Mass and Collision in Quantum Kinetic Theory

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YITP workshop: Quantum kinetic theories in magnetic and vortical fields

Outline

Motivation

The kinetic equations for massive fermions are known, but the distribution function(s) are not.
Possible hints:

- Decomposition of A^μ.
- Massless limit.
- Frame dependence.
- Collision

Outline

Kinetic equation:

$$\Pi^\mu V_\mu = mF$$
$$\frac{\hbar}{2}\mathrm{D}^\mu A_\mu = mP$$
$$\Pi_\mu F - \frac{1}{2}\hbar \mathrm{D}^\nu S_{\nu\mu} = mV_\mu$$
$$-\hbar \mathrm{D}_\mu P + \epsilon_{\mu\nu\sigma\rho}\Pi^\nu S^{\sigma\rho} = 2mA_\mu$$
$$\frac{1}{2}\hbar(\mathrm{D}_\mu V_\nu - \mathrm{D}_\nu V_\mu) + \epsilon_{\mu\nu\sigma\rho}\Pi^\sigma A^\rho = mS_{\mu\nu}$$

and

$$\hbar \mathrm{D}^\mu V_\mu = 0$$
$$\Pi^\mu A_\mu = 0$$
$$\frac{1}{2}\hbar D_\mu F + \Pi^\nu S_{\nu\mu} = 0$$
$$\Pi_\mu P + \frac{\hbar}{4}\epsilon_{\mu\nu\sigma\rho}\mathrm{D}^\nu S^{\sigma\rho} = 0$$
$$\Pi_\mu V_\nu - \Pi_\nu V_\mu - \frac{\hbar}{2}\epsilon_{\mu\nu\sigma\rho}\mathrm{D}^\sigma A^\rho = 0$$

0th Order Solution

$$\begin{aligned}
P^{(0)} &= 0 \\
F^{(0)} &= m f_V^{(0)} \delta(p^2 - m^2) \\
V_\mu^{(0)} &= p_\mu f_V^{(0)} \delta(p^2 - m^2) \\
A_\mu^{(0)} &= (p_\mu f_A^{(0)} - \theta_\mu^{(0)}) \delta(p^2 - m^2)^1 \\
S_{\mu\nu}^{(0)} &= -\frac{1}{m} \epsilon_{\mu\nu\sigma\rho} p^\sigma \theta^{(0)\rho}
\end{aligned}$$

[1] See Yu-Chen Liu's poster at QM2019

Axial Current

$$A_\mu^{(0)} = (p_\mu f_A^{(0)} - \theta_\mu^{(0)})\delta(p^2 - m^2)$$

Related to an arbitrary time-like vector n_μ:

$$\theta_\mu^{(0)}\delta(p^2 - m^2) = -\frac{m}{2p \cdot n}\epsilon_{\mu\nu\sigma\rho}n^\nu S^{\sigma\rho}$$

$$f_A^{(0)}\delta(p^2 - m^2) = \frac{p \cdot \theta^{(0)}}{m^2}\delta(p^2 - m^2) = \frac{A \cdot n}{p \cdot n}$$

Assumption: Any component of W will not diverge in the massless limit.

Then $\theta^{(0)} \to 0$ when $m \to 0$.

Transport Equation

$$
\begin{aligned}
(p\cdot \mathrm{D})f_V^{(0)} &= 0\\
(p\cdot \mathrm{D}\theta_\mu^{(0)} - qF_{\mu\nu}\theta^{(0)\nu}) - p_\mu p\cdot \mathrm{D}f_A^{(0)} &= 0
\end{aligned}
$$

Some interesting possible solutions:

- Only constant *B* field: $\theta_\mu^{(0)} = a\epsilon_{\mu\nu\sigma\rho}\beta^\nu F^{\sigma\rho}, f_A^{(0)} = 0$.
- Only vorticity: $\theta_\mu^{(0)} = a\epsilon_{\mu\nu\sigma\rho}\beta^\nu p^\sigma \omega^{\rho\lambda}p_\lambda, f_A^{(0)} = 0$

Even though we may not expect non-zero axial charge in classical equilibrium, there is every possibility to have non-zero average spin.

1st Order Solutions

$$
\begin{aligned}
P^{(1)} &= \frac{1}{2m}\mathrm{D}^{\mu}A^{(0)}_{\mu} \\
F^{(1)} &= mf^{(1)}\delta(p^2-m^2) - \frac{1}{2m(p^2-m^2)}\epsilon_{\mu\nu\sigma\rho}p^{\mu}\mathrm{D}^{\nu}p^{\sigma}A^{(0)\rho} \\
V^{(1)}_{\mu} &= p_{\mu}f^{(1)}_{V}\delta(p^2-m^2) + \frac{qp_{\mu}}{2m^2(p^2-m^2)}\epsilon_{\alpha\beta\sigma\rho}F^{\sigma\rho}p^{\alpha}A^{(0)\beta} \\
&\quad +\frac{1}{2m^2}\epsilon_{\mu\nu\sigma\rho}\mathrm{D}^{\nu}p^{\sigma}A^{(0)\rho} \\
A^{(1)}_{\mu} &= (p_{\mu}f^{(1)}_{A} - \theta^{(1)}_{\mu})\delta(p^2-m^2) - \frac{1}{2(p^2-m^2)}\epsilon_{\mu\nu\sigma\rho}p^{\nu}\mathrm{D}^{\sigma}V^{(0)\rho} \\
S^{(1)}_{\mu\nu} &= \frac{1}{2m}(\mathrm{D}_{\mu}V^{(0)}_{\nu} - \mathrm{D}_{\nu}V^{(0)}_{\mu}) + \frac{1}{m}\epsilon_{\mu\nu\sigma\rho}p^{\sigma}A^{(1)\rho}
\end{aligned}
$$

$V_\mu^{(1)}$ can be rewritten as

$$V_\mu^{(1)} = \Big[p_\mu f_V^{(1)} - \frac{p_\mu}{2p^2 p\cdot n}\epsilon_{\alpha\beta\sigma\rho}p^\alpha n^\beta (\mathrm{D}^\sigma \theta^{(0)\rho}) + \frac{q}{2p^2}\epsilon_{\mu\nu\sigma\rho}F^{\sigma\rho}p^\nu f_A^{(0)} + \frac{1}{2p\cdot n}\epsilon_{\mu\nu\sigma\rho}n^\nu (\mathrm{D}^\rho \theta^{(0)\sigma}) - \frac{1}{2p\cdot n}\epsilon_{\mu\nu\sigma\rho}n^\nu p^\sigma (\mathrm{D}^\rho f_A^{(0)})\Big]\delta(p^2-m^2)$$

In massless limit, $\mathrm{D}^\sigma\theta^{(0)\rho}$ vanishes, so the expression goes back to the corresponding CKT one. Also,

$$f_V^{(1)} = \frac{V^{(1)}\cdot n}{p\cdot n} - \frac{q}{2p^2 p\cdot n}\epsilon^{\mu\nu\sigma\rho}n^\mu p^\nu F^{\sigma\rho} f_A^{(0)}\delta(p^2-m^2)$$

Similarly,

$$
\begin{aligned}
A^{(1)}_{\mu} &= \left[(p_\mu f^{(1)}_A - \tilde{\theta}^{(1)}_{\mu}) - \frac{1}{2p\cdot n}\epsilon_{\mu\nu\sigma\rho} n^{\nu} p^{\sigma} (\mathrm{D}^{\rho} f^{(0)}_V)\right]\delta(p^2 - m^2) \\
&\quad - \frac{1}{2(p^2-m^2)}\epsilon_{\mu\nu\sigma\rho} p^{\nu} \mathrm{D}^{\sigma} V^{(0)\rho}
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\theta}^{(1)}_{\mu}\delta(p^2 - m^2) &= -\frac{m}{2p\cdot n}\epsilon_{\mu\nu\sigma\rho} n^{\nu} S^{(1)\sigma\rho} \\
&\quad - \frac{m^2 q}{2p\cdot n(p^2-m^2)} n^{\nu} F^{\sigma\rho} f^{(0)}_V \delta(p^2 - m^2) \\
f^{(1)}_A \delta(p^2 - m^2) &= \frac{A^{(1)}\cdot n}{p\cdot n} \\
&\quad - \frac{q}{2p\cdot n(p^2-m^2)}\epsilon_{\mu\nu\sigma\rho} n^{\mu} p^{\nu} F^{\sigma\rho} f^{(0)}_V \delta(p^2 - m^2)
\end{aligned}
$$

Outline

Collision Term

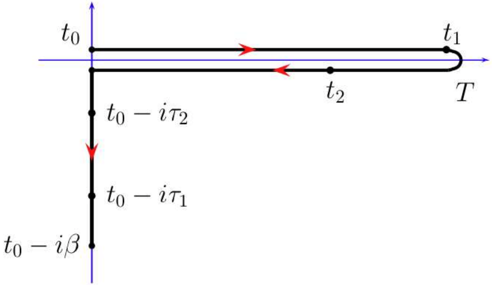
'..., the general quantum equations derived here should provide a natural starting point for further work on deriving generalized quantum collision terms. '

D. Vasak, M. Gyulassy and H. T. Elze, Annals Phys. **173**, 462 (1987).

For simplicity, we consider massive Dirac fermion with four-Fermion interaction and no EM field.
For Weyl fermion case, see: Y. Hidaka, S. Pu and D. L. Yang, Phys. Rev. D **95**, no. 9, 091901 (2017).
The definition of Wigner function is related to the the lesser Green's function defined on the Keldysh contour

$$
\begin{aligned}
W_{ab}(x,p) &= \int \mathrm{d}^4 y e^{ipy} \langle \bar{\Psi}_b(x-\frac{y}{2}) \Psi_a(x+\frac{y}{2}) \rangle \\
&= \int \mathrm{d}^4 y e^{ipy} \mathcal{T}_c G^{<}(t_1, \vec{x}_1, t_2, \vec{x}_2)|_{1=x+\frac{y}{2}, 2=x-\frac{y}{2}}
\end{aligned}
$$

Keldysh Contour



- t_2 is always ‘in front of‘ t_1 by this definition.
- The imaginary part is related to initial time correlation.
- In our case, $t_0 \to -\infty$ and $T \to \infty$ limits are taken.

Lagrangian:

$$\mathcal{L} = \bar{\Psi}(\gamma^\mu \partial_\mu - m)\Psi + \lambda(\bar{\Psi}\Psi)^2$$

Now the kinetic equation for Wigner function is

$$\begin{aligned}
&(\gamma^\mu K_\mu - m) W_{ab} = \\
&-2\lambda \int \mathrm{d}^4 y e^{ipy} \langle \mathcal{T}_c \bar{\Psi}_b(x - \frac{y}{2}) \Psi_a(x + \frac{y}{2}) \bar{\Psi}(x + \frac{y}{2}) \Psi(x + \frac{y}{2}) \rangle \\
&\sim \int_c \mathrm{d}z_0 \int \mathrm{d}^3 z \Sigma(x_1, z) G(z, x_2)
\end{aligned}$$

Σ is the fermion self-energy. $\int_c$ means integration on the Keldysh contour.

$$\Sigma_{ij} = \frac{\partial \Phi}{\partial G_{ij}}$$

Φ is the Luttinger-Ward functional. In our case, up to the next to the leading order,

$$\Sigma(x,y) = \lambda \langle \mathcal{T}_c \Psi_I(x) \bar{\Psi}_I(y) \rangle \delta^{(4)}(x-y)$$
$$+ i\hbar\lambda^2 \langle \mathcal{T}_c \Psi_I(x) \bar{\Psi}_I(x) \Psi_I(x) \bar{\Psi}_I(y) \Psi_I(y) \bar{\Psi}_I(y) \rangle$$

- Ψ_I stands for free field operator.
- The second term corresponds to the collision contribution.

Integration on the Keldysh contour can be divided into three parts

$$\int_c \mathrm{d}z_0 \int \mathrm{d}^3 z \Sigma(x_1, z) G(z, x_2) = \int_{t_0}^{t_1} \mathrm{d}z_0 \mathrm{d}^3 z \Sigma^>(x_1, z) G^<(z, x_2)$$

$$+ \int_{t_1}^{t_2} \mathrm{d}z_0 \mathrm{d}^3 z \Sigma^<(x_1, z) G^<(z, x_2) - \int_{t_0}^{t_2} \mathrm{d}z_0 \mathrm{d}^3 z \Sigma^<(x_1, z) G^>(z, x_2)$$

$$= \int \mathrm{d}^4 p_1 \mathrm{d}^4 k_1 \mathrm{d}^4 k_2 \{(C(p_1) - W(x,p_1))_a W(x,p)_b \mathrm{Tr}[W(x,k_1)(C(k_2) - W(x,k_2))]$$

$$-(C(p_1) - W(x,p_1)) W(x,k_1)(C(p_2) - W(x,k_2)) W(x,p)_b\}$$

$$- \int \mathrm{d}^4 p_1 \mathrm{d}^4 k_1 \mathrm{d}^4 k_2 \{W(x,p_1)_a (C(p) - W(x,p))_b \mathrm{Tr}[(C(k_1) - W(x,k_1)(W(x,k_2))]$$

$$- W(x,p_1)(C(k_1) - W(x,k_1)) W(x,k_2)(C(p) - W(x,p))_b\}$$

where $C(p) = 2E_p \delta(p^2 - E_p)$

$$C[W]\sim -i\hbar\lambda^2 \int \mathrm{d}^4p_1\mathrm{d}^4k_1\mathrm{d}^4k_2\{(C(p_1)-W(x,p_1))_a W(x,p)_b \mathrm{Tr}[W(x,k_1)(C(k_2)-W(x,k_2))]$$
$$-(C(p_1)-W(x,p_1))W(x,k_1)(C(p_2)-W(x,k_2))W(x,p)_b\}$$
$$+i\hbar\lambda^2 \int \mathrm{d}^4p_1\mathrm{d}^4k_1\mathrm{d}^4k_2\{W(x,p_1)_a(C(p)-W(x,p))_b \mathrm{Tr}[(C(k_1)-W(x,k_1)(W(x,k_2)]$$
$$-W(x,p_1)(C(k_1)-W(x,k_1))W(x,k_2)(C(p)-W(x,p))_b\}$$

Sanity check:

$$K_\mu = p_\mu + \frac{i\hbar}{2}\partial_\mu$$

So

$$p\cdot\partial f_V \sim \lambda^2 \int \mathrm{d}^3p_1\mathrm{d}^3k_1\mathrm{d}^3k_2 f(\vec{k}_1)f(\vec{k}_2)(1-f(\vec{p}))(1-f(\vec{p}_1))$$
$$-\lambda^2 \int \mathrm{d}^3p_1\mathrm{d}^3k_1\mathrm{d}^3k_2 f(\vec{p})f(\vec{p}_1)(1-f(\vec{k}_1))(1-f(\vec{k}_2)) + \dots$$

Non-local interaction

In general, the effective four-fermion interaction may take the form

$$\sigma(x_1 - x_2, x_1 - x_3, x_1 - x_4)\bar{\Psi}(x_1)A\Psi(x_2)\bar{\Psi}(x_3)B\Psi(x_4)$$

- The basic routine is still the same.
- Plane wave vs. finite interaction range.
- Spin-orbital coupling possible?

Summary

- We showed that the Wigner function for massive fermions can goes to massless one when taking $m \to 0$ limit (up to $\hbar$ order.
- We gave the general form of collision term for Dirac fermions using Schwinger-Keldysh formalism.
- Outlook:
  - Spin structure
  - Equilibrium distribution
  - Non-local case