# Quantum Kinetic Theory of Spin Polarization of Massive Quark in Perturbative QCD

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# Why density matrix for spin $1/2$?

**Two spin states are almost degenerate**
$\Delta E \sim \boldsymbol{S} \cdot \omega + \boldsymbol{S} \cdot \boldsymbol{B} \sim \hbar$.
**Quantum correlation time is classical**

$$\tau_q \sim \frac{\hbar}{\Delta E} \sim \mathcal{O}(\hbar^0)$$

**We need to keep $2 \times 2$ spin density matrix in the kinetic theory of time evolution $\Delta t \sim \mathcal{O}(\hbar^0)$.**
**quantum kinetic theory**

## We expect a Lindblad-type of kinetic equation

$$
\begin{aligned}
\frac{d\hat{\rho}}{dt} &= -\frac{i}{\hbar}[H_0, \hat{\rho}] - L\hat{\rho}L^\dagger + \frac{1}{2}L^\dagger L\hat{\rho} + \frac{1}{2}\hat{\rho}L^\dagger L \\
&= -\frac{i}{\hbar}[H_0, \hat{\rho}] - \Gamma \cdot \hat{\rho}
\end{aligned}
$$

**The first term contains free streaming advective flow and background EM field, and has been worked out in Refs:Gao-Liang, Weickgenannt-Sheng-Speranza-Wang-Rischke, Hattori-Hidaka-Yang**

**The $\Gamma \cdot \hat{\rho}$ is the collision term, that we aim to construct in perturbative QCD framework (to leading log in QCD coupling constant $g$)**

**Should eventually be compared with the works of Hidaka-Pu-Yang, and Hattori-Hidaka-Yang**

**We consider the case of dilute (Boltzmann), massive quarks (strange, bottom), interacting with the background thermal QGP**

**The problem reduces to the 1-particle quantum mechanics of a single quark moving in QGP**

**Further simplification: Neglect EM field and vorticity in the collision term**

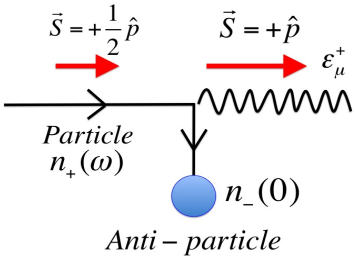
$$\Gamma = \Gamma_0 + \mathcal{O}(\mathcal{B}, \omega)$$

**The equilibrium density matrix is expected to be Boltzmann $\hat{\rho}_{\rm eq} = z e^{-\beta E_p} \mathbf{1}_{2\times 2}$, where $E_p = \sqrt{p^2 + m^2}$.**

$$\Gamma_0 \cdot \hat{\rho}_{\rm eq} = 0$$

**$\Gamma_0 \sim g^4 \log(1/g) T$ gives the relaxation of spin polarization to equilibrium**

**We consider an arbitrary hard scale quark mass $m \gg gT$. This justifies neglecting quark-gluon conversion process in our leading log computation, since the t-channel fermion exchange momentum becomes hard $q \gtrsim m \gg gT$, and leading log is absent**




**For light quarks, this conversion process means that we need to consider both quark and gluon spins together in leading log**

**The difficult part is the multi-particle fermion/boson statistics with spin density matrix**

# Relativistic massive fermion

## Field quantization

$$\psi(\boldsymbol{x}) = \int_{\boldsymbol{p}} \frac{1}{\sqrt{2E_p}} \sum_s u(p,s) e^{i\boldsymbol{p}\cdot\boldsymbol{x}} a_{\boldsymbol{p},s} + \mathrm{anti\,quark}$$

**1-quark states:** $|\boldsymbol{p}, s\rangle = a^\dagger_{\boldsymbol{p},s}|0\rangle$

**It is most convenient to use the helicity basis,**

$$u(\boldsymbol{p}, s) \sim \begin{pmatrix} \sqrt{E_p - sp}\,\xi_s(\boldsymbol{p}) \\ \sqrt{E_p + sp}\,\xi_s(\boldsymbol{p}) \end{pmatrix}, \quad (\boldsymbol{\sigma} \cdot \boldsymbol{p})\xi_s(\boldsymbol{p}) = s|\boldsymbol{p}|\xi_s(\boldsymbol{p})$$

**Spin polarization operator in canonical decomposition**

$$S^i = \frac{1}{V} \int d\boldsymbol{x} \epsilon^{ijk} \bar{\psi}(\boldsymbol{x}) \gamma^0 [\gamma^j, \gamma^k] \psi(\boldsymbol{x}) \sim \int_{\boldsymbol{p}} (\xi^\dagger_{\boldsymbol{p},s'} \sigma^i \xi_{\boldsymbol{p},s}) a^\dagger_{\boldsymbol{p},s'} a_{\boldsymbol{p},s}$$

**For 1-quark states, this is the same as in usual QM of spin $1/2$ particle, where $|\boldsymbol{p}, s\rangle \sim \xi_s(\boldsymbol{p})$**

**The interaction vertices with the background gluon fields $A^a_\mu$, through the relativistic spinors $u(\boldsymbol{p}, s)$**

$$
\begin{aligned}
H_I(t) &= g \int d\boldsymbol{x}\, \bar{\psi}(\boldsymbol{x})\gamma^\mu t^a \psi(\boldsymbol{x}) A^a_\mu(\boldsymbol{x}, t) \\
&\sim \int_{\boldsymbol{p},\boldsymbol{p}',\boldsymbol{q}} \bar{u}(\boldsymbol{p}', s')\gamma^\mu u(\boldsymbol{p}, s) A_\mu(\boldsymbol{q}) a^\dagger_{\boldsymbol{p}',s'} a_{\boldsymbol{p},s} \delta_{\boldsymbol{p}'-\boldsymbol{p}-\boldsymbol{q}}
\end{aligned}
$$

**The amplitudes $\bar{u}(\boldsymbol{p}', s')\gamma^\mu u(\boldsymbol{p}, s) A_\mu(\boldsymbol{q})$ give the matrix elements of $H_I$ in the QM of 1-quark Hilbert space of spin 1/2.**

# Position and momentum operators

**Define the position and the momentum operators, that become identical to the usual QM in 1-quark Hilbert space**

$$\hat{\boldsymbol{P}} \equiv \int_{\boldsymbol{p}} \boldsymbol{p} a^\dagger_{\boldsymbol{p},s} a_{\boldsymbol{p},s}, \quad \hat{\boldsymbol{X}} \equiv \int_{\boldsymbol{x}} \boldsymbol{x} a^\dagger_{\boldsymbol{x},s} a_{\boldsymbol{x},s}$$

$$a_{\boldsymbol{x},s} \equiv \int_{\boldsymbol{p}} e^{i\boldsymbol{p}\cdot\boldsymbol{x}} a_{\boldsymbol{p},s} \quad (\mathrm{Note}: a(\boldsymbol{x}) \neq \psi(\boldsymbol{x}))$$

**We have**

$$[\hat{\boldsymbol{X}}, \hat{\boldsymbol{P}}] = i\hbar \hat{\boldsymbol{N}} = i\hbar$$

**where $\hat{N}$ is the quark number operator.**

# The density matrix in momentum basis

$$\hat{\rho} = \int_{\boldsymbol{p},\boldsymbol{p}'} \sum_{s,s'} \rho_{s,s'}(\boldsymbol{p},\boldsymbol{p}')|\boldsymbol{p},s\rangle\langle\boldsymbol{p}',s'|, \quad |\boldsymbol{p},s\rangle \equiv a^\dagger_{\boldsymbol{p},s}|0\rangle$$

**We map it to a state in $\mathcal{H}\otimes\mathcal{H}^*$ ("Thermo-field theory")**

$$\hat{\rho} = \int_{\boldsymbol{p},\boldsymbol{p}'} \sum_{s,s'} \rho_{s,s'}(\boldsymbol{p},\boldsymbol{p}')|\boldsymbol{p},s\rangle \otimes |\boldsymbol{p}',s'\rangle^*$$

**where $\mathcal{H}^*$ is the conjugate space to $\mathcal{H}$, that is, the time-reversed (T) space.**
**Under time-reversal (T), we have $\hat{\boldsymbol{X}}^* \to \hat{\boldsymbol{X}}$ and $\hat{\boldsymbol{P}}^* \to -\hat{\boldsymbol{P}}$, and**

$$[\hat{\boldsymbol{X}}^*, \hat{\boldsymbol{P}}^*] = -i\hbar$$

**The $\mathcal{H}$ and $\mathcal{H}^*$ are naturally described by the Schwinger-Keldysh contours: the forward-time and the backward-time contours (labeled by 1 and 2 respectively)**

$$[x_1, p_1] = i\hbar, \quad [x_2, p_2] = -i\hbar$$

**Define $x_r = \frac{1}{2}(x_1 + x_2)$ and $x_a = x_1 - x_2$, and similarly $p_{r/a}$, then we have**

$$[x_r, p_a] = i\hbar, \quad [x_a, p_r] = i\hbar, \quad [x_r, p_r] = 0$$

**This means we can have the basis of simultaneous eigenstates of $(x_r, p_r)$ in $\mathcal{H} \otimes \mathcal{H}^*$, that is closest to the classical phase space**

$$\hat{\rho} = \int_{\boldsymbol{x}_r} \int_{\boldsymbol{p}_r} \hat{\rho}_{2\times 2}(\boldsymbol{x}_r, \boldsymbol{p}_r) |\boldsymbol{x}_r, \boldsymbol{p}_r\rangle$$

Since $[x_r, p_a] = i\hbar$, we have

$$\hat{\rho}_{2\times 2}(\boldsymbol{x}_r, \boldsymbol{p}_r) = \int_{\boldsymbol{p}_a} e^{i\boldsymbol{x}_r\cdot\boldsymbol{p}_a}\hat{\rho}_{2\times 2}(\boldsymbol{p}_1, \boldsymbol{p}_2)$$

in terms of the previous density matrix in momentum space.

We restrict to the spatially homogeneous case, that is, $x_r$-independent case

This means that the density matrix in momentum space is diagonal, $\boldsymbol{p}_a = 0$,

$$\begin{aligned}\hat{\rho} &= \int_{\boldsymbol{p}} \rho_{s,s'}(\boldsymbol{p})|\boldsymbol{p}, s\rangle\langle\boldsymbol{p}, s'| \\ &= \int_{\boldsymbol{p}} \rho_{s,s'}(\boldsymbol{p}) a^\dagger_{\boldsymbol{p},s}|0\rangle\langle 0|a_{\boldsymbol{p},s'}\end{aligned}$$

**From the spin operator $S^i \sim \frac{1}{2}\int_{\boldsymbol{p}}(\xi^\dagger_{\boldsymbol{p},s'}\sigma^i\xi_{\boldsymbol{p},s})a^\dagger_{\boldsymbol{p},s'}a_{\boldsymbol{p},s}$, the spin polarization is**

$$\langle S^i\rangle = \mathrm{Tr}(S^i\hat{\rho}) \sim \frac{1}{2}\int_{\boldsymbol{p}}(\xi^\dagger_{\boldsymbol{p},s'}\sigma^i\xi_{\boldsymbol{p},s})\rho_{s,s'}(\boldsymbol{p}) = \frac{1}{2}\mathrm{Tr}(\sigma^i\hat{\rho}_{2\times 2}(\boldsymbol{p}))$$

**with the explicit $2\times 2$ spin density matrix in momentum space**

$$\hat{\rho}_{2\times 2}(\boldsymbol{p}) \equiv \sum_{s,s'}\xi_{\boldsymbol{p},s}\rho_{s,s'}(\boldsymbol{p})\xi^\dagger_{\boldsymbol{p},s'}$$

**This object is unambiguous under a phase redefinition of $\xi_{\boldsymbol{p},s}$, since $\langle S^i\rangle$ is physical.**
**We are going to derive the evolution equation for this physical object**
**N.B. : The spin-traced object $f(\boldsymbol{p}) = \mathrm{Tr}(\hat{\rho}(\boldsymbol{p}))$ is the usual number distribution**

# Relation to the field theory Wigner function
## (for example, Q. Wang's talk)

**It is easy to show that $\hat{\rho}(\boldsymbol{x}_r, \boldsymbol{p}_r)$ is the Wigner transform of $a(\boldsymbol{x})$**

$$\hat{\rho}_{2\times 2}(\boldsymbol{x}_r, \boldsymbol{p}_r) \;=\; \int_{\boldsymbol{x}_a} e^{-i\boldsymbol{x}_a\cdot\boldsymbol{p}_r} \langle a(\boldsymbol{x}_r - \boldsymbol{x}_a/2) a^\dagger(\boldsymbol{x}_r + \boldsymbol{x}_a/2)\rangle_{\hat{\rho}}$$

**Recall $\psi(\boldsymbol{x}) = \int_{\boldsymbol{p}} \frac{1}{\sqrt{2E_p}} e^{i\boldsymbol{p}\cdot\boldsymbol{x}} u(\boldsymbol{p}) a_{\boldsymbol{p}}$ and $a(\boldsymbol{x}) = \int_{\boldsymbol{p}} e^{i\boldsymbol{p}\cdot\boldsymbol{x}} a_{\boldsymbol{p}}$, so $\psi(\boldsymbol{x})$ and $a(\boldsymbol{x})$ are non-locally related, and $\hat{\rho}(\boldsymbol{x}_r, \boldsymbol{p}_r)$ is not equal to the Wigner transform of $\psi(\boldsymbol{x})$ field.**

**However, for the spatially homogeneous case, they are related by**

$$\int d\boldsymbol{x}_a \langle \psi_\alpha(x - x_a/2)\psi^\dagger_\beta(x + x_a/2)\rangle e^{i\boldsymbol{p}\cdot\boldsymbol{x}_a}$$

$$= \sum_{s,s'} \frac{1}{2E_p} u_\alpha(\boldsymbol{p}, s) u^\dagger_\beta(\boldsymbol{p}, s')\rho_{s,s'}(\boldsymbol{p})$$

# Time evolution of density matrix

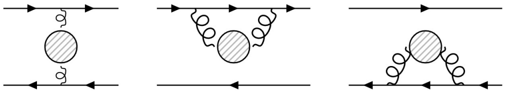
$$\hat{\rho}(t+\Delta t) = U_1(\Delta t)\hat{\rho}(t)U_2^\dagger(\Delta t)$$

**where $U_{1,2}$ are unitary evolution operators with QCD gluons in SK contours 1 or 2, with the Hamiltonians**

$$H_{1,2} = H_{\rm kinetic} + g\int d\boldsymbol{x}\bar{\psi}(\boldsymbol{x})\gamma^\mu\psi(\boldsymbol{x})A_\mu^{(1)/(2)}$$

**where $A_\mu^{(i)}$ are the gluon fields on the SK contour $i = 1, 2$. Note $U_1 \neq U_2$, and $H_{1,2}$ are time-dependent due to time-dependent gluon fields.**
**We average over quantum/thermal fluctuating SK gluon fields $A_\mu^{(i)}$, given by equilibrium two-point functions of $\langle A_\mu^{(i)}(\boldsymbol{p})A_\nu^{(j)}(\boldsymbol{p})\rangle = G_{\mu\nu}^{(ij)}(\boldsymbol{p})$, satisfying thermal KMS relations**

# Working out second order perturbation theory in the interaction picture

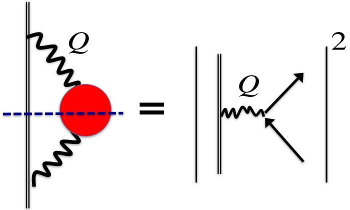


(a) (b) (c)

$$
\begin{aligned}
\hat{\rho}(\Delta t) &= U_0(\Delta t)\hat{\rho}(0)U_0^\dagger(\Delta t) \\
&+ \int_0^{\Delta t} dt_1 \int_0^{\Delta t} dt_2\, U_0(\Delta t)\langle H_I^{\mathrm{int}(1)}(t_1)\hat{\rho}(0)H_I^{\mathrm{int}(2)}(t_2)\rangle_A U_0^\dagger(\Delta t) \\
&+ (-i)^2 U_0(\Delta t)\int_0^{\Delta t} dt_1 \int_0^{t_1} dt_1' \langle H_I^{\mathrm{int}(1)}(t_1)H_I^{\mathrm{int}(1)}(t_1')\rangle_A \hat{\rho}(0)U_0^\dagger(\Delta t) \\
&+ (+i)^2 U_0(\Delta t)\hat{\rho}(0)\int_0^{\Delta t} dt_2 \int_0^{t_2} dt_2' \langle H_I^{\mathrm{int}(2)}(t_2')H_I^{\mathrm{int}(2)}(t_2)\rangle_A U_0^\dagger(\Delta t)
\end{aligned}
$$

**N.B. Compare this with the Lindblad form**

Gluon two-point functions $G^{(ij)}_{\mu\nu}$ include HTL self-energy. These contributions represent interactions with background hard thermal particles with t-channel gluon exchange. We keep quantum correlations in $G^{(ij)}_{\mu\nu}$, that is beyond the simple scattering picture.



**$G^{(ij)}(t,t')$ have correlation time of $\tau_c \sim (gT)^{-1}$ because the leading log contribution comes from soft t-channel momentum exchange $gT \ll q \ll T$**
**When $\Delta t \gg \tau_c$ (but $\Delta t \ll 1/\Gamma \sim 1/(g^4 \log(1/g) T)$ to neglect multi-interactions within $\Delta t$), we have linear terms in $\Delta t$**

$$\int_0^{\Delta t} dt \int_0^{\Delta t} dt' \, G^{(ij)}_{\mu\nu}(t-t') e^{i\omega(t-t')} \sim G^{(ij)}(\omega)\Delta t + \cdots$$

**that gives the evolution equation first order in time.**

**N.B. In diagrammatic language, this corresponds to the ladder approximation, which is justified because of the scale separation**
$\tau_c \sim 1/gT \ll 1/\Gamma \sim 1/g^4 \log(1/g) T$

# Two regimes

**Depending on how $\tau_q = \frac{\hbar}{\Delta E}$ compares with $\tau_c$, we have two different physics**

**Fermi golden rule (scattering picture) regime: When $\tau_q \ll \tau_c$, off-diagonal components of density matrix are time-averaged zero, and only the diagonal components of probabilities make sense. The transition rates are given by the Fermi golden rule.**

**Quantum kinetic regime: When $\tau_c \ll \tau_q$, full components of density matrix are quantum correlated and have to be kept in time evolution. If also $\tau_q \gtrsim 1/\Gamma$, quantum correlations are kept even in ladder multi-scatterings. This is the regime of the LPM effect. Our case belongs to this regime.**

# Explicitly

$$\frac{d}{dt}\rho_{s,s'}(\boldsymbol{p},t) = g^2 C_2(F)(\Gamma_{\rm cross} + \Gamma_{\rm self\ energy})$$

$$\Gamma_{\rm cross} = \int \frac{d^3\boldsymbol{p}'}{(2\pi)^3}\frac{1}{4E_pE_{p'}}\sum_{s'',s'''}[\bar{u}(\boldsymbol{p},s)\gamma^\mu u(\boldsymbol{p}',s'')]\rho_{s'',s'''}(\boldsymbol{p}')[\bar{u}(\boldsymbol{p}',s''')\gamma^\nu u(\boldsymbol{p},s')]G^{(1}_{\mu\nu}$$

$$\Gamma_{\rm self\ energy} = -\gamma\,\hat{\rho}(\boldsymbol{p})$$

## where

$$\gamma = \frac{1}{2}\int \frac{d^3\boldsymbol{q}}{(2\pi)^3}\frac{1}{4E_pE_{p'}}\sum_{s,s''}[\bar{u}(\boldsymbol{p},s)\gamma^\mu u(\boldsymbol{p}',s'')][\bar{u}(\boldsymbol{p}',s'')\gamma^\nu u(\boldsymbol{p},s)]G^{(21)}_{\mu\nu}(E_p - E_{p'},\boldsymbol{q})$$

**The spin sum is challenging, but can be done with great effort**

**N.B. Detailed balance is achieved by KMS relation:**

$$G^{(12)}(q^0) = n_B(q^0)/(n_B(q^0)+1)G^{(21)}(q^0)$$

**The self energy involves $G^{21}(q^0) = (n_B(q^0) + 1)\rho(q^0)$, which only depends on the gluon spectral densities $\rho_{L/T}$**

**N.B. This will no longer be true when the density matrix is off-diagonal in momentum space. The real part of $G^R$ will shift the dispersion relation, which should be absorbed into the kinetic $H_0$**

**Expanding in soft momentum exchange $q = p - p' \sim gT$ ("diffusion approximation"), we need to compute typically**

$$J_n^{L/T} = \int_{q^0_{\rm min}}^{q^0_{\rm max}} \frac{dq^0}{(2\pi)} (q^0)^{2n-1} \rho_{L/T}(q^0, q)$$

**Express $J_n^{L/T}$ as**

$$J_n^{L/T} = \frac{m_D^2}{q^{(4-2n)}} j_n^{L/T} \quad (n = \text{integer}) \,, J_n^{L/T} = \frac{m_D^2}{q^{(3-2n)}} \frac{m^2}{E_p^3} j_n^{L/T} \; (n = \text{half integer})$$

**N.B. The half-integer $n$ appear only in the massive case**

## Table of $j_n^{L/T}$

| | |
|---|---|
| $j_0^L = \frac{p}{E_p}$ | $j_0^T = \frac{\eta_p}{2}$ |
| $j_{1/2}^L = -\frac{p}{2E_p}$ | $j_{1/2}^T = -\frac{pE_p}{4m^2}$ |
| $j_1^L = \frac{p^3}{3E_p^3}$ | $j_1^T = \frac{\eta_p}{2} - \frac{p}{2E_p}$ |
| $j_{3/2}^L = -\frac{2p^3}{E_p^3}$ | $j_{3/2}^T = -\frac{p^3}{4m^2 E_p}$ |
| $j_2^L = \frac{p^5}{5E_p^5}$ | $j_2^T = \frac{\eta_p}{2} - \frac{p}{2E_p} - \frac{p^3}{6E_p^3}$ |

$\eta_p = \frac{1}{2}\ln\frac{E_p+p}{E_p-p}$ **is kinetic rapidity**

# Result

**Write $\hat{\rho}(\boldsymbol{p}) = \frac{1}{2} f(\boldsymbol{p}) \mathbf{1}_{2\times 2} + \boldsymbol{\sigma} \cdot \boldsymbol{S}(\boldsymbol{p})$, so that the spin polarization and number density is given simply by**

$$\langle \boldsymbol{S}^i \rangle = \int_{\boldsymbol{p}} K^{ij}(\boldsymbol{p}) \boldsymbol{S}^j(\boldsymbol{p}), \quad n = \int_p f(\boldsymbol{p})$$

**where** $K^{ij}(\boldsymbol{p}) = \frac{1}{4E_p} \mathrm{Tr}(\sqrt{p \cdot \sigma} \sigma^i \sqrt{p \cdot \sigma} \sigma^j + \sqrt{p \cdot \bar{\sigma}} \sigma^i \sqrt{p \cdot \bar{\sigma}} \sigma^j)$

**(thank to Di-Lun)**

$$\frac{\partial f(\boldsymbol{p}, t)}{\partial t} = C_2(F) \frac{m_D^2 g^2 \log(1/g)}{(4\pi)} \frac{1}{2pE_p} \Gamma_f$$

$$\frac{\partial \boldsymbol{S}(\boldsymbol{p}, t)}{\partial t} = C_2(F) \frac{m_D^2 g^2 \log(1/g)}{(4\pi)} \frac{1}{2pE_p} \Gamma_S$$

$$
\begin{aligned}
\frac{\Gamma_f}{2pE_p} &= \nabla_{p^i}\Big(T(\frac{3}{4}-\frac{E_p^2}{4p^2}+\frac{\eta_p m^4}{4p^3E_p})\nabla_{p^i} f(\boldsymbol{p}) + \frac{\boldsymbol{p}^i}{2p^2}(E_p-\frac{\eta_p m^2}{p})f(\boldsymbol{p}) \\
&+ \boldsymbol{p}^i\frac{Tm^2}{4p^3E_p}(\eta_p+\frac{3E_p}{p}-\frac{3\eta_p E_p^2}{p^2})\boldsymbol{p}\cdot\nabla_p f(\boldsymbol{p})\Big)
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\Gamma}_S^i &= \left(2p+\frac{TE_p}{p}-\frac{\eta_p m^2 T}{p^2}\right)\boldsymbol{S}^i(p) + \left(pTE_p - \frac{m^2TE_p}{2p}+\frac{\eta_p m^4 T}{2p^2}\right)\nabla_p^2\boldsymbol{S}^i(\boldsymbol{p}) \\
&+ \left(\frac{\eta_p m^2 T}{2p^2}\left(1-\frac{3E_p^2}{p^2}\right)+\frac{3m^2TE_p}{2p^3}\right)(\boldsymbol{p}\cdot\nabla_p)^2\boldsymbol{S}^i(\boldsymbol{p}) \\
&+ \frac{1}{p^2}\left(pE_p^2-\frac{3m^2TE_p}{2p}+\eta_p m^2\left(-E_p-\frac{T}{2}+\frac{3TE_p^2}{2p^2}\right)\right)(\boldsymbol{p}\cdot\nabla_p)\boldsymbol{S}^i(\boldsymbol{p}) \\
&+ 2T\left(\eta_p\left(\frac{1}{2}-\frac{E_p^2}{p^2}+\frac{mE_p}{2p^2}+\frac{E_p^3}{2p^2(E_p+m)}\right)+\frac{E_p}{p}-\frac{m}{2p}-\frac{m^2}{2p(E_p+m)}\right)\boldsymbol{p}^i \\
&- 2T\left(\eta_p\left(\frac{1}{2}-\frac{E_p^2}{p^2}+\frac{mE_p}{2p^2}+\frac{E_p^3}{2p^2(E_p+m)}\right)+\frac{E_p}{p}-\frac{m}{2p}-\frac{m^2}{2p(E_p+m)}\right)\nabla \\
&- \frac{T}{p^2}\left(\frac{E_p(E_p+2m)}{p(E_p+m)}+\frac{\eta_p m E_p}{E_p+m}\left(-\frac{3E_p}{p^2}+\frac{1}{E_p+m}\right)\right)\boldsymbol{p}^i(\boldsymbol{p}\cdot\boldsymbol{S}(\boldsymbol{p}))
\end{aligned}
$$

**These results pass very non-trivial tests of**
**1) Detailed balance:** $f(\boldsymbol{p}) = ze^{-E_p/T}$ **is equilibrium, that is,** $\Gamma_f = 0$ **for this**
**2) Chirality in massless limit: When formally** $m = 0$**, the density matrix factorizes as**

$$\hat{\rho}(\boldsymbol{p}) = f_+(\boldsymbol{p})\mathcal{P}_+(\boldsymbol{p}) + f_-(\boldsymbol{p})\mathcal{P}_-(\boldsymbol{p})$$

**where** $\mathcal{P}_\pm(\boldsymbol{p}) = \frac{1}{2}(\mathbf{1} \pm \hat{\boldsymbol{p}} \cdot \boldsymbol{\sigma})$ **are the chirality projection operators, and** $f_\pm(\boldsymbol{p})$ **satisfy the same equation in parity-even background. This means that it should admit the consistent Ansatz** $S(\boldsymbol{p}) = f_s(\boldsymbol{p})\hat{\boldsymbol{p}} \cdot \boldsymbol{\sigma}$**, and moreover** $f(\boldsymbol{p})$ **and** $f_s(\boldsymbol{p})$ **should satisfy the same evolution equation. Also,** $f_s(\boldsymbol{p}) = ze^{-|p|/T}$ **should be the equilibrium solution of** $\Gamma_S = 0$**.**
**All these are true in the above result**

# Future direction

**Go beyond the spatial homogeneous limit**

**The density matrix $\hat{\rho}(\boldsymbol{p}, \boldsymbol{p}')$ with $\boldsymbol{p} \neq \boldsymbol{p}'$ will give us the kinetic equation in the phase space $(\boldsymbol{x}_r, \boldsymbol{p}_r)$.**

**Free streaming example of non-relativistic particle:**

$H_0 = \frac{p_1^2}{2m} - \frac{p_2^2}{2m} = \frac{p_r}{m} \cdot p_a$ **in the $\mathcal{H} \otimes \mathcal{H}^*$ language. Since $p_a \sim -i\hbar\frac{\partial}{\partial x_r}$ we obtain the free streaming kinetic theory**

$$\frac{\partial}{\partial t}\hat{\rho}(\boldsymbol{x}_r, \boldsymbol{p}_r) = -\frac{i}{\hbar} H_0 \hat{\rho} = -\frac{p_r}{m}\frac{\partial}{\partial x_r}\hat{\rho}(\boldsymbol{x}_r, \boldsymbol{p}_r)$$

**Include external slowly varying EM field in $H_0$ to obtain a Vlasov-type equation with spin density matrix. Should reproduce the result by Hattori-Hidaka-Yang**

**Working out the non-local collision term is planned with Shiyong Li**

**Thank you very much !**