol{r}\times(\boldsymbol{r}\times\boldsymbol{\omega})]\right)\Bigg\}\Bigg]\psi_L. \end{aligned} \tag{5.5}$$

The details of the calculations are given in Appendix J. It is sometimes convenient to rewrite the Hamiltonian in Eq. (5.5) in the following form:

$$H_L = -\left[\left\{1 + \frac{1}{c^2}\left(\phi + \overline{\boldsymbol{p}}\cdot\phi\overline{\boldsymbol{p}}\frac{1}{\overline{p}^2} + 2\frac{GM}{r^3}\boldsymbol{L}\cdot\boldsymbol{S}\frac{1}{\overline{p}^2}\right)\right\}c\overline{p}\,\frac{\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}}{\overline{p}}\right.$$

$$
\begin{aligned}
&- \frac{1}{c^2}\left(\frac{4GMR^2}{5r^3}\boldsymbol{\omega}\cdot(\boldsymbol{L}+\boldsymbol{S}) + \frac{6GMR^2}{5r^5}\boldsymbol{S}\cdot[\boldsymbol{r}\times(\boldsymbol{r}\times\boldsymbol{\omega})]\right)\\
&+ \left\{1 + \frac{1}{c^2}\frac{1}{4}\left(\phi - \frac{1}{\overline{p}^2}\phi\overline{p}^2 + \frac{1}{\overline{p}^2}\overline{\boldsymbol{p}}\cdot\phi\overline{\boldsymbol{p}} - \overline{\boldsymbol{p}}\cdot\phi\overline{\boldsymbol{p}}\frac{1}{\overline{p}^2}\right.\right.\\
&\qquad\qquad \left.\left. + 2\frac{1}{\overline{p}^2}\frac{GM}{r^3}\boldsymbol{L}\cdot\boldsymbol{S} - 2\frac{GM}{r^3}\boldsymbol{L}\cdot\boldsymbol{S}\frac{1}{\overline{p}^2}\right)\right\}\frac{m^2c^3}{2\overline{p}}\ \frac{\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}}{\overline{p}}\\
&+ \frac{1}{c^2}\frac{1}{8}m^2c^2\left\{\left(\frac{4GMR^2}{5r^3}\boldsymbol{\omega}\cdot(\boldsymbol{L}+\boldsymbol{S}) + \frac{6GMR^2}{5r^5}\boldsymbol{S}\cdot[\boldsymbol{r}\times(\boldsymbol{r}\times\boldsymbol{\omega})]\right)\frac{1}{\overline{p}^2}\right.\\
&\qquad\qquad - 2\frac{\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}}{\overline{p}^2}\left(\frac{4GMR^2}{5r^3}\boldsymbol{\omega}\cdot(\boldsymbol{L}+\boldsymbol{S}) + \frac{6GMR^2}{5r^5}\boldsymbol{S}\cdot[\boldsymbol{r}\times(\boldsymbol{r}\times\boldsymbol{\omega})]\right)\frac{\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}}{\overline{p}^2}\\
&\qquad\qquad \left.\left. + \frac{1}{\overline{p}^2}\left(\frac{4GMR^2}{5r^3}\boldsymbol{\omega}\cdot(\boldsymbol{L}+\boldsymbol{S}) + \frac{6GMR^2}{5r^5}\boldsymbol{S}\cdot[\boldsymbol{r}\times(\boldsymbol{r}\times\boldsymbol{\omega})]\right)\right\}\right].
\end{aligned}
\tag{5.6}
$$

From this, we find how the spin-orbit coupling, the coupling between the total angular momentum and the rotation of the gravitational source, or the coupling between the spin and the rotation is coupled to the infinitesimal mass. In radial propagation, the orbital angular momentum vanishes. Therefore, in this case the spin effects coupled only to the rotation appear. If we set $\boldsymbol{\omega} = \boldsymbol{0}$, then there is no spin effect in radial propagation. This consequence is consistent with the previous work [26].

5.2 Gravitational effects on neutrino oscillations

In this section, we consider the gravitational effects on neutrino oscillations, and investigate the gravitationally induced neutrino oscillation phases. First, we shall reconsider the neutrino oscillations in flat space-time. Second, we shall discuss the gravitational effects, and derive the phase shift directly from the ultra-relativistic Hamiltonian.

5.2.1 Neutrino oscillations in flat space-time

Now, we briefly review neutrino oscillations in flat space-time. (See, e.g., Refs. [30], [31], [32].)

If neutrinos are not massless, then their mass matrix will be nondiagonal and complex as in the case for quarks. This means that the flavor eigenstates, which are denoted by

$|\nu_\alpha\rangle$, can be represented as linear superpositions of the mass eigenstates denoted by $|\nu_i\rangle$:

$$|\nu_\alpha\rangle = \sum_i U_{\alpha i} |\nu_i\rangle \,, \tag{5.7}$$

where U is a unitary matrix, by which we can transform the mass matrix into a diagonal form. (See Appendix K for the details of possible neutrino mixing schemes.) For three interacting neutrinos, U can be parametrized like the Kobayashi-Maskawa (KM) matrix [33] for quark mixing angles:

$$U = \begin{pmatrix} \mathrm{c}_1 & \mathrm{s}_1\mathrm{c}_3 & \mathrm{s}_1\mathrm{s}_3 \\ -\mathrm{s}_1\mathrm{c}_2 & \mathrm{c}_1\mathrm{c}_2\mathrm{c}_3 - \mathrm{s}_2\mathrm{s}_3 e^{i\delta} & \mathrm{c}_1\mathrm{c}_2\mathrm{s}_3 - \mathrm{s}_2\mathrm{c}_3 e^{i\delta} \\ -\mathrm{s}_1\mathrm{s}_2 & \mathrm{c}_1\mathrm{s}_2\mathrm{s}_3 - \mathrm{c}_2\mathrm{s}_3 e^{i\delta} & \mathrm{c}_1\mathrm{s}_2\mathrm{s}_3 - \mathrm{c}_2\mathrm{c}_3 e^{i\delta} \end{pmatrix}, \tag{5.8}$$

where $c_i \equiv \cos\theta_i$ and $s_i \equiv \sin\theta_i$.

If at time $t = 0$, a beam of pure ν_α states is produced, the initial state is a superposition of the mass eigenstates as

$$|\nu_\alpha(0)\rangle = \sum_i U_{\alpha i} |\nu_i\rangle \,. \tag{5.9}$$

The time evolution of a mass eigenstate $|\nu_i\rangle$ is determined by the Dirac equation for a freely propagating neutrino with definite mass m_i. From the Dirac equation, we can obtain

$$i\hbar\frac{\partial}{\partial t}\psi_{iL}(\boldsymbol{x},t) = -\sqrt{\overline{p}^2c^2 + m_i^2c^4}\,\frac{\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}}{\overline{p}}\psi_{iL}(\boldsymbol{x},t)\,, \tag{5.10}$$

where $\psi_{iL}(\boldsymbol{x},t) = \langle\boldsymbol{x}|\nu_i\rangle_t$. (The "phenomenological" neutrinos are left-handed.) The details of the calculations are given in Appendix L. In the ultra-relativistic limit ($mc^2/pc \ll 1$), we have

$$\sqrt{\overline{p}^2c^2 + m_i^2c^4} \simeq \overline{p}c + \frac{m_i^2c^3}{2\overline{p}}. \tag{5.11}$$

Hence, the Dirac equation is written as

$$i\hbar\frac{\partial}{\partial t}\psi_{iL}(\boldsymbol{x},t) = -\left[\overline{p}c + \frac{m_i^2c^3}{2\overline{p}}\right]\frac{\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}}{\overline{p}}\psi_{iL}(\boldsymbol{x},t)\,. \tag{5.12}$$

We now regard the second term in the square brackets in Eq. (5.12) as a perturbation, and assume that the spinor ψ_{iL} is written as

$$\psi_{iL}(\boldsymbol{x},t) = e^{i\Phi(t)}\,\psi_{0iL}(\boldsymbol{x},t)\,, \tag{5.13}$$

where ψ_{0iL} is the unperturbed quantity satisfying the equation

$$i\hbar\frac{\partial}{\partial t}\psi_{0iL}\left(\boldsymbol{x},t\right)=-c\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}\,\psi_{0iL}\left(\boldsymbol{x},t\right). \tag{5.14}$$

Furthermore, we assume

$$\frac{\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}}{\overline{p}}\psi_{0iL}\left(\boldsymbol{x},t\right)=-\psi_{0iL}\left(\boldsymbol{x},t\right), \tag{5.15}$$

and

$$\overline{\boldsymbol{p}}\,\psi_{0iL}\left(\boldsymbol{x},t\right)=\boldsymbol{p}\,\psi_{0iL}\left(\boldsymbol{x},t\right). \tag{5.16}$$

(For simplicity, each state ψ_{0iL} is assumed to have same momentum $\boldsymbol{p}$.) Under these assumptions, substituting Eq. (5.13) into Eq. (5.12), we obtain

$$\begin{aligned}\Phi(t) &= -\frac{1}{\hbar}\int_0^t\frac{m_i^2c^3}{2p}dt\\ &= -\frac{m_i^2c^3}{2\hbar p}t\end{aligned} \tag{5.17}$$

Therefore, we derive

$$\langle\boldsymbol{x}|\nu_i\rangle_t=\psi_{iL}\left(\boldsymbol{x},t\right)=e^{-i\frac{m_i^2c^3}{2\hbar p}t}\,\psi_{0iL}\left(\boldsymbol{x},t\right). \tag{5.18}$$

This is equivalent to

$$|\nu_i\rangle_t=e^{-i\frac{m_i^2c^3}{2\hbar p}t}\,e^{-i\frac{pc}{\hbar}t}\,|\nu_i\rangle. \tag{5.19}$$

Consequently, we obtain

$$|\nu_\alpha(t)\rangle=\sum_i U_{\alpha i}\,e^{-i\frac{m_i^2c^3}{2\hbar p}t}\,e^{-i\frac{pc}{\hbar}t}\,|\nu_i\rangle. \tag{5.20}$$

The amplitude for observing an initially created flavor eigenstate $|\nu_\alpha\rangle$ as the (different or same) flavor eigenstate $|\nu_\beta\rangle$ at some future time t is

$$\langle\nu_\beta|\nu_\alpha(t)\rangle=\sum_i U_{\alpha i}U_{\beta i}^*\,e^{-i\frac{m_i^2c^3}{2\hbar p}t}\,e^{-i\frac{pc}{\hbar}t}. \tag{5.21}$$

Hence, the probability for a transition from the state $|\nu_\alpha\rangle$ to the state $|\nu_\beta\rangle$ is

$$P_{\nu_\alpha\to\nu_\beta}(t)=\left|\langle\nu_\beta|\nu_\alpha(t)\rangle\right|^2=\sum_{i,j}U_{\alpha i}U_{\beta i}^*U_{\alpha j}^*U_{\beta j}\ \exp\left[-i\frac{\left(m_i^2-m_j^2\right)c^3}{2\hbar p}t\right]. \tag{5.22}$$

For example, the probability of an electron neutrino remaining an electron neutrino after a time t, or after traveling a distance $d \simeq ct$, is

$$\begin{aligned} P_{\nu_e \to \nu_e}(t) &= 1 - 4\cos^2\theta_1 \sin^2\theta_1 \cos^2\theta_3 \sin^2\left[\frac{(m_1^2 - m_2^2)\,c^3}{4\hbar p}t\right] \\ &\quad - 4\cos^2\theta_1 \sin^2\theta_1 \sin^2\theta_3 \sin^2\left[\frac{(m_1^2 - m_3^2)\,c^3}{4\hbar p}t\right] \\ &\quad - 4\sin^4\theta_1 \sin^2\theta_3 \cos^2\theta_3 \sin^2\left[\frac{(m_2^2 - m_3^2)\,c^3}{4\hbar p}t\right]. \end{aligned} \tag{5.23}$$

It is convenient to define the oscillation lengths

$$l_{ij} = \frac{4\pi\hbar pc}{\Delta m_{ij}^2 c^3} \simeq 2.5\left(\frac{pc}{\text{MeV}}\right)\left(\frac{\text{eV}^2/c^4}{\Delta m_{ij}^2}\right)\text{m}, \tag{5.24}$$

where $\Delta m_{ij}^2 = \left|m_i^2 - m_j^2\right|$. When we use the oscillation lengths, the probability of observing an electron neutrino at a distance $d \simeq ct$ from the source is given by

$$\begin{aligned} P_{\nu_e \to \nu_e}(t) &= 1 - 4\cos^2\theta_1 \sin^2\theta_1 \cos^2\theta_3 \sin^2\left(\pi\frac{d}{l_{12}}\right) \\ &\quad - 4\cos^2\theta_1 \sin^2\theta_1 \sin^2\theta_3 \sin^2\left(\pi\frac{d}{l_{13}}\right) \\ &\quad - 4\sin^4\theta_1 \sin^2\theta_3 \cos^2\theta_3 \sin^2\left(\pi\frac{d}{l_{23}}\right). \end{aligned} \tag{5.25}$$

Similarly, we can obtain the probability for the other transitions.

5.2.2 Gravitationally induced neutrino oscillation phases

Next, we discuss the gravitationally induced neutrino oscillation phases. As we saw in the last subsection, the most important one in the neutrino oscillations is the phase difference of the two different mass eigenstates. Hence we restrict our attention to the phase shifts of the mass eigenstates.

We now regard terms arising from the small mass and the gravitational field as perturbations. Then the equation for the left-handed component obtained in the last section is considered as

$$i\hbar\frac{\partial}{\partial t}\psi_L = (H_{0L} + \Delta H_L)\,\psi_L, \tag{5.26}$$

where H_{0L} denotes the unperturbed Hamiltonian $H_{0L} = -c\boldsymbol{\sigma} \cdot \overline{\boldsymbol{p}}$, and ΔH_L the corrections arising from the small mass and the gravitational field. This equation corresponds to Eq. (5.12). From the perturbative point of view, it is plausible that the particle trajectory is taken in the unperturbed system. Then we can consider the radial propagation, in which the spin-orbit coupling vanishes.

Following the discussion of the last subsection, we again assume that the spinor ψ_L is given by

$$\psi_L(\boldsymbol{x}, t) = e^{i\Phi(t)}\, \psi_{0L}(\boldsymbol{x}, t)\,, \tag{5.27}$$

where ψ_{0L} satisfies Eq. (5.14), that is, the equation

$$i\hbar\frac{\partial}{\partial t}\psi_{0L}(\boldsymbol{x}, t) = H_{0L}\psi_{0L}(\boldsymbol{x}, t)\,. \tag{5.28}$$

Substituting Eq. (5.27) into Eq. (5.26) and using Eq. (5.28), we obtain

$$\Phi = -\frac{1}{\hbar}\int^t \Delta H_L dt. \tag{5.29}$$

In order to derive the phase practically, we use the assumptions (5.15) and (5.16) again. Furthermore, we here replace the remaining q-numbers with the c-numbers. This is a kind of semi-classical approximation. From this, except for the spin effects, we derive

$$\Delta H_L = 2\frac{\phi}{c^2}cp + \frac{m^2c^3}{2p}. \tag{5.30}$$

Now, let us consider the case that the neutrino is produced at time $t = t_A$, and detected at time $t = t_B$. Then the phase the neutrino acquires is

$$\Phi = -\frac{1}{\hbar}\int_{t_A}^{t_B}\left(2\frac{\phi}{c^2}cp + \frac{m^2c^3}{2p}\right) dt. \tag{5.31}$$

We pay attention to the term related to m^2, because neutrino oscillations take place as a result of the mass square difference. This term reduces to

$$-\frac{m^2c^3}{2\hbar p}\left(t_B - t_A\right). \tag{5.32}$$

Furthermore, let the two different mass eigenstates have common momentum p and propagate along a same path. Then the relative phase $\Delta\Phi_{ij}$ of the two different mass eigenstates, $|\nu_i\rangle$ and $|\nu_j\rangle$, is given by

$$\Delta\Phi_{ij} = \frac{\Delta m_{ij}^2 c^3}{2\hbar p}\left(t_B - t_A\right), \tag{5.33}$$

where p is interpreted as the momentum the neutrinos have at initial time.

Finally, we shall show that the result obtained above is consistent with the result which would be derived from the classical action. Using the classical action, the quantum phase is given by

$$\begin{aligned}\Phi &= -\frac{1}{\hbar}\int_A^B p_\mu dx^\mu \\ &= -\frac{1}{\hbar}\int_A^B \left(Edt + p_i dx^i\right), \end{aligned} \tag{5.34}$$

where $E \equiv p_t$. Since we now consider the case that two different mass eigenstates have same momentum, we are particularly interested in the first term. (The phase difference between the two mass eigenstates arises from the first term.)

Considering the metric (3.13), we find that the space-time has a Killing vector $X^\alpha \partial_\alpha = \partial/\partial t$. Indeed, using $X^\alpha = (1,0,0,0)$, we derive

$$\begin{aligned} L_X g_{\alpha\beta} &= X^\mu \frac{\partial}{\partial x^\mu} g_{\alpha\beta} + g_{\alpha\mu}\frac{\partial}{\partial x^\beta} X^\mu + g_{\beta\mu}\frac{\partial}{\partial x^\alpha} X^\mu \\ &= \nabla_\beta X_\alpha + \nabla_\alpha X_\beta \\ &= 0, \end{aligned} \tag{5.35}$$

where L_X denotes the Lie derivative with respect to X^α. Furthermore, since $\nabla_\beta X_\alpha + \nabla_\alpha X_\beta = 0$, along the geodesic ($p^\alpha \nabla_\alpha p^\beta = 0$) we obtain

$$\begin{aligned} p^\alpha \nabla_\alpha \left(p^\beta X_\beta\right) &= p^\alpha \left(\nabla_\alpha p^\beta\right) X_\beta + p^\alpha p^\beta \nabla_\alpha X_\beta \\ &= \frac{1}{2} p^\alpha p^\beta \left(\nabla_\beta X_\alpha + \nabla_\alpha X_\beta\right) \\ &= 0. \end{aligned} \tag{5.36}$$

Hence, $p^\alpha X_\alpha$ is constant and, therefore, $p_t = E$ is also constant along the geodesic.

In order to derive Eq. (5.33) again, it will be sufficient to consider the isotropic form of the Schwarzschild metric (because the effects arising from the rotation of the gravitational source is coupled to the spin). We now consider the radial propagation (say the x direction). Then, from the mass shell condition

$$g^{\alpha\beta} p_\alpha p_\beta = m^2 c^2, \tag{5.37}$$

we obtain

$$E = \frac{1 + \frac{\phi}{2c^2}}{\left(1 - \frac{\phi}{2c^2}\right)^3} p_x(x) c \left[1 + \left(1 - \frac{\phi}{2c^2}\right)^4 \frac{m^2 c^2}{p_x^2(x)}\right]^{\frac{1}{2}}. \tag{5.38}$$

If we assume that for any x, $p_x(x) \gg mc$, then we have

$$E \simeq \frac{1+\frac{\phi}{2c^2}}{\left(1-\frac{\phi}{2c^2}\right)^3}\, p_x(x)c + \left(1+\frac{\phi}{2c^2}\right)\left(1-\frac{\phi}{2c^2}\right)\frac{m^2c^3}{2p_x(x)} + \cdots . \tag{5.39}$$

Moreover, the weak field approximation gives

$$E \simeq p_x(x)c + 2\,\frac{\phi}{c^2}\,p_x(x)c + \frac{m^2c^3}{2p_x(x)} + \cdots . \tag{5.40}$$

Here, since the left-hand side of Eq. (5.40) is constant, we can evaluate the right-hand side at an arbitrary point. Therefore, We derive

$$E \simeq p_x(x_A)c + 2\,\frac{\phi}{c^2}\,p_x(x_A)c + \frac{m^2c^3}{2p_x(x_A)} + \cdots . \tag{5.41}$$

From this, we can obtain the same result as Eq. (5.33) (where $p = p_x(x_A)$) again.

Chapter 6

Summary and conclusion

In this thesis, we have studied the gravitational effects on Dirac particles. In particular, we have considered the particles propagating in the Kerr geometry, and restricted ourselves to the slowly rotating, weak field limit. First, following the discussion of Ref. [14], we have summarized the effects of the general relativistic gravity on a non-relativistic particle. There we have obtained the Schrödinger equation with the general relativistic corrections, and from this we have derived the gravitationally induced phase difference in a quantum interferometer. Next, we have discussed the gravitational effects on a Dirac particle with infinitesimal mass. By performing a unitary transformation similar to the FWT transformation, we have obtained the two-component Weyl equations with the corrections arising from the small mass and the gravitational field from the covariant Dirac equation. Thereby, it has become clear how the spin-orbit coupling, the coupling between the total angular momentum and the rotation of the gravitational source, or the coupling between the spin and the rotation is coupled to the infinitesimal mass. Furthermore, we have discussed the gravitationally induced neutrino oscillation phases, and derived the phase difference of the two different mass eigenstates in radial propagation except for the spin effects.

We have not pursued the spin effects on the neutrino oscillation phases. However, it is interesting to investigate the effects of the spin-orbit coupling (which is associated with the non-radial propagation) and the dragging of inertia on the neutrino oscillations. To add to this, the relation to experiments and the quantitative estimation will be the subjects of further investigation.

Although it seems to be difficult to provide the verification of these effects with current experimental detectability, we think that the investigation of the topics in which both quantum effects and gravitational effects come into play is important. Progress in technology may make the verification of the effects possible.

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Appendix A

Representations of the Lorentz group

We now investigate the Lorentz transformations in a more general way. Under the general Lorentz transformation rule, a general field denoted by ψ_m transforms under a Lorentz transformation $\Lambda^a{}_b$ according to

$$\psi'_m = \sum_n [U(\Lambda)]_{mn} \psi_n. \tag{A.1}$$

In order for a Lorentz transformation Λ_1 followed by a Lorentz transformation Λ_2 to give the same result as the Lorentz transformation $\Lambda_1\Lambda_2$, it is necessary that the matrices $U(\Lambda)$ should furnish a representation of the Lorentz group, that is,

$$U(\Lambda_1) U(\Lambda_2) = U(\Lambda_1\Lambda_2). \tag{A.2}$$

In fact, the U-matrices given by Eqs. (2.15) and (2.16) satisfy the group multiplication rule (A.2).

Next, we consider the infinitesimal Lorentz group which consists of Lorentz transformations infinitesimally close to the identity, that is,

$$\Lambda^a{}_b = \delta^a{}_b + \omega^a{}_b, \quad |\omega^a{}_b| \ll 1, \tag{A.3}$$

where

$$\omega_{ab} = -\omega_{ba}. \tag{A.4}$$

For such a transformation, the matrix representation $U(\Lambda)$ must be infinitesimally close to the identity:

$$U(1+\omega) = \mathbf{1} + \frac{1}{2}\,\omega^{ab}\sigma_{ab}, \tag{A.5}$$

where σ_{ab} are antisymmetric in a and b.

The matrices σ_{ab} must be constrained by the group multiplication rule (A.2). It is convenient first to apply this rule to the product $\Lambda(1+\omega)\Lambda^{-1}$:

$$U(\Lambda)U(1+\omega)U(\Lambda^{-1}) = U\left(1+\Lambda\omega\Lambda^{-1}\right). \tag{A.6}$$

Up to first order in ω, this reduces to

$$\mathbf{1} + \frac{1}{2}\,\omega^{ab}U(\Lambda)\sigma_{ab}U(\Lambda^{-1}) = \mathbf{1} + \frac{1}{2}\,\omega^{ab}\sigma_{cd}\Lambda^{c}{}_{a}\Lambda^{d}{}_{b}. \tag{A.7}$$

Thus we have

$$U(\Lambda)\sigma_{ab}U(\Lambda^{-1}) = \sigma_{cd}\,\Lambda^{c}{}_{a}\Lambda^{d}{}_{b}. \tag{A.8}$$

If we now set $\Lambda = 1+\omega$ and $\Lambda^{-1} = 1-\omega$, then we have

$$\omega^{cd}\left(\sigma_{ab}\sigma_{cd} - \sigma_{cd}\sigma_{ab}\right) = \omega^{cd}\left(2\,\eta_{cb}\sigma_{ad} + 2\,\eta_{ca}\sigma_{db}\right), \tag{A.9}$$

that is,

$$\omega^{cd}\left[\sigma_{ab},\sigma_{cd}\right] = \omega^{cd}\left(\eta_{cb}\sigma_{ad} - \eta_{ca}\sigma_{bd} + \eta_{db}\sigma_{ca} - \eta_{da}\sigma_{cb}\right). \tag{A.10}$$

This will be satisfied provided that σ satisfies the commutation relations

$$\left[\sigma_{ab},\sigma_{cd}\right] = \eta_{cb}\sigma_{ad} - \eta_{ca}\sigma_{bd} + \eta_{db}\sigma_{ca} - \eta_{da}\sigma_{cb}. \tag{A.11}$$

The problem of finding the general representations of the infinitesimal homogeneous Lorentz group is equivalent to finding all matrices that satisfy the commutation relations (A.11).

Finally, we shall consider the Dirac spinors. In flat space-time, the Dirac spinors satisfy the Dirac equation

$$\left(i\hbar\gamma^{(a)}\partial_a - mc\right)\Psi = 0. \tag{A.12}$$

In order to investigate the behavior of this equation under the Lorentz group, let us define the transformation rule of the spinors under a Lorentz transformation Λ as follows:

$$\Psi'(x') = S(\Lambda)\Psi(x), \tag{A.13}$$

where $S(\Lambda)$ is a matrix representation of the Lorentz group. Using this definition, Eq. (A.12) is written as

$$\Big(i\hbar\, S(\Lambda)\gamma^{(a)}S^{-1}(\Lambda)\, (\Lambda^{-1})_a{}^b\partial_{b'} - mc\Big)\, \Psi' = 0, \tag{A.14}$$

where we have multiplied the Dirac equation by $S(\Lambda)$ on the left. In order for the Dirac equation to be covariant under the Lorentz transformations, the following relations must be satisfied:

$$S(\Lambda)\gamma^{(a)}S^{-1}(\Lambda)\, (\Lambda^{-1})_a{}^b = \gamma^{(b)}, \tag{A.15}$$

that is,

$$S(\Lambda)\gamma^{(a)}S^{-1}(\Lambda) = (\Lambda^{-1})^a{}_b\, \gamma^{(b)}. \tag{A.16}$$

Here, we consider the infinitesimal Lorentz transformations (A.3) again. Then $S(\Lambda)$ must be written in the form

$$S(\Lambda) = \mathbf{1} + \frac{1}{2}\, \omega^{ab}\sigma_{ab}. \tag{A.17}$$

From Eq. (A.16), we derive the following condition:

$$\frac{1}{2}\, \omega_{ab} \left[\, \sigma^{ab}\, ,\, \gamma^{(c)} \,\right] = -\omega^c{}_d \gamma^{(d)}. \tag{A.18}$$

This condition is satisfied by setting

$$\sigma^{ab} = \frac{1}{4}\, \left[\, \gamma^{(a)}\, ,\, \gamma^{(b)} \,\right]. \tag{A.19}$$

(See also Appendix D for this derivation.) It is easy to check that the matrices (A.19) do satisfy Eq. (A.11).

Appendix B

Non-minimal coupling

Now, we shall derive Eq. (2.57). The left-hand side of Eq. (2.57) is written as

$$\begin{aligned}
&\frac{1}{4}\left[\gamma^{\alpha},\gamma^{\beta}\right][\mathcal{D}_{\alpha},\mathcal{D}_{\beta}] \\
&\quad = \frac{1}{4}\left[\gamma^{\alpha},\gamma^{\beta}\right][\nabla_{\alpha}-\Gamma_{\alpha},\nabla_{\beta}-\Gamma_{\beta}] \\
&\quad = \frac{1}{4}\left[\gamma^{\alpha},\gamma^{\beta}\right](\nabla_{\beta}\Gamma_{\alpha}-\nabla_{\alpha}\Gamma_{\beta}+\Gamma_{\alpha}\Gamma_{\beta}-\Gamma_{\beta}\Gamma_{\alpha})\,,
\end{aligned} \tag{B.1}$$

where Γ_{α} is given by Eq. (2.52). Using Eq. (2.52) and the relation

$$\nabla_{\alpha}\nabla_{\beta}\, e_{(a)\mu}-\nabla_{\beta}\nabla_{\alpha}\, e_{(a)\mu} = -R^{\lambda}{}_{\mu\alpha\beta}\, e_{(a)\lambda}, \tag{B.2}$$

we can write

$$\nabla_{\beta}\Gamma_{\alpha}-\nabla_{\alpha}\Gamma_{\beta}+\Gamma_{\alpha}\Gamma_{\beta}-\Gamma_{\beta}\Gamma_{\alpha} = \frac{1}{4}\,\gamma^{\mu}\gamma^{\nu}R_{\mu\nu\alpha\beta}, \tag{B.3}$$

where $R_{\mu\nu\alpha\beta}$ is the Riemann tensor. Thus we derive

$$\frac{1}{4}\left[\gamma^{\alpha},\gamma^{\beta}\right][\mathcal{D}_{\alpha},\mathcal{D}_{\beta}] = \frac{1}{8}\,\gamma^{\alpha}\gamma^{\beta}\gamma^{\mu}\gamma^{\nu}R_{\alpha\beta\mu\nu}. \tag{B.4}$$

Furthermore, if we utilize the identity

$$R_{\alpha\beta\mu\nu}+R_{\alpha\mu\nu\beta}+R_{\alpha\nu\beta\mu} = 0, \tag{B.5}$$

then we have

$$\gamma^{\alpha}\gamma^{\beta}\gamma^{\mu}\gamma^{\nu}R_{\alpha\beta\mu\nu} = -\,6R-2\,\gamma^{\alpha}\gamma^{\beta}\gamma^{\mu}\gamma^{\nu}R_{\alpha\beta\mu\nu}, \tag{B.6}$$

that is,

$$\frac{1}{8}\,\gamma^{\alpha}\gamma^{\beta}\gamma^{\mu}\gamma^{\nu}R_{\alpha\beta\mu\nu} = -\,\frac{1}{4}\,R. \tag{B.7}$$

Therefore, we can derive Eq. (2.57):

$$\frac{1}{4}\left[\gamma^{\alpha},\gamma^{\beta}\right]\left[\mathcal{D}_{\alpha},\mathcal{D}_{\beta}\right] = -\,\frac{1}{4}\,R. \tag{B.8}$$

Appendix C

Conformal invariance

Let us consider conformal transformations:

$$g_{\alpha\beta} \to \bar{g}_{\alpha\beta} = \Omega^2 g_{\alpha\beta}. \tag{C.1}$$

An equation for a field ψ is said to be conformally invariant if there exists a number s such that ψ is a solution with the metric $g_{\alpha\beta}$ if and only if $\bar{\psi} = \Omega^s \psi$ is a solution with the metric $\bar{g}_{\alpha\beta}$. Then s is called the conformal weight of the field. (See, e.g., Ref. [34].)

Under these transformations, the spin connection, the Christoffel symbol, the Riemann tensor, the Ricci tensor and the Ricci scalar transform, respectively, according to

$$\Gamma_\alpha \;\to\; \bar{\Gamma}_\alpha = \Gamma_\alpha - \frac{1}{2}\gamma_\alpha \gamma^\beta \nabla_\beta \ln\Omega + \frac{1}{2}\nabla_\alpha \ln\Omega, \tag{C.2}$$

$$\Gamma^\alpha{}_{\mu\nu} \;\to\; \bar{\Gamma}^\alpha{}_{\mu\nu} = \Gamma^\alpha{}_{\mu\nu} + 2\,\delta^\alpha{}_{(\mu}\nabla_{\nu)} \ln\Omega - g_{\mu\nu} g^{\alpha\beta}\nabla_\beta \ln\Omega, \tag{C.3}$$

$$\begin{aligned} R^\alpha{}_{\beta\mu\nu} \quad\to\quad \bar{R}^\alpha{}_{\beta\mu\nu} = {} & R^\alpha{}_{\beta\mu\nu} + 2\,\delta^\alpha{}_{[\nu}\nabla_{\mu]}\nabla_\beta \ln\Omega - 2\,g^{\alpha\sigma} g_{\beta[\nu}\nabla_{\mu]}\nabla_\sigma \ln\Omega \\ & + 2\,\delta^\alpha{}_{[\mu}\nabla_{\nu]}\ln\Omega \cdot \nabla_\beta \ln\Omega - 2\,g_{\beta[\mu}\nabla_{\nu]}\ln\Omega \cdot g^{\alpha\sigma}\nabla_\sigma \ln\Omega \\ & - 2\,\delta^\alpha{}_{[\mu}\,g_{\nu]\beta} \cdot g^{\sigma\lambda}\,\nabla_\sigma \ln\Omega\,\nabla_\lambda \ln\Omega, \end{aligned} \tag{C.4}$$

$$\begin{aligned} R_{\beta\nu} \quad\to\quad \bar{R}_{\beta\nu} = {} & R_{\beta\nu} - 4\,\nabla_\nu\nabla_\beta \ln\Omega - g_{\beta\nu}\,g^{\sigma\lambda}\nabla_\sigma\nabla_\lambda \ln\Omega \\ & + 2\,\nabla_\beta \ln\Omega\,\nabla_\nu \ln\Omega - 2\,g_{\beta\nu}\,g^{\sigma\lambda}\nabla_\sigma \ln\Omega\,\nabla_\lambda \ln\Omega, \end{aligned} \tag{C.5}$$

$$R \;\to\; \bar{R} = \Omega^{-2}\left[R - 6\,g^{\alpha\beta}\nabla_\alpha\nabla_\beta \ln\Omega - 6\,g^{\alpha\beta}\nabla_\alpha \ln\Omega \nabla_\beta \ln\Omega\right]. \tag{C.6}$$

We shall now show that for a massless particle, Eq. (2.58) is conformally invariant. If $\bar{\Psi} = \Omega^s \Psi$, using Eqs. (C.2) and (C.6) we have

$$\begin{aligned}
&\left[\hbar^2 \bar{g}^{\alpha\beta} \bar{\mathcal{D}}_\alpha \bar{\mathcal{D}}_\beta - \frac{1}{4}\,\hbar^2 \bar{R}\right] \bar{\Psi} \\
&\quad = \Omega^{s-2} \left[\hbar^2 g^{\alpha\beta} \mathcal{D}_\alpha \mathcal{D}_\beta - \frac{1}{4}\,\hbar^2 R \right. \\
&\qquad + 2\left(s + \frac{3}{2}\right) \hbar^2 g^{\alpha\beta} (\nabla_\alpha \ln \Omega) \mathcal{D}_\beta + \left(s + \frac{3}{2}\right) \hbar^2 g^{\alpha\beta} (\nabla_\alpha \nabla_\beta \ln \Omega) \\
&\qquad \left. + \left\{\left(s + \frac{3}{2}\right)^2 - \left(s + \frac{3}{2}\right)\right\} \hbar^2 g^{\alpha\beta} (\nabla_\alpha \ln \Omega)\, (\nabla_\beta \ln \Omega)\right] \Psi.
\end{aligned} \tag{C.7}$$

Therefore, for a massless particle, Eq. (2.58) becomes conformally invariant provided that $s = -3/2$.

Next, we consider scalar fields. In curved space-time, the equation for a scalar field Φ is

$$\left[\hbar^2 g^{\alpha\beta} \nabla_\alpha \nabla_\beta + m^2 c^2\right] \Phi = 0. \tag{C.8}$$

However, this equation for a massless particle is not conformally invariant. In fact, we have

$$\begin{aligned}
&\bar{g}^{\alpha\beta} \bar{\nabla}_\alpha \bar{\nabla}_\beta \bar{\Phi} \\
&\quad = \Omega^{s-2} \left[g^{\alpha\beta} \nabla_\alpha \nabla_\beta + 2\,(s+1)\,\Omega^{-1} g^{\alpha\beta} (\nabla_\alpha \Omega)\, \nabla_\beta \right. \\
&\qquad \left. + s\,\Omega^{-1} g^{\alpha\beta} (\nabla_\alpha \nabla_\beta \Omega) + s\,(s+1)\,\Omega^{-2} g^{\alpha\beta} (\nabla_\alpha \Omega)\, (\nabla_\beta \Omega)\right] \Phi.
\end{aligned} \tag{C.9}$$

Hence no choice of s will make $\bar{g}^{\alpha\beta} \bar{\nabla}_\alpha \bar{\nabla}_\beta \bar{\Phi}$ vanish whenever $g^{\alpha\beta} \nabla_\alpha \nabla_\beta \Phi$ vanishes. (We now consider a 4-dimensional manifold.)

However, it is possible to modify Eq. (C.8) so that it becomes conformally invariant. First, if we choose $s = -1$, then the $(\nabla_\alpha \Omega) \nabla_\beta \Phi$ term and the $(\nabla_\alpha \Omega)(\nabla_\beta \Omega) \Phi$ term will vanish. Using this choice, we find

$$\begin{aligned}
&\left[\bar{g}^{\alpha\beta} \bar{\nabla}_\alpha \bar{\nabla}_\beta - c\bar{R}\right] \bar{\Phi} \\
&\quad = \Omega^{-3} \left[g^{\alpha\beta} \nabla_\alpha \nabla_\beta - cR - (1 - 6c)\,\Omega^{-1} g^{\alpha\beta} (\nabla_\alpha \nabla_\beta \Omega)\right] \Phi.
\end{aligned} \tag{C.10}$$

Thus, if we choose $c = 1/6$, then the $(\nabla_\alpha \nabla_\beta \Omega)$ term is eliminated. Therefore, the equation for the scalar field Φ,

$$\left[\hbar^2 g^{\alpha\beta} \nabla_\alpha \nabla_\beta - \frac{1}{6}\,\hbar^2 R\right] \Phi = 0, \tag{C.11}$$

is conformally invariant.

Appendix D

Derivation of spin connection

We now provide the derivation of Eq. (2.69). For the purpose of this, we utilize the fact that the following 16 matrices are linearly independent:

$$\Gamma^A = \left\{ I \ , \ \gamma_{(a)} \ , \ \tilde{\sigma}_{ab} \ , \ \gamma_{(5)}\gamma_{(a)} \ , \ \gamma_{(5)} \right\} , \tag{D.1}$$

where $\tilde{\sigma}_{ab}$ and $\gamma_{(5)}$ are given by Eqs. (2.71) and (2.72), respectively. (See Appendix E.) From this, any 4×4 matrix is expressed by a linear combination of these 16 matrices. Hence the matrices Γ_α satisfying Eq. (2.68) are also expressed in terms of these matrices. What we have to do is to find Γ^B satisfying the following relations:

$$\left[\Gamma^B \ , \ \gamma_{(a)} \right] \ \propto \ \gamma_{(b)}. \tag{D.2}$$

It is convenient to classify these lineally independent 16 matrices into five groups.

$$\left\{ \begin{array}{cccc} \text{(A)} & I & \cdots & 1 \\ \text{(B)} & \gamma_{(a)} & \cdots & 4 \\ \text{(C)} & \tilde{\sigma}_{ab} & \cdots & 6 \\ \text{(D)} & \gamma_{(5)}\gamma_{(a)} & \cdots & 4 \\ \text{(E)} & \gamma_{(5)} & \cdots & 1 \end{array} \right. \tag{D.3}$$

Let us calculate the commutator for each case.

- In the case of the group (A), we have

$$\left[I \ , \ \gamma_{(a)} \right] = 0. \tag{D.4}$$

- In the case of the group (B), we have

$$\left[\, \gamma_{(a)} \,,\, \gamma_{(b)} \,\right] = 2\, \gamma_{(a)}\gamma_{(b)} - 2\, \eta_{ab} I. \tag{D.5}$$

- In the case of the group (C), we have

$$\left[\, \tilde{\sigma}_{ab} \,,\, \gamma_{(c)} \,\right] = 2i\, \eta_{bc}\gamma_{(a)} - 2i\, \eta_{ac}\gamma_{(b)}. \tag{D.6}$$

- In the case of the group (D), we have

$$\left[\, \gamma_{(5)}\gamma_{(a)} \,,\, \gamma_{(b)} \,\right] = 2\, \gamma_{(5)}\eta_{ab}. \tag{D.7}$$

- In the case of the group (E), we have

$$\left[\, \gamma_{(5)} \,,\, \gamma_{(a)} \,\right] = 2\, \gamma_{(5)}\gamma_{(a)}. \tag{D.8}$$

Therefore, the matrices Γ_α satisfying the relations (2.68) are written as linear combinations of the matrices belonging to the groups (A) and (C):

$$\Gamma_\alpha = \mathcal{A}_\alpha{}^{ab}\, \tilde{\sigma}_{ab} + a_\alpha I, \tag{D.9}$$

where $\mathcal{A}_\alpha{}^{ab}$ is always chosen to be antisymmetric in a and b. Considering Eq. (D.4), we find that a_α is arbitrary.

Next, we shall derive $\mathcal{A}_\alpha{}^{ab}$ introduced in Eq. (D.9). From Eqs. (2.68) and (D.6), we have

$$-\, 4i\, \eta_{ac}\, \gamma_{(d)}\, \mathcal{A}_\alpha{}^{cd} = \gamma_{(b)}\, e_{(a)}{}^{\beta}\, \nabla_\alpha\, e^{(b)}{}_{\beta}. \tag{D.10}$$

If we multiply this by $\gamma_{(e)}$ on the left, then we have

$$-\, 4i\, \eta_{ac}\, \gamma_{(e)}\gamma_{(d)}\, \mathcal{A}_\alpha{}^{cd} = \gamma_{(e)}\gamma_{(b)}\, e_{(a)}{}^{\beta}\, \nabla_\alpha\, e^{(b)}{}_{\beta}, \tag{D.11}$$

whereas multiplying by $\gamma_{(e)}$ on the right, we have

$$-\, 4i\, \eta_{ac}\, \gamma_{(d)}\gamma_{(e)}\, \mathcal{A}_\alpha{}^{cd} = \gamma_{(b)}\gamma_{(e)}\, e_{(a)}{}^{\beta}\, \nabla_\alpha\, e^{(b)}{}_{\beta}. \tag{D.12}$$

By adding these equations, we derive

$$-\, 4i\, \eta_{ac}\, \eta_{bd}\, \mathcal{A}_\alpha{}^{cd} = e_{(a)}{}^{\beta}\, \nabla_\alpha\, e_{(b)\beta}. \tag{D.13}$$

Hence we obtain

$$\mathcal{A}_\alpha{}^{ab} = \frac{i}{4}\, \eta^{ac}\eta^{bd}\, e_{(c)}{}^{\beta}\, \nabla_\alpha\, e_{(d)\beta}. \tag{D.14}$$

Therefore, we can derive Eq. (2.69):

$$\Gamma_\alpha = -\frac{1}{8}\left[\gamma^{(a)}, \gamma^{(b)}\right] e_{(a)}{}^{\beta}\, \nabla_\alpha\, e_{(b)\beta} + a_\alpha I. \tag{D.15}$$

Appendix E

Proof of linear independence

Here, we prove the following 16 matrices to be linearly independent:

$$\Gamma^A = \left\{ I \ , \ \gamma_{(a)} \ , \ \tilde{\sigma}_{ab} \ , \ \gamma_{(5)}\gamma_{(a)} \ , \ \gamma_{(5)} \right\} , \tag{E.1}$$

where $\tilde{\sigma}_{ab}$ and $\gamma_{(5)}$ are given by Eqs. (2.71) and (2.72), respectively. In this Appendix, we use units in which $c = 1$.

To begin with, we investigate the trace of Γ^A. It is convenient to use the relations

$$\gamma_{(5)}\gamma_{(5)} = I, \tag{E.2}$$

$$\gamma_{(5)}\gamma_{(a)} + \gamma_{(a)}\gamma_{(5)} = 0. \tag{E.3}$$

Using these properties, we can derive

$$\mathrm{Tr}\,(I) \;=\; 4, \tag{E.4}$$
$$\mathrm{Tr}\left(\gamma_{(a)}\right) \;=\; 0, \tag{E.5}$$
$$\mathrm{Tr}\,(\sigma_{ab}) \;=\; 0, \tag{E.6}$$
$$\mathrm{Tr}\left(\gamma_{(5)}\gamma_{(a)}\right) \;=\; 0, \tag{E.7}$$
$$\mathrm{Tr}\left(\gamma_{(5)}\right) \;=\; 0, \tag{E.8}$$

that is,

$$\Gamma^A = \begin{cases} 4 & (\Gamma^A = I) \\ 0 & (\Gamma^A \neq I) \end{cases} . \tag{E.9}$$

Furthermore, we can show

$$\Gamma^A\Gamma^B = \begin{cases} I & \text{or} \quad -I & (A=B) \\ \Gamma^C(\neq I) & \text{or} \quad -\Gamma^C(\neq -I) & (A\neq B) \end{cases} . \tag{E.10}$$

Next, we assume that the following relation exists:

$$\sum_A c_A\Gamma^A = 0, \tag{E.11}$$

where c_A are numbers. When we multiply this by Γ^B on the left and take the trace, we derive

$$\mathrm{Tr}\left(\Gamma^B\sum_A c_A\Gamma^A\right) = \pm\, 4c_B. \tag{E.12}$$

From this, we find that for any B,

$$c_B = 0. \tag{E.13}$$

Therefore, these 16 matrices must be linearly independent.

Appendix F

Components of Christoffel symbol

We here show the components of the Christoffel symbol derived from the metric (3.13). Up to the order of our interest, we obtain

$$
\begin{aligned}
\Gamma^0{}_{00} &= 0 + O(1/c^4), && \text{(F.1)}\\
\Gamma^0{}_{01} &= \frac{1}{c^2}\phi_{,1} - \frac{1}{c^4}\frac{6GMR^2}{5r^5}\omega_O\omega_S x\left(x^2+y^2\right) + O(1/c^6), && \text{(F.2)}\\
\Gamma^0{}_{02} &= \frac{1}{c^2}\phi_{,2} - \frac{1}{c^4}\frac{6GMR^2}{5r^5}\omega_O\omega_S y\left(x^2+y^2\right) + O(1/c^6), && \text{(F.3)}\\
\Gamma^0{}_{03} &= \frac{1}{c^2}\phi_{,3} - \frac{1}{c^4}\frac{6GMR^2}{5r^5}\omega_O\omega_S z\left(x^2+y^2\right) + O(1/c^6), && \text{(F.4)}\\
\Gamma^0{}_{11} &= \frac{1}{c^4}\frac{12GMR^2}{5r^5}\omega_S xy + O(1/c^6), && \text{(F.5)}\\
\Gamma^0{}_{12} &= -\frac{1}{c^4}\frac{6GMR^2}{5r^5}\omega_S\left(x^2-y^2\right) + O(1/c^6), && \text{(F.6)}\\
\Gamma^0{}_{13} &= \frac{1}{c^4}\frac{6GMR^2}{5r^5}\omega_S yz + O(1/c^6), && \text{(F.7)}\\
\Gamma^0{}_{22} &= -\frac{1}{c^4}\frac{12GMR^2}{5r^5}\omega_S xy + O(1/c^6), && \text{(F.8)}\\
\Gamma^0{}_{23} &= -\frac{1}{c^4}\frac{6GMR^2}{5r^5}\omega_S zx + O(1/c^6), && \text{(F.9)}\\
\Gamma^0{}_{33} &= 0 + O(1/c^6), && \text{(F.10)}
\end{aligned}
$$

$$
\Gamma^1{}_{00} = \phi_{,1} - \omega_O^2 x
$$

$$
\begin{aligned}
& +\frac{1}{c^2}\left[4\phi\phi_{,1}+\omega_O^2 x\left(x\phi_{,1}+y\phi_{,2}\right)+\frac{8GMR^2}{5r^3}\omega_O\omega_S x-\frac{12GMR^2}{5r^5}\omega_O\omega_S x\left(x^2+y^2\right)\right] \\
& + O(1/c^4), \qquad \text{(F.11)}
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^1{}_{01} &= \frac{1}{c^2}\omega_O y\phi_{,1}+O(1/c^4), && \text{(F.12)}\\
\Gamma^1{}_{02} &= -\omega_O+\frac{1}{c^2}\left[\omega_O x\phi_{,1}+2\omega_O y\phi_{,2}+\frac{4GMR^2}{5r^3}\omega_S-\frac{6GMR^2}{5r^5}\omega_S\left(x^2+y^2\right)\right]+O(1/c^4), && \text{(F.13)}\\
\Gamma^1{}_{03} &= \frac{1}{c^2}\left[2\omega_O y\phi_{,3}-\frac{6GMR^2}{5r^5}\omega_S yz\right]+O(1/c^4), && \text{(F.14)}\\
\Gamma^1{}_{11} &= -\frac{1}{c^2}\phi_{,1}+O(1/c^4), && \text{(F.15)}\\
\Gamma^1{}_{12} &= -\frac{1}{c^2}\phi_{,2}+O(1/c^4), && \text{(F.16)}\\
\Gamma^1{}_{13} &= -\frac{1}{c^2}\phi_{,3}+O(1/c^4), && \text{(F.17)}\\
\Gamma^1{}_{22} &= \frac{1}{c^2}\phi_{,1}+O(1/c^4), && \text{(F.18)}\\
\Gamma^1{}_{23} &= 0+O(1/c^4), && \text{(F.19)}\\
\Gamma^1{}_{33} &= \frac{1}{c^2}\phi_{,1}+O(1/c^4), && \text{(F.20)}
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^2{}_{00} &= \phi_{,2}-\omega_O^2 y \\
&\quad +\frac{1}{c^2}\left[4\phi\phi_{,2}+\omega_O^2 y\left(x\phi_{,1}+y\phi_{,2}\right)+\frac{8GMR^2}{5r^3}\omega_O\omega_S y-\frac{12GMR^2}{5r^5}\omega_O\omega_S y\left(x^2+y^2\right)\right] \\
&\quad + O(1/c^4), && \text{(F.21)}\\
\Gamma^2{}_{01} &= \omega_O+\frac{1}{c^2}\left[-2\omega_O x\phi_{,1}-\omega_O y\phi_{,2}-\frac{4GMR^2}{5r^3}\omega_S+\frac{6GMR^2}{5r^5}\omega_S\left(x^2+y^2\right)\right]+O(1/c^4), && \text{(F.22)}\\
\Gamma^2{}_{02} &= -\frac{1}{c^2}\omega_O x\phi_{,2}+O(1/c^4), && \text{(F.23)}\\
\Gamma^2{}_{03} &= \frac{1}{c^2}\left[-2\omega_O x\phi_{,3}+\frac{6GMR^2}{5r^5}\omega_S zx\right]+O(1/c^4), && \text{(F.24)}\\
\Gamma^2{}_{11} &= \frac{1}{c^2}\phi_{,2}+O(1/c^4), && \text{(F.25)}\\
\Gamma^2{}_{12} &= -\frac{1}{c^2}\phi_{,1}+O(1/c^4), && \text{(F.26)}
\end{aligned}
$$

$$
\begin{aligned}
{\Gamma^2}_{13} &= 0 + O(1/c^4), && \text{(F.27)}\\
{\Gamma^2}_{22} &= -\frac{1}{c^2}\phi_{,2} + O(1/c^4), && \text{(F.28)}\\
{\Gamma^2}_{23} &= -\frac{1}{c^2}\phi_{,3} + O(1/c^4), && \text{(F.29)}\\
{\Gamma^2}_{33} &= \frac{1}{c^2}\phi_{,2} + O(1/c^4), && \text{(F.30)}
\end{aligned}
$$

$$
\begin{aligned}
{\Gamma^3}_{00} &= \phi_{,2} + \frac{1}{c^2}\left[4\phi\phi_{,3} + \omega_O^2\left(x^2+y^2\right)\phi_{,3} - \frac{12GMR^2}{5r^5}\omega_O\omega_S z\left(x^2+y^2\right)\right] + O(1/c^4), && \text{(F.31)}\\
{\Gamma^3}_{01} &= \frac{1}{c^2}\left[-\omega_O y\phi_{,3} + \frac{6GMR^2}{5r^5}\omega_S yz\right] + O(1/c^4), && \text{(F.32)}\\
{\Gamma^3}_{02} &= \frac{1}{c^2}\left[\omega_O x\phi_{,3} - \frac{6GMR^2}{5r^5}\omega_S zx\right] + O(1/c^4), && \text{(F.33)}\\
{\Gamma^3}_{03} &= 0 + O(1/c^4), && \text{(F.34)}\\
{\Gamma^3}_{11} &= \frac{1}{c^2}\phi_{,3} + O(1/c^4), && \text{(F.35)}\\
{\Gamma^3}_{12} &= 0 + O(1/c^4), && \text{(F.36)}\\
{\Gamma^3}_{13} &= -\frac{1}{c^2}\phi_{,1} + O(1/c^4), && \text{(F.37)}\\
{\Gamma^3}_{22} &= \frac{1}{c^2}\phi_{,3} + O(1/c^4), && \text{(F.38)}\\
{\Gamma^3}_{23} &= -\frac{1}{c^2}\phi_{,2} + O(1/c^4), && \text{(F.39)}\\
{\Gamma^3}_{33} &= -\frac{1}{c^2}\phi_{,3} + O(1/c^4). && \text{(F.40)}
\end{aligned}
$$

Appendix G

Components of spin connection

We now calculate the components of the spin connection. The spin connection is given by Eq. (2.52):

$$\Gamma_\alpha = -\frac{1}{8}\,\left[\gamma^{(a)}, \gamma^{(b)}\right] g_{\mu\nu}\, e_{(a)}^{\ \ \mu}\, \nabla_\alpha\, e_{(b)}^{\ \ \nu}. \tag{G.1}$$

Using the tetrad (3.28)–(3.32) and the components of the Christoffel symbol derived in Appendix F, up to the order of our interest, we obtain

$$\begin{aligned}
\Gamma_0 \;&=\; -\frac{1}{8}\,\varepsilon_{ijk}\,\omega_{_O}^{\ \,i}\,\gamma^{[jk]} - \frac{1}{4}\,\gamma^{[0i]}\phi_{,i} \\
&\quad - \frac{1}{8c^2}\left[\frac{GM}{r^3}\varepsilon_{ijk}\left[\boldsymbol{r}\times(\boldsymbol{r}\times\boldsymbol{\omega}_{_O})\right]^i\gamma^{[jk]}\right. \\
&\qquad \left. - \frac{4GMR^2}{5r^3}\varepsilon_{ijk}\,\omega_{_S}^{\ \,i}\,\gamma^{[jk]} - \frac{6GMR^2}{5r^5}\varepsilon_{ijk}\left[\boldsymbol{r}\times(\boldsymbol{r}\times\boldsymbol{\omega}_S)\right]^i\gamma^{[jk]}\right], \qquad (\mathrm{G.2}) \\
\Gamma_1 \;&=\; -\frac{1}{8c^2}\left(\phi_{,2}\,\varepsilon_{3jk} - \phi_{,3}\,\varepsilon_{2jk}\right)\gamma^{[jk]} \\
&\quad + \frac{3GMR^2}{10c^2r^5}\omega_{_S}\left[-2xy\,\gamma^{[01]} + \left(x^2-y^2\right)\gamma^{[02]} - yz\,\gamma^{[03]}\right], \qquad (\mathrm{G.3}) \\
\Gamma_2 \;&=\; -\frac{1}{8c^2}\left(\phi_{,3}\,\varepsilon_{1jk} - \phi_{,1}\,\varepsilon_{3jk}\right)\gamma^{[jk]} \\
&\quad + \frac{3GMR^2}{10c^2r^5}\omega_{_S}\left[\left(x^2-y^2\right)\gamma^{[01]} + 2xy\,\gamma^{[02]} + zx\,\gamma^{[03]}\right], \qquad (\mathrm{G.4}) \\
\Gamma_3 \;&=\; -\frac{1}{8c^2}\left(\phi_{,1}\,\varepsilon_{2jk} - \phi_{,2}\,\varepsilon_{1jk}\right)\gamma^{[jk]} \\
&\quad + \frac{3GMR^2}{10c^2r^5}\omega_{_S}\left[-yz\,\gamma^{[01]} + zx\,\gamma^{[02]}\right], \qquad (\mathrm{G.5})
\end{aligned}$$

where ε_{ijk} is the Levi-Civita antisymmetric tensor ($\varepsilon_{123} = +1$), $\gamma^{[ab]}$ is defined as

$$\gamma^{[ab]} = \left[\gamma^{(a)}, \gamma^{(b)}\right], \tag{G.6}$$

and the angular velocity vectors $\boldsymbol{\omega}_{O}$, $\boldsymbol{\omega}_{S}$ are, respectively,

$$\begin{aligned} \boldsymbol{\omega}_{O} &= (0, 0, \omega_{O}), && \text{(G.7)} \\ \boldsymbol{\omega}_{S} &= (0, 0, \omega_{S}). && \text{(G.8)} \end{aligned}$$

It is convenient to introduce the following 4×4 matrices similar to the Pauli spin matrices:

$$\rho_1 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}, \quad \rho_3 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{G.9}$$

where I is the 2×2 unit matrix. These matrices satisfy the relations

$$\rho_i \rho_j = \delta_{ij} + i\varepsilon_{ijk}\rho_k. \tag{G.10}$$

We now adopt the standard representation as the Dirac matrices:

$$\gamma^{(0)} = \frac{1}{c}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^{(i)} = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \tag{G.11}$$

where σ_i are the well-known Pauli matrices. Then we have

$$\begin{aligned} \gamma^{[0i]} &= \frac{2}{c}\,\rho_1\,\sigma_i, && \text{(G.12)} \\ \gamma^{[ij]} &= -\,2i\,\varepsilon_{ijk}\,\sigma_k. && \text{(G.13)} \end{aligned}$$

Hence we obtain

$$\begin{aligned} i\hbar\Gamma_0 &= -\,\boldsymbol{\omega}_{O}\cdot\boldsymbol{S} + \frac{1}{2c}\,\rho_1\,\boldsymbol{\sigma}\cdot(\overline{\boldsymbol{p}}\phi) \\ &\quad + \frac{1}{c^2}\left[\frac{4GMR^2}{5r^3}\boldsymbol{\omega}_{S}\cdot\boldsymbol{S} - \frac{GM}{r^3}\boldsymbol{S}\cdot[\boldsymbol{r}\times(\boldsymbol{r}\times\boldsymbol{\omega}_{O})] + \frac{6GMR^2}{5r^5}\boldsymbol{S}\cdot[\boldsymbol{r}\times(\boldsymbol{r}\times\boldsymbol{\omega}_{S})]\right], \end{aligned} \tag{G.14}$$

$$\begin{aligned} i\hbar\Gamma_1 &= -\,\frac{\hbar}{2c^2}\left(\phi_{,2}\,\sigma_3 - \phi_{,3}\,\sigma_2\right) \\ &\quad + \frac{i\hbar}{c^3}\,\rho_1\frac{3GMR^2}{5r^5}\omega_{S}\left[-2xy\,\sigma_1 + \left(x^2 - y^2\right)\sigma_2 - yz\,\sigma_3\right], \end{aligned} \tag{G.15}$$

$$
\begin{aligned}
i\hbar\Gamma_2 &= -\frac{\hbar}{2c^2}\left(\phi_{,3}\,\sigma_1-\phi_{,1}\,\sigma_3\right) \\
&\quad+\frac{i\hbar}{c^3}\,\rho_1\frac{3GMR^2}{5r^5}\omega_S\left[\left(x^2-y^2\right)\sigma_1+2xy\,\sigma_2+zx\,\sigma_3\right], &\text{(G.16)}\\
i\hbar\Gamma_3 &= -\frac{\hbar}{2c^2}\left(\phi_{,1}\,\sigma_2-\phi_{,2}\,\sigma_1\right) \\
&\quad+\frac{i\hbar}{c^3}\,\rho_1\frac{3GMR^2}{5r^5}\omega_S\left[-yz\,\sigma_1+zx\,\sigma_2\right], &\text{(G.17)}
\end{aligned}
$$

where $\overline{\boldsymbol{p}}$ is the momentum operator in flat space-time, and $\boldsymbol{S}=\hbar\boldsymbol{\sigma}/2$ is the spin of the particle.

On the other hand, if we adopt the Weyl representation as the Dirac matrices:

$$
\gamma^{(0)}=\frac{1}{c}\begin{pmatrix}0 & I\\ I & 0\end{pmatrix},\quad \gamma^{(i)}=\begin{pmatrix}0 & -\sigma_i\\ \sigma_i & 0\end{pmatrix}, \tag{G.18}
$$

then we have

$$
\begin{aligned}
\gamma^{[0i]} &= \frac{2}{c}\,\rho_3\,\sigma_i, &\text{(G.19)}\\
\gamma^{[ij]} &= -2i\,\varepsilon_{ijk}\,\sigma_k. &\text{(G.20)}
\end{aligned}
$$

Hence we obtain

$$
\begin{aligned}
i\hbar\Gamma_0 &= -\boldsymbol{\omega}_O\cdot\boldsymbol{S}+\frac{1}{2c}\,\rho_3\,\boldsymbol{\sigma}\cdot(\overline{\boldsymbol{p}}\phi) \\
&\quad+\frac{1}{c^2}\left[\frac{4GMR^2}{5r^3}\boldsymbol{\omega}_S\cdot\boldsymbol{S}-\frac{GM}{r^3}\boldsymbol{S}\cdot[\boldsymbol{r}\times(\boldsymbol{r}\times\boldsymbol{\omega}_O)]+\frac{6GMR^2}{5r^5}\boldsymbol{S}\cdot[\boldsymbol{r}\times(\boldsymbol{r}\times\boldsymbol{\omega}_S)]\right], \\
& &\text{(G.21)}\\
i\hbar\Gamma_1 &= -\frac{\hbar}{2c^2}\left(\phi_{,2}\,\sigma_3-\phi_{,3}\,\sigma_2\right) \\
&\quad+\frac{i\hbar}{c^3}\,\rho_3\frac{3GMR^2}{5r^5}\omega_S\left[-2xy\,\sigma_1+\left(x^2-y^2\right)\sigma_2-yz\,\sigma_3\right], &\text{(G.22)}\\
i\hbar\Gamma_2 &= -\frac{\hbar}{2c^2}\left(\phi_{,3}\,\sigma_1-\phi_{,1}\,\sigma_3\right) \\
&\quad+\frac{i\hbar}{c^3}\,\rho_3\frac{3GMR^2}{5r^5}\omega_S\left[\left(x^2-y^2\right)\sigma_1+2xy\,\sigma_2+zx\,\sigma_3\right], &\text{(G.23)}\\
i\hbar\Gamma_3 &= -\frac{\hbar}{2c^2}\left(\phi_{,1}\,\sigma_2-\phi_{,2}\,\sigma_1\right) \\
&\quad+\frac{i\hbar}{c^3}\,\rho_3\frac{3GMR^2}{5r^5}\omega_S\left[-yz\,\sigma_1+zx\,\sigma_2\right]. &\text{(G.24)}
\end{aligned}
$$

Appendix H

Non-relativistic Hamiltonian

We now use the standard representation (G.11) for the Dirac matrices. Then, the components of the spin connection are given by Eqs. (G.14) – (G.17). From this, we obtain

$$\begin{aligned} H &= \rho_3 mc^2 + c\rho_1 \boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}} + \rho_3 m\phi - \boldsymbol{\omega}\cdot(\boldsymbol{L}+\boldsymbol{S}) \\ &\quad + \frac{1}{c}\rho_1\left[-\frac{1}{2}\boldsymbol{\sigma}\cdot(\overline{\boldsymbol{p}}\phi) + 2\phi\,\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}\right] \\ &\quad + \frac{1}{c^2}\left[\frac{1}{2}\rho_3 m\phi^2 + \frac{4GMR^2}{5r^3}\boldsymbol{\omega}\cdot(\boldsymbol{L}+\boldsymbol{S}) + \frac{6GMR^2}{5r^5}\boldsymbol{S}\cdot[\boldsymbol{r}\times(\boldsymbol{r}\times\boldsymbol{\omega})]\right], \end{aligned} \tag{H.1}$$

where ρ_i are defined by Eq. (G.9). Moreover, the Hamiltonian H' redefined by Eq. (4.2) is then

$$\begin{aligned} H' &= \rho_3 mc^2 + c\rho_1 \boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}} + \rho_3 m\phi - \boldsymbol{\omega}\cdot(\boldsymbol{L}+\boldsymbol{S}) \\ &\quad + \frac{1}{c}\rho_1\left(\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}\,\phi + \phi\,\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}\right) \\ &\quad + \frac{1}{c^2}\left[\frac{1}{2}\rho_3 m\phi^2 + \frac{4GMR^2}{5r^3}\boldsymbol{\omega}\cdot(\boldsymbol{L}+\boldsymbol{S}) + \frac{6GMR^2}{5r^5}\boldsymbol{S}\cdot[\boldsymbol{r}\times(\boldsymbol{r}\times\boldsymbol{\omega})]\right]. \end{aligned} \tag{H.2}$$

Next, we perform the FWT transformation to derive the non-relativistic Hamiltonian for the "large" component. First, we use the unitary operator

$$U_1 = \exp\left(i\rho_2\frac{\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}}{2mc}\right), \tag{H.3}$$

so that we can eliminate the odd term of $O(c)$. Using the useful formula

$$e^{iS}He^{-iS} = H + i\,[S,H] + \frac{i^2}{2!}\,[S,[S,H]] + \frac{i^3}{3!}\,[S,[S,[S,H]]] + \cdots \tag{H.4}$$

and the relation (G.10), we obtain the transformed Hamiltonian $U_1 H' U_1^\dagger$:

$$\begin{aligned} U_1 H' U_1^\dagger \;\; &= \;\; \rho_3 mc^2 + \rho_3 \left(\frac{\overline{\boldsymbol{p}}^2}{2m} + m\phi \right) - \boldsymbol{\omega} \cdot (\boldsymbol{L} + \boldsymbol{S}) \\ &\quad + \frac{1}{c}\rho_1 \left[\frac{1}{2} \left(\boldsymbol{\sigma} \cdot \overline{\boldsymbol{p}}\, \phi + \phi\, \boldsymbol{\sigma} \cdot \overline{\boldsymbol{p}} \right) - \frac{1}{3m^2} \left(\boldsymbol{\sigma} \cdot \overline{\boldsymbol{p}} \right)^3 \right] \\ &\quad + \frac{1}{c^2} \left[\rho_3 \left(\frac{1}{2} m\phi^2 - \frac{\overline{\boldsymbol{p}}^4}{8m^3} + \frac{3}{2m} \overline{\boldsymbol{p}} \cdot \phi \overline{\boldsymbol{p}} + \frac{3GM}{2mr^3} \boldsymbol{L} \cdot \boldsymbol{S} \right) \right. \\ &\qquad \left. + \frac{4GMR^2}{5r^3} \boldsymbol{\omega} \cdot (\boldsymbol{L} + \boldsymbol{S}) + \frac{6GMR^2}{5r^5} \boldsymbol{S} \cdot [\boldsymbol{r} \times (\boldsymbol{r} \times \boldsymbol{\omega})] \right] . \end{aligned} \tag{H.5}$$

Second, we use

$$U_2 = \exp\left(i\rho_2 \frac{3m^2 \left(\boldsymbol{\sigma} \cdot \overline{\boldsymbol{p}}\, \phi + \phi\, \boldsymbol{\sigma} \cdot \overline{\boldsymbol{p}} \right) - 2 \left(\boldsymbol{\sigma} \cdot \overline{\boldsymbol{p}} \right)^3}{12(mc)^3} \right), \tag{H.6}$$

which makes the odd terms of $O(1/c)$ vanish. Using this unitary operator, we finally obtain

$$\begin{aligned} U H' U^\dagger \;\; &= \;\; \rho_3 mc^2 + \rho_3 \left(\frac{\overline{\boldsymbol{p}}^2}{2m} + m\phi \right) - \boldsymbol{\omega} \cdot (\boldsymbol{L} + \boldsymbol{S}) \\ &\quad + \frac{1}{c^2} \left[\rho_3 \left(\frac{1}{2} m\phi^2 - \frac{\overline{\boldsymbol{p}}^4}{8m^3} + \frac{3}{2m} \overline{\boldsymbol{p}} \cdot \phi \overline{\boldsymbol{p}} + \frac{3GM}{2mr^3} \boldsymbol{L} \cdot \boldsymbol{S} \right) \right. \\ &\qquad \left. + \frac{4GMR^2}{5r^3} \boldsymbol{\omega} \cdot (\boldsymbol{L} + \boldsymbol{S}) + \frac{6GMR^2}{5r^5} \boldsymbol{S} \cdot [\boldsymbol{r} \times (\boldsymbol{r} \times \boldsymbol{\omega})] \right] , \end{aligned} \tag{H.7}$$

where U is given by $U = U_2 U_1$.

Appendix I

Canonical quantization

Now, we shall follow the canonical quantization procedure to derive the Schrödinger equation involving general relativistic corrections for a non-relativistic particle.

I.1 Classical Hamiltonian

Let us consider a particle which has the mass m and propagates in the gravitational field described by Eqs. (3.20) – (3.24) with the condition (4.1). The relativistic Lagrangian for this particle is

$$\begin{aligned} L &= -mc\,\frac{ds}{dt} \\ &= -mc\sqrt{g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}} \\ &= -mc\sqrt{N^2 - \gamma_{ij}\left(N^i + \dot{x}^i\right)\left(N^j + \dot{x}^j\right)}\,, \end{aligned} \tag{I.1}$$

where the dot over x^{μ} denotes the differentiation with respect to t. We can now define the canonical momentum as

$$p_i = \frac{\partial L}{\partial \dot{x}^i}. \tag{I.2}$$

Using this, we obtain the reduced classical Hamiltonian in the form

$$\begin{aligned} H &= p_i\dot{x}^i - L - mc^2 \\ &= N\sqrt{m^2c^2 + \gamma^{ij}p_ip_j} - N^ip_i - mc^2. \end{aligned} \tag{I.3}$$

As mentioned previously, we consider a non-relativistic particle, whose rest energy is much larger than the kinematic one. Then we have

$$\gamma^{ij}p_ip_j \ll m^2c^2. \tag{I.4}$$

The non-relativistic Hamiltonian up to the order of our interest is

$$H = -N^ip_i + \left(\frac{N}{c} - 1\right)mc^2 + \frac{N}{c}\left[\frac{\gamma^{ij}p_ip_j}{2m} - \frac{\left(\gamma^{ij}p_ip_j\right)^2}{8m^3c^2}\right] + O\left(\frac{1}{c^4}\right). \tag{I.5}$$

I.2 Quantum Hamiltonian

We have obtained the non-relativistic classical Hamiltonian in the last section. Next, we follow the canonical quantization procedure to derive the quantum Hamiltonian. What we have to do is to replace the momentum p_i in the classical Hamiltonian with the momentum operator $\hat{p}_i$. The canonical variables, x^i and $\hat{p}_i$, satisfy the canonical commutation relation

$$\left[x^i, \hat{p}_j\right] = i\hbar\delta^i{}_j. \tag{I.6}$$

In addition, the momentum operator $\hat{p}_i$ is hermitian and, therefore, satisfies the relation

$$(\hat{p}_i\psi, \varphi) = (\psi, \hat{p}_i\varphi), \tag{I.7}$$

where the round brackets denote the inner product which is invariant under the spatial coordinate transformations:

$$(\psi, \varphi) \equiv \int \psi^*\varphi\sqrt{\gamma}d^3x. \tag{I.8}$$

Considering the definition of the inner product, we adopt the momentum operator

$$\hat{p}_i = -i\hbar\gamma^{-1/4}\frac{\partial}{\partial x^i}\gamma^{1/4} \equiv \gamma^{-1/4}\overline{p}_i\gamma^{1/4}. \tag{I.9}$$

This momentum operator $\hat{p}_i$, of course, satisfies the commutation relation (I.6).

By replacing p_i in the classical Hamiltonian (I.5) with $\hat{p}_i$, we obtain the quantum Hamiltonian

$$\begin{aligned} H \;=\; \gamma^{-1/4}\Bigg[&\frac{\overline{\boldsymbol{p}}^2}{2m} + m\phi - \boldsymbol{\omega}\cdot\boldsymbol{L} \\ &+ \frac{1}{c^2}\left(\frac{4GMR^2}{5r^3}\boldsymbol{\omega}\cdot\boldsymbol{L} - \frac{\overline{\boldsymbol{p}}^4}{8m^3} + \frac{1}{2}m\phi^2 + \frac{3}{2m}\overline{\boldsymbol{p}}\cdot\phi\overline{\boldsymbol{p}}\right)\Bigg]\gamma^{1/4}, \end{aligned} \tag{I.10}$$

where we have taken the appropriate ordering to make the Hamiltonian hermitian. (Although there can be various ways to the ordering, we have particularly chosen the one which leads to the result derived from the covariant Dirac equation.) The Schrödinger equation is then

$$i\hbar\frac{\partial}{\partial t}\Phi = H\Phi. \tag{I.11}$$

We now follow the same discussions as in Sec. 4.1, and redefine the wave function and the Hamiltonian in the following way:

$$\Phi' = \gamma^{1/4}\Phi, \quad H' = \gamma^{1/4}H\gamma^{-1/4}. \tag{I.12}$$

Under this redefinition, we obtain

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\Phi' &= H'\Phi' \\ &= \left[\frac{\overline{\boldsymbol{p}}^2}{2m} + m\phi - \boldsymbol{\omega}\cdot\boldsymbol{L} \right. \\ &\quad \left. + \frac{1}{c^2}\left(\frac{4GMR^2}{5r^3}\boldsymbol{\omega}\cdot\boldsymbol{L} - \frac{\overline{\boldsymbol{p}}^4}{8m^3} + \frac{1}{2}m\phi^2 + \frac{3}{2m}\overline{\boldsymbol{p}}\cdot\phi\overline{\boldsymbol{p}}\right)\right]\Phi'. \end{aligned} \tag{I.13}$$

Appendix J

Ultra-relativistic Hamiltonian

Now, we shall obtain the ultra-relativistic Hamiltonian in Eq. (5.5).

If we adopt the Weyl representation (G.18) as the Dirac matrices, then the components of the spin connection are given by Eqs. (G.21) – (G.24). Using this result, up to the order of our interest, we obtain

$$\begin{aligned} H &= \rho_3 c\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}} + \rho_3\left[-\frac{1}{2}c\boldsymbol{\sigma}\cdot\left(\overline{\boldsymbol{p}}\frac{\phi}{c^2}\right) + 2\frac{\phi}{c^2}\,c\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}\right] \\ &\quad + \frac{1}{c^2}\left[\frac{4GMR^2}{5r^3}\boldsymbol{\omega}\cdot(\boldsymbol{L}+\boldsymbol{S}) + \frac{6GMR^2}{5r^5}\boldsymbol{S}\cdot[\boldsymbol{r}\times(\boldsymbol{r}\times\boldsymbol{\omega})]\right] + \rho_1 mc^2 + \rho_1 mc^2\frac{\phi}{c^2}, \end{aligned} \tag{J.1}$$

where ρ_i are defined by Eq. (G.9). Moreover, the Hamiltonian redefined by Eq. (5.2) is then

$$\begin{aligned} H' &= \rho_3 c\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}} + \rho_3\left(c\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}\,\frac{\phi}{c^2} + \frac{\phi}{c^2}\,c\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}\right) \\ &\quad + \frac{1}{c^2}\left[\frac{4GMR^2}{5r^3}\boldsymbol{\omega}\cdot(\boldsymbol{L}+\boldsymbol{S}) + \frac{6GMR^2}{5r^5}\boldsymbol{S}\cdot[\boldsymbol{r}\times(\boldsymbol{r}\times\boldsymbol{\omega})]\right] + \rho_1 mc^2 + \rho_1 mc^2\frac{\phi}{c^2}. \end{aligned} \tag{J.2}$$

By performing a unitary transformation similar to the FWT transformation, we shall derive the ultra-relativistic Hamiltonian for the left-handed component. We here divide the unitary transformation into several steps. First, we use the unitary operator

$$U_1 = \exp\left(i\rho_2\frac{1}{2}mc^2\frac{c\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}}{c^2\overline{p}^2}\right), \tag{J.3}$$

which is introduced to eliminate the odd term proportional to mc^2. Using the formula (H.4) and the relation (G.10), we obtain the transformed Hamiltonian

$$\begin{aligned}
U_1 H' U_1^\dagger \;=\;& \rho_3 c\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}} + \rho_3\left(c\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}\,\frac{\phi}{c^2} + \frac{\phi}{c^2}\,c\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}\right) + A + \rho_1 mc^2\frac{\phi}{c^2} \\
& - \rho_1\frac{1}{2}mc^2\left[c\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}\left(\frac{c\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}}{c^2\overline{p}^2}\,\frac{\phi}{c^2} + \frac{\phi}{c^2}\,\frac{c\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}}{c^2\overline{p}^2}\right) + \left(\frac{c\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}}{c^2\overline{p}^2}\,\frac{\phi}{c^2} + \frac{\phi}{c^2}\,\frac{c\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}}{c^2\overline{p}^2}\right)c\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}\right] \\
& + i\rho_2\frac{1}{2}mc^2\left(\frac{c\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}}{c^2\overline{p}^2}A - A\frac{c\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}}{c^2\overline{p}^2}\right) + \rho_3\frac{1}{2}m^2c^4\frac{c\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}}{c^2\overline{p}^2} \\
& - \rho_3\frac{1}{8}m^2c^4\left[\frac{1}{c^2\overline{p}^2}\frac{\phi}{c^2}c\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}} + c\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}\frac{\phi}{c^2}\frac{1}{c^2\overline{p}^2} - \left(\frac{c\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}}{c^2\overline{p}^2}\frac{\phi}{c^2} + \frac{\phi}{c^2}\frac{c\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}}{c^2\overline{p}^2}\right)\right] \\
& - \frac{1}{8}m^2c^4\left(A\frac{1}{c^2\overline{p}^2} - 2\frac{c\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}}{c^2\overline{p}^2}A\frac{c\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}}{c^2\overline{p}^2} + \frac{1}{c^2\overline{p}^2}A\right), \qquad \text{(J.4)}
\end{aligned}$$

where A is given by

$$A = \frac{1}{c^2}\left[\frac{4GMR^2}{5r^3}\boldsymbol{\omega}\cdot(\boldsymbol{L}+\boldsymbol{S}) + \frac{6GMR^2}{5r^5}\boldsymbol{S}\cdot[\boldsymbol{r}\times(\boldsymbol{r}\times\boldsymbol{\omega})]\right]. \qquad \text{(J.5)}$$

Second, in order to eliminate the fifth term in Eq. (J.4), we use the unitary operator

$$U_2 = \exp\left[-i\rho_2\frac{1}{2}mc^2\left(\frac{c\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}}{c^2\overline{p}^2}\frac{\phi}{c^2} + \frac{\phi}{c^2}\frac{c\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}}{c^2\overline{p}^2}\right)\right]. \qquad \text{(J.6)}$$

Using this unitary operator, we obtain

$$\begin{aligned}
U_2U_1H'U_1^\dagger U_2^\dagger \;=\;& \rho_3 c\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}} + \rho_3\left(c\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}\,\frac{\phi}{c^2} + \frac{\phi}{c^2}\,c\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}\right) + A + \rho_1 mc^2\frac{\phi}{c^2} \\
& + i\rho_2\frac{1}{2}mc^2\left(\frac{c\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}}{c^2\overline{p}^2}A - A\frac{c\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}}{c^2\overline{p}^2}\right) + \rho_3\frac{1}{2}m^2c^4\frac{c\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}}{c^2\overline{p}^2} \\
& - \rho_3\frac{1}{8}m^2c^4\left[\frac{1}{c^2\overline{p}^2}\frac{\phi}{c^2}c\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}} + c\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}\frac{\phi}{c^2}\frac{1}{c^2\overline{p}^2} - \left(\frac{c\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}}{c^2\overline{p}^2}\frac{\phi}{c^2} + \frac{\phi}{c^2}\frac{c\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}}{c^2\overline{p}^2}\right)\right] \\
& - \frac{1}{8}m^2c^4\left(A\frac{1}{c^2\overline{p}^2} - 2\frac{c\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}}{c^2\overline{p}^2}A\frac{c\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}}{c^2\overline{p}^2} + \frac{1}{c^2\overline{p}^2}A\right). \qquad \text{(J.7)}
\end{aligned}$$

Finally, we use the two unitary operators $U_3 = e^{iS_3}$ and $U_4 = e^{iS_4}$ where S_3 and S_4 satisfy, respectively, the relations

$$i\left[S_3, \rho_3 c\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}\right] = -\,\rho_1 mc^2\frac{\phi}{c^2}, \qquad \text{(J.8)}$$

$$i\left[S_4, \rho_3 c\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}\right] = -\, i\rho_2 \frac{1}{2} mc^2 \left(\frac{c\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}}{c^2\overline{p}^2} A - A \frac{c\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}}{c^2\overline{p}^2} \right). \tag{J.9}$$

We here assume the existence of these unitary operators, which make the remaining odd terms vanish. (We need not find the concrete forms of these unitary operators, because the extra terms arising from these unitary transformations are higher oder terms.) Using these unitary operators, we obtain the transformed Hamiltonian $UH'U^\dagger$ which is even up to the order of our interest:

$$\begin{aligned} UH'U^\dagger \;\; = \;\; & \rho_3 c\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}} + \rho_3 \left(c\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}\, \frac{\phi}{c^2} + \frac{\phi}{c^2}\, c\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}} \right) + A \\ & + \rho_3 \frac{1}{2} m^2 c^4 \frac{c\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}}{c^2\overline{p}^2} \\ & - \rho_3 \frac{1}{8} m^2 c^4 \left[\frac{1}{c^2\overline{p}^2} \frac{\phi}{c^2} c\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}} + c\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}} \frac{\phi}{c^2} \frac{1}{c^2\overline{p}^2} - \left(\frac{c\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}}{c^2\overline{p}^2} \frac{\phi}{c^2} + \frac{\phi}{c^2} \frac{c\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}}{c^2\overline{p}^2} \right) \right] \\ & - \frac{1}{8} m^2 c^4 \left(A \frac{1}{c^2\overline{p}^2} - 2 \frac{c\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}}{c^2\overline{p}^2} A \frac{c\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}}{c^2\overline{p}^2} + \frac{1}{c^2\overline{p}^2} A \right), \end{aligned} \tag{J.10}$$

where U is given by $U = U_4 U_3 U_2 U_1$.

Appendix K

Neutrino mixing schemes

Now, we shall consider possible neutrino mixing schemes. In particular, we consider the general case of n neutrino flavors. There exist several distinct schemes for neutrino mixing, whereas only one scheme for quark mixing is possible. This arises from the fact that neutrinos are electrically neutral. In contrast to quarks, neutrinos which have definite masses can be of Majorana-type as well as Dirac-type. Moreover, the number of massive Majorana neutrinos can exceed the number of lepton flavors.

Let us classify neutrino mixing schemes according to the type of mass terms. For the purpose of this, we now introduce the following columns:

$$\nu_L = (\nu_{lL}) = \begin{pmatrix} \nu_{eL} \\ \nu_{\mu L} \\ \nu_{\tau L} \\ \vdots \end{pmatrix}, \quad \nu_R = (\nu_{l'R}) = \begin{pmatrix} \nu_{eR} \\ \nu_{\mu R} \\ \nu_{\tau R} \\ \vdots \end{pmatrix}, \tag{K.1}$$

where l and l' run over n values: $e, \mu, \tau, \cdots$. Although the right-handed fields $\nu_{l'R}$ do not enter the interaction Lagrangian of the standard electroweak theory, these fields may be present in the mass terms.

Before constructing the possible neutrino mass terms, we briefly review the charge conjugation of spinor fields. The Dirac equation describing a spinor field ψ with electric charge $-e$ (in flat space-time) is given by

$$\left(i\gamma^{(a)}\frac{\partial}{\partial x^a} - e\gamma^{(a)}A_a - m\right)\psi = 0, \tag{K.2}$$

where A_a denote electromagnetic potentials. (In this Appendix, we use units in which $\hbar = c = 1$.) If we write the spinor field with the opposite charge as ψ_c. then we have

$$\left(i\gamma^{(a)}\frac{\partial}{\partial x^a} + e\gamma^{(a)}A_a - m\right)\psi_c = 0. \tag{K.3}$$

In order to investigate the relation between ψ and ψ_c, we take the hermitian conjugate and then the transpose of Eq. (K.2). From this, we obtain

$$\left[-\gamma^{(a)T}\left(i\frac{\partial}{\partial x^a} + eA_a\right) - m\right]\left(\gamma^{(0)T}\psi^*\right) = 0, \tag{K.4}$$

where we have used the relation

$$\gamma^{(0)}\gamma^{(a)\dagger}\gamma^{(0)} = \gamma^{(a)}. \tag{K.5}$$

It can be shown that for any representation of the Dirac matrices, there exists a matrix C such that

$$C\gamma^{(a)T}C^{-1} = -\,\gamma^{(a)}. \tag{K.6}$$

If we use the standard representation for the Dirac matrices, then we find the following solution for C:

$$C = i\gamma^{(2)}\gamma^{(0)} = \begin{pmatrix} 0 & -i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix}, \tag{K.7}$$

which satisfies the relations

$$-\,C = C^{-1} = C^T = C^\dagger. \tag{K.8}$$

Using Eq. (K.6), we derive the following equation from Eq. (K.4):

$$\left(i\gamma^{(a)}\frac{\partial}{\partial x^a} + e\gamma^{(a)}A_a - m\right)C\left(\gamma^{(0)T}\psi^*\right) = 0. \tag{K.9}$$

Comparing this with the equation for ψ_c, up to a phase, we find

$$\psi_c = e^{i\phi}C\left(\gamma^{(0)T}\psi^*\right) = e^{i\phi}C\overline{\psi}^T, \tag{K.10}$$

where $\overline{\psi} = \psi^\dagger\gamma^{(0)}$. From now on, however, we ignore the phase factor $e^{i\phi}$. Finally, it should be noted that applying the C matrix to a particle field, we obtain the antiparticle field.

Let us consider the charge conjugation of ν_L and ν_R:

$$(\nu_L)_c \equiv C\overline{\nu}_L^{\,T}, \qquad (\nu_R)_c \equiv C\overline{\nu}_R^{\,T}. \tag{K.11}$$

Using Eqs. (K.8) and (K.11), we derive

$$\overline{(\nu_L)_c} = -\,\nu_L^T C^{-1}, \qquad \overline{(\nu_R)_c} = -\,\nu_R^T C^{-1}. \tag{K.12}$$

Moreover, we can show that $(\nu_L)_c$ is a right-handed field, and $(\nu_R)_c$ a left-handed one. In fact, using the projection operator $P_R = \frac{1}{2}\left(1+\gamma_{(5)}\right)$, we have

$$\begin{aligned} P_R\,(\nu_L)_c &= \frac{1}{2}\left(1+\gamma_{(5)}\right)(\nu_L)_c \\ &= C\left[\overline{\nu}_L\,\frac{1}{2}\left(1+\gamma_{(5)}\right)\right]^T \\ &= C\overline{\nu}_L^{\,T} \\ &= (\nu_L)_c\,, \end{aligned} \tag{K.13}$$

where we have used the relations

$$C^{-1}\gamma_{(5)}C = \gamma_{(5)}^T, \tag{K.14}$$

$$\overline{\nu}_L\,\frac{1}{2}\left(1+\gamma_{(5)}\right) = \overline{\nu}_L. \tag{K.15}$$

Similarly, we can obtain

$$P_L\,(\nu_R)_c = (\nu_R)_c\,, \tag{K.16}$$

where $P_L = \frac{1}{2}\left(1-\gamma_{(5)}\right)$.

Let us proceed to the construction of the possible neutrino mass terms in terms of ν_L, $(\nu_L)_c$, ν_R, and $(\nu_R)_c$. First, using only the fields ν_L and ν_R, we can construct the mass term in the form

$$\mathcal{L}^D = -\,\overline{\nu}_R M^D \nu_L + \text{h.c.}, \tag{K.17}$$

where (h.c.) denotes the hermitian conjugate of foregoing terms. This mass term is invariant under the global gauge transformations $\nu_L \to e^{i\Lambda}\nu_L$, $\nu_R \to e^{i\Lambda}\nu_R$. This invariance will lead to the conservation of the lepton charge $L = \sum_{l=e,\mu,\tau} L_l$. The mass term $\mathcal{L}^D$ is called a Dirac mass term. Second, if we use the fields ν_L and $(\nu_L)_c$, then we have

$$\mathcal{L}^M = -\,\frac{1}{2}\,\overline{(\nu_L)_c} M^M \nu_L + \text{h.c.} \tag{K.18}$$

In this case, the mass term is not invariant under the global gauge transformations. The mass term $\mathcal{L}^M$ is called a Majorana mass term. Finally, the most general neutrino mass term is given by

$$\mathcal{L}^{D+M} = -\,\frac{1}{2}\,\overline{(\nu_L)_c} M_L^M \nu_L - \frac{1}{2}\,\overline{\nu}_R M_R^M\,(\nu_R)_c - \overline{\nu}_R M_1^D \nu_L + \text{h.c.}, \tag{K.19}$$

where the possible term $\overline{(\nu_L)_c} M_2^D (\nu_R)_c$ can be reduced to the third term, because we have

$$\begin{aligned}\overline{(\nu_L)_c} M_2^D (\nu_R)_c &= -\nu_L{}^T C^{-1} M_2^D C \overline{\nu}_R{}^T \\ &= \overline{\nu}_R \left(M_2^D\right)^T \nu_L. \end{aligned} \tag{K.20}$$

(The fields ν are anticommuting fields.) As in the case of the Lagrangian $\mathcal{L}^M$, no global gauge transformations under which the Lagrangian $\mathcal{L}^{D+M}$ would be invariant exist. The Lagrangian $\mathcal{L}^{D+M}$ is called a Dirac-Majorana mass term. Furthermore, the matrices M^D, M^M, M_L^M, M_R^M, and M_1^D introduced above are $n \times n$ complex matrices.

Let us discuss the neutrino mixing arising from these three types of neutrino mass terms in order.

K.1 Dirac mass term

We now consider the Dirac mass term

$$\begin{aligned}\mathcal{L}^D &= -\overline{\nu}_R M^D \nu_L + \text{h.c.} \\ &= -\sum_{l,l'=e,\mu,\tau,\cdots} \overline{\nu}_{l'R} M^D_{l'l} \nu_{lL} + \text{h.c.}\end{aligned} \tag{K.21}$$

Let us diagonalize the matrix M^D to make the mass term $\mathcal{L}^D$ the standard form. For this purpose, we shall show that an arbitrary complex matrix M can always be diagonalized by a biunitary transformation. To show this, we consider the matrix $MM^\dagger$, which is evidently hermitian. Considering the eigenvalue equation

$$M^\dagger \overrightarrow{x}_i = \mu_i \overrightarrow{x}_i, \tag{K.22}$$

we have

$$\begin{aligned}\overrightarrow{x}_i^\dagger M M^\dagger \overrightarrow{x}_i &= \left(M^\dagger \overrightarrow{x}_i\right)^\dagger M^\dagger \overrightarrow{x}_i \\ &= |\mu_i|^2 \, \overrightarrow{x}_i^\dagger \overrightarrow{x}_i,\end{aligned} \tag{K.23}$$

Hence the matrix $MM^\dagger$ has positive eigenvalues. (For simplicity, we assume $|\mu_i|^2 \equiv m_i^2 > 0$.) From this, it follows that the matrix $MM^\dagger$ can be written as

$$MM^\dagger = Vm^2V^\dagger, \tag{K.24}$$

where $V^\dagger V = VV^\dagger = \mathbf{1}$ and $(m^2)_{ij} = m_i^2 \delta_{ij}$. Furthermore, if we set $m_{ij} = +\left(m_i^2\right)^{1/2} \delta_{ij}$, then we obtain

$$M = VmU^\dagger, \tag{K.25}$$

where $U^\dagger = m^{-1} V^\dagger M$. Using Eq. (K.25), we find that U is a unitary matrix:

$$U^\dagger U = m^{-1} V^\dagger M M^\dagger V m^{-1} = \mathbf{1}. \tag{K.26}$$

Thus the matrix M^D is also written as

$$M^D = VmU^\dagger. \tag{K.27}$$

Inserting Eq. (K.27) into the Lagrangian $\mathcal{L}^D$, we derive

$$\begin{aligned} \mathcal{L}^D &= -\overline{\nu'}_R m \nu'_L + \text{h.c.} \\ &= -\overline{\nu'} m \nu' \\ &= -\sum_{k=1}^{n} m_k \overline{\nu}_k \nu_k, \end{aligned} \tag{K.28}$$

where

$$\nu'_L = U^\dagger \nu_L, \quad \nu'_R = V^\dagger \nu_R, \quad \nu' = \nu'_L + \nu'_R = \begin{pmatrix} \nu_1 \\ \nu_2 \\ \vdots \\ \nu_n \end{pmatrix}. \tag{K.29}$$

From this, we see that ν_k is a field with the definite mass m_k. Indeed, since for freely propagating neutrinos the total Lagrangian $\mathcal{L}$ is given by

$$\begin{aligned} \mathcal{L} &= \overline{\nu}_L \, i\gamma^{(a)} \frac{\partial}{\partial x^a} \nu_L + \overline{\nu}_R \, i\gamma^{(a)} \frac{\partial}{\partial x^a} \nu_R - \overline{\nu}_R M^D \nu_L - \overline{\nu}_L \left(M^D\right)^\dagger \nu_R \\ &= \overline{\nu'} \left(i\gamma^{(a)} \frac{\partial}{\partial x^a} - m \right) \nu' \\ &= \sum_{k=1}^{n} \overline{\nu}_k \left(i\gamma^{(a)} \frac{\partial}{\partial x^a} - m_k \right) \nu_k, \end{aligned} \tag{K.30}$$

we can obtain the Dirac equations

$$\left(i\gamma^{(a)} \frac{\partial}{\partial x^a} - m_k \right) \nu_k = 0. \tag{K.31}$$

It follows from the unitarity of the matrix U that the left-handed field ν_L is written as

$$\nu_L = U\nu'_L. \tag{K.32}$$

Therefore, we obtain

$$\nu_{lL} = \sum_{k=1}^{n} U_{lk}\nu_{kL}, \quad l = e, \mu, \tau, \cdots. \tag{K.33}$$

Thus, in the case of the Dirac mass term $\mathcal{L}^D$, the left-handed fields of flavor neutrinos are linear superpositions of the left-handed fields of neutrinos with definite masses. The unitary matrix U is called a mixing matrix.

K.2 Majorana mass term

Next, we shall consider the Majorana mass term

$$\begin{aligned}
\mathcal{L}^M &= -\frac{1}{2}\overline{(\nu_L)_c} M^M \nu_L + \text{h.c.} \\
&= -\frac{1}{2}\sum_{l,l'=e,\mu,\tau,\cdots} \overline{(\nu_{l'L})_c} M^M_{l'l}\nu_{lL} + \text{h.c.}
\end{aligned} \tag{K.34}$$

In order to reduce the Lagrangian $\mathcal{L}^M$ to the standard form, we diagonalize the matrix M^M. Here, it should be noted that the matrix M^M is a symmetric matrix. In fact, we have

$$\begin{aligned}
\overline{(\nu_L)_c} M^M \nu_L &= -\,{\nu_L}^T C^{-1} M^M \nu_L \\
&= -\left({\nu_L}^T C^{-1} M^M \nu_L\right)^T \\
&= {\nu_L}^T C \left(M^M\right)^T \nu_L \\
&= -\,{\nu_L}^T C^{-1} \left(M^M\right)^T \nu_L \\
&= \overline{(\nu_L)_c} \left(M^M\right)^T \nu_L.
\end{aligned} \tag{K.35}$$

Hence, it follows that

$$\left(M^M\right)^T = M^M. \tag{K.36}$$

We now use the fact that a complex symmetric matrix M can always be written as

$$M = \left(U^\dagger\right)^T m U^\dagger, \tag{K.37}$$

where U is a unitary matrix and $m_{ij} = m_i \delta_{ij}\,(m_i \geq 0)$. To see this, we recall that an arbitrary matrix M is expressed in the form

$$M = VmU^{\dagger}, \tag{K.38}$$

where $VV^{\dagger} = \mathbf{1}$, $UU^{\dagger} = \mathbf{1}$, and $m_{ij} = m_i \delta_{ij}\,(m_i \geq 0)$. For simplicity, we assume that $m_i \neq m_j$ for $i \neq j$, and that $m_i > 0$. From Eq. (K.38), we have

$$MM^{\dagger} = Vm^2V^{\dagger}. \tag{K.39}$$

On the other hand, using the relation

$$M = M^T = \left(U^{\dagger}\right)^T mV^T, \tag{K.40}$$

we have

$$MM^{\dagger} = \left(U^{\dagger}\right)^T m^2U^T. \tag{K.41}$$

Therefore, from Eqs. (K.39) and (K.41), we derive

$$U^TVm^2 = m^2U^TV. \tag{K.42}$$

Since m^2 is a diagonal matrix and $m_i \neq m_j$ for $i \neq j$, U^TV is also a diagonal matrix. Furthermore, U^TV is a unitary matrix. If we set $S = U^TV$, then S can be written as

$$S_{ij} = e^{2i\alpha_i}\delta_{ij}, \tag{K.43}$$

where α_i are real constants. Using the expression $V = \left(U^{\dagger}\right)^T S$, we obtain

$$\begin{aligned} M &= \left(U^{\dagger}\right)^T SmU^{\dagger} \\ &= \left(U^{\dagger}\right)^T S^{1/2}mS^{1/2}U^{\dagger} \\ &= \left(S^{1/2}U^{\dagger}\right)^T mS^{1/2}U^{\dagger}, \end{aligned} \tag{K.44}$$

where $\left(S^{1/2}\right)_{ij} = e^{i\alpha_i}\delta_{ij}$ Therefore, if we redefine $U^{\dagger}$ as $U'^{\dagger} = S^{1/2}U^{\dagger}$, then we can obtain Eq. (K.37).

Inserting the expression

$$M^M = \left(U^{\dagger}\right)^T mU^{\dagger} \tag{K.45}$$

into the Lagrangian $\mathcal{L}^M$, we obtain

$$\mathcal{L}^M = -\frac{1}{2}\,\overline{(n_L)_c} m n_L - \frac{1}{2}\,\overline{n}_L m \left(n_L\right)_c , \tag{K.46}$$

where

$$n_L = U^\dagger \nu_L, \quad \left(n_L\right)_c = C \overline{n}_L^{\,T}. \tag{K.47}$$

Furthermore, if we define

$$\chi = n_L + \left(n_L\right)_c = \begin{pmatrix} \chi_1 \\ \chi_2 \\ \vdots \\ \chi_n \end{pmatrix}, \tag{K.48}$$

then the neutrino mass term $\mathcal{L}^M$ is given by

$$\mathcal{L}^M = -\frac{1}{2}\,\overline{\chi} m \chi = -\frac{1}{2}\sum_{k=1}^{n} m_k \overline{\chi}_k \chi_k. \tag{K.49}$$

From this, we conclude that χ_k is the field of a neutrino with definite mass m_k. Indeed, the total Lagrangian for freely propagating neutrinos is

$$\begin{aligned} \mathcal{L} &= \overline{\nu}_L\, i\gamma^{(a)} \frac{\partial}{\partial x^a} \nu_L - \frac{1}{2}\,\overline{(\nu_L)_c} M^M \nu_L - \frac{1}{2}\,\overline{\nu}_L \left(M^M\right)^\dagger \left(\nu_L\right)_c \\ &= \overline{n}_L\, i\gamma^{(a)} \frac{\partial}{\partial x^a} n_L - \frac{1}{2}\,\overline{(n_L)_c} m n_L - \frac{1}{2}\,\overline{n}_L m \left(n_L\right)_c \\ &= \frac{1}{2}\,\overline{n}_L\, i\gamma^{(a)} \frac{\partial}{\partial x^a} n_L + \frac{1}{2}\,\overline{(n_L)_c}\, i\gamma^{(a)} \frac{\partial}{\partial x^a} \left(n_L\right)_c - \frac{1}{2}\,\overline{(n_L)_c} m n_L - \frac{1}{2}\,\overline{n}_L m \left(n_L\right)_c \\ &= \frac{1}{2}\,\overline{\chi} \left(i\gamma^{(a)} \frac{\partial}{\partial x^a} - m \right) \chi \\ &= \frac{1}{2}\sum_{k=1}^{n} \overline{\chi}_k \left(i\gamma^{(a)} \frac{\partial}{\partial x^a} - m_k \right) \chi_k. \end{aligned} \tag{K.50}$$

Furthermore, it should be noted that the fields χ_k satisfy the relations

$$\chi_k = C \overline{\chi}_k^{\,T} = \left(\chi_k\right)_c , \quad k = 1, 2, \cdots, n. \tag{K.51}$$

This implies that χ_k are the fields of Majorana neutrinos.

Considering Eqs. (K.47) and (K.48), we obtain

$$\nu_L = U \chi_L, \tag{K.52}$$

that is,

$$\nu_{lL} = \sum_{k=1}^{n} U_{lk}\chi_{kL}. \tag{K.53}$$

Thus, in the case of the Majorana mass term, the left-handed fields of flavor neutrinos are linear superpositions of the left-handed fields of Majorana neutrinos with definite masses. Moreover, it should be emphasized that the $2n$ states with different helicity of the n massive Majorana neutrinos correspond to the $2n$ neutrinos and antineutrinos $(\nu_e, \nu_\mu, \nu_\tau, \cdots, \overline{\nu}_e, \overline{\nu}_\mu, \overline{\nu}_\tau, \cdots)$.

K.3 Dirac-Majorana mass term

Finally, we consider the Dirac-Majorana mass term given by

$$\mathcal{L}^{D+M} = -\frac{1}{2}\overline{(n_L)_c} M^{D+M} n_L + \text{h.c.}, \tag{K.54}$$

where M^{D+M} is a complex $2n \times 2n$ matrix expressed as

$$M^{D+M} = \begin{pmatrix} M_L^M & \left(M_1^D\right)^T \\ M_1^D & M_R^M \end{pmatrix}, \tag{K.55}$$

and

$$n_L = \begin{pmatrix} \nu_L \\ (\nu_R)_c \end{pmatrix}. \tag{K.56}$$

From Eq. (K.55), it follows that the matrix M^{D+M} is symmetric:

$$\left(M^{D+M}\right)^T = M^{D+M}. \tag{K.57}$$

As in the last subsection, we assume that the eigenvalues of M^{D+M} are not degenerate. Then we have

$$M^{D+M} = \left(U^\dagger\right)^T m U^\dagger, \tag{K.58}$$

where U is a $2n \times 2n$ unitary matrix, and $m_{ij} = m_i\delta_{ij}\,(m_i \geq 0)$. Using Eq. (K.58), we obtain

$$\mathcal{L}^{D+M} = -\frac{1}{2}\overline{(n'_L)_c} m n'_L + \text{h.c.}, \tag{K.59}$$

where

$$n'_L = U^\dagger n_L. \tag{K.60}$$

Furthermore, if we use

$$\chi = n'_L + (n'_L)_c = \begin{pmatrix} \chi_1 \\ \chi_2 \\ \vdots \\ \chi_{2n} \end{pmatrix}, \tag{K.61}$$

then we have

$$\begin{aligned} \mathcal{L}^{D+M} &= -\frac{1}{2}\,\overline{\chi} m \chi \\ &= -\frac{1}{2}\sum_{k=1}^{2n} m_k \overline{\chi}_k \chi_k. \end{aligned} \tag{K.62}$$

The fields χ_k satisfy the relations

$$\chi_k = C\overline{\chi}_k^{\,T} = (\chi_k)_c \,. \tag{K.63}$$

Therefore, χ_k are the fields of Majorana neutrinos with definite masses.

From Eqs. (K.60) and (K.61), we find

$$n_L = U n'_L = U \chi_L. \tag{K.64}$$

It follows that

$$\nu_{lL} = \sum_{k=1}^{2n} U_{lk}\chi_{kL}, \quad (\nu_{l'R})_c = \sum_{k=1}^{2n} U_{l'k}\chi_{kL}, \tag{K.65}$$

where the index l runs over n values: $e, \mu, \tau, \cdots$, whereas the index l' takes the n lower values: $e, \mu, \tau, \cdots$. It should be emphasized that the left-handed fields of flavor neutrinos are linear superpositions of the left-handed components of $2n$ Majorana fields.

Appendix L

Dirac equation in flat space-time

We now give the derivation of Eq. (5.10). For the purpose of this, we use the Weyl representation (G.18) for the Dirac matrices. In flat space-time, the Dirac equation is given by

$$i\hbar\frac{\partial}{\partial t}\Psi = \left(\rho_3 c\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}} + \rho_1 mc^2\right)\Psi, \tag{L.1}$$

where ρ_i are defined by Eq. (G.9).

Next, we shall show that there exists a unitary operator U which satisfies the relation

$$U\left(\rho_3 c\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}} + \rho_1 mc^2\right)U^{\dagger} = \rho_3\sqrt{\overline{p}^2c^2 + m^2c^4}\,\frac{\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}}{\overline{p}}. \tag{L.2}$$

In order to derive the unitary operator $U = e^{iS}$ satisfying Eq. (L.2), we consider

$$U^{\dagger}\rho_3\sqrt{\overline{p}^2c^2 + m^2c^4}\,\frac{\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}}{\overline{p}}U. \tag{L.3}$$

Using the useful formula

$$e^{-\frac{i}{2}a\rho_2}\rho_3 e^{\frac{i}{2}a\rho_2} = \rho_3\cos a + \rho_1\sin a, \tag{L.4}$$

we have

$$e^{-\frac{i}{2}a\rho_2}\,\rho_3\sqrt{\overline{p}^2c^2 + m^2c^4}\,\frac{\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}}{\overline{p}}\,e^{\frac{i}{2}a\rho_2} = \left(\rho_3\cos a + \rho_1\sin a\right)\sqrt{\overline{p}^2c^2 + m^2c^4}\,\frac{\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}}{\overline{p}}, \tag{L.5}$$

where we have assumed that a can be represented as a power series of $\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}$ in the following way:

$$a = \sum_n a_n\left(\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}\right)^n. \tag{L.6}$$

The right-hand side of Eq. (L.5) must be equal to

$$\rho_3 c\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}} + \rho_1 mc^2, \tag{L.7}$$

and, therefore, we have

$$\cos a = \frac{\overline{p}c}{\sqrt{\overline{p}^2c^2 + m^2c^4}}, \tag{L.8}$$

$$\sin a = \frac{mc^2}{\sqrt{\overline{p}^2c^2 + m^2c^4}}\frac{\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}}{\overline{p}}, \tag{L.9}$$

$$\tan a = \frac{mc^2}{\overline{p}c}\frac{\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}}{\overline{p}}. \tag{L.10}$$

From this, we derive

$$a = \tan^{-1}\left(\frac{mc^2}{\overline{p}c}\frac{\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}}{\overline{p}}\right). \tag{L.11}$$

Therefore, using the unitary operator

$$U = \exp\left[i\rho_2\frac{1}{2}\tan^{-1}\left(\frac{mc^2}{\overline{p}c}\frac{\boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}}}{\overline{p}}\right)\right], \tag{L.12}$$

we obtain Eq. (L.2). Hence, there exists a unitary operator U satisfying Eq. (L.2).

If we set

$$\Psi' = U\Psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}, \tag{L.13}$$

then we can obtain Eq. (5.10) for the left-handed component.

Bibliography

[1] S. Weinberg, Phys. Rev. Lett. **19** (1967), 1264.

[2] A. Salam, *Elementary Particle Theory* (Almquist and Forlag, Stockholm, 1968)

[3] H. Georgi and S.L. Glashow, Phys. Rev. Lett. **32** (1974), 438.

[4] R. Colella, A.W. Overhauser and S.A. Werner, Phys. Rev. Lett. **34** (1975), 1472.

[5] D.M. Greenberger, Rev. Mod. Phys. **55** (1983), 875.

[6] Y. Aharonov and D. Bohm, Phys. Rev. **115** (1959), 485.

[7] A.W. Overhauser and R. Colella, Phys. Rev. Lett. **33** (1974), 1237.

[8] L.A. Page, Phys. Rev. Lett. **35** (1975), 543.

[9] S.A. Werner, J.-L. Staudenmann and R. Colella, Phys. Rev. Lett. **42** (1979), 1103.

[10] J. Anandan, Phys. Rev. D **15** (1977), 1448.

[11] M. Dresden and C.N. Yang, Phys. Rev. D **20** (1979), 1846.

[12] J.J. Sakurai, Phys. Rev. D **21** (1980), 2993.

[13] J. Kuroiwa, M. Kasai and T. Futamase, Phys. Lett. A **182** (1993), 330.

[14] S. Wajima, M. Kasai and T. Futamase, Phys. Rev. D **55** (1997), 1964.

[15] D.V. Ahluwalia and C. Burgard, Gen. Rel. and Grav. **28** (1996), 1161.

[16] D.V. Ahluwalia and C. Burgard, preprint: gr-qc/9606031.

[17] T. Bhattacharya, S. Habib and E. Mottola, preprint: gr-qc/9605074.

[18] Y. Kojima, Mod. Phys. Lett. A **11** (1996), 2965.

[19] Y. Grossman and H.J. Lipkin, Phys. Rev. D **55** (1997), 2760.

[20] C.Y. Cardall and G.M. Fuller, Phys. Rev. D **55** (1997), 7960.

[21] N. Fornengo, C. Giunti, C.W. Kim and J. Song, Phys. Rev. D **56** (1997), 1895.

[22] L. Stodolsky, Gen. Rel. and Grav. **11** (1979), 391.

[23] Y. Totsuka (SuperKamiokande Collaboration), in the Proceedings of the 28th International Symposium on Lepton Photon Interactions, Hamburg, Germany, 1997.

[24] S. Weinberg, *Gravitation and Cosmology* (John Wiley & Sons, USA, 1972)

[25] N.D. Birrel and P.C.W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, 1984)

[26] D.R. Brill and J.A. Wheeler, Rev. Mod. Phys. **29** (1957), 465.

[27] L.L. Foldy and S.A. Wouthuysen, Phys. Rev. **78** (1950), 29.

[28] S. Tani, Prog. Theor. Phys. **6** (1951), 267.

[29] F. Riehle *et al.*, Phys. Rev. Lett. **67** (1991), 177.

[30] B. Kayser, Phys. Rev. D **24** (1981), 110.

[31] S.M. Bilenky and S.T. Petcov, Rev. Mod. Phys. **59** (1987), 671.

[32] J.N. Bahcall, *Neutrino Astrophysics* (Cambridge University Press, USA, 1989)

[33] M. Kobayashi and T. Maskawa, Prog. Theor. Phys. **49** (1973), 652.

[34] R.M. Wald, *General Relativity* (The University of Chicago Press, Chicago, 1984)