Chiral kinetic theory in curved spacetime

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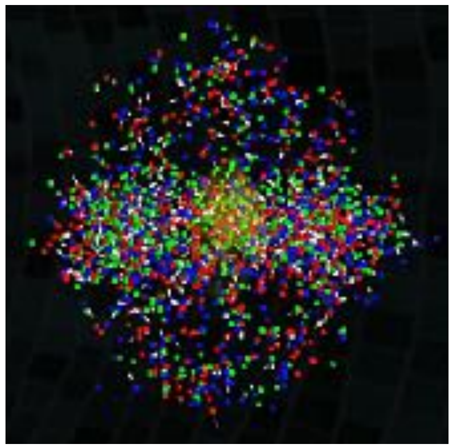
Y.C.Liu, L.L.Gao, KM, X.G.Huang, PRD 99.085014 (2019)
T. Hayata, Y. Hidaka, KM, arXiv : 19xx.xxxxx

Chiral Transport Phenomena

nuclear physics



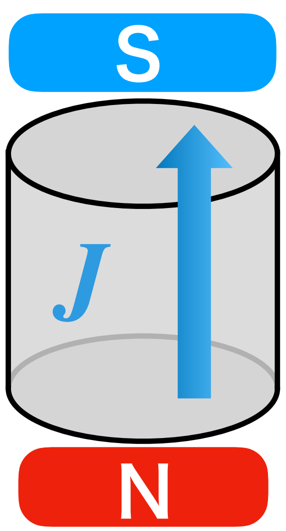
QGP

condensed matter

Dirac/Weyl semimetal

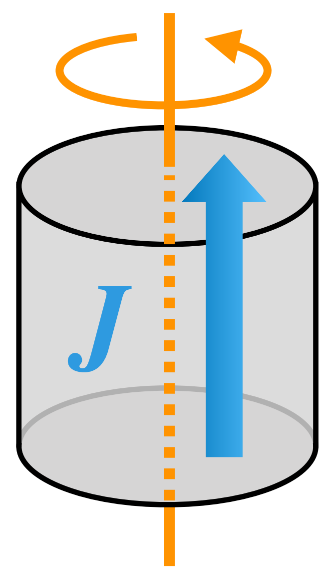
astrophysics

supernovae



chiral magnetic effect

$$\vec{J} = \sigma_{\mathrm{CME}}\,\vec{B}$$



chiral vortical effect

$$\vec{J} = \sigma_{\mathrm{CVE}}\,\vec{\omega}$$

Anomaly-induced ⇒ universal phenomena in chiral matter

Kinetic Theory for Anomalous Transport

equilibrium

$$\sigma^0_{\rm CME} = \frac{e^2\mu_5}{2\pi^2}$$

nonequilibrium (with AC mag. field)

$$\sigma^{\rm noneq}_{\rm CME}(\omega) = \sigma^0_{\rm CME}\left(1 + \frac{2}{3}\frac{\omega}{\omega + i\tau^{-1}}\right)$$

Kharzeev, Stephanov, Yee (2017)

Boltzmann kinetic theory + anomaly = Chiral Kinetic Theory

Stephanov, Yin (2012)

$$\left[(1+\hbar\#)\partial_t + (\boldsymbol{v}+\hbar\#)\cdot\boldsymbol{\nabla} + (\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}+\hbar\#)\cdot\boldsymbol{\nabla}_p\right] f = I_{\rm coll}[f]$$

$$\Rightarrow \quad \partial_\mu J^\mu = \hbar\#\boldsymbol{E}\cdot\boldsymbol{B}$$

$$J^0 = \int_{\boldsymbol{p}} (1+\hbar\#) f \qquad \boldsymbol{J} = \int_{\boldsymbol{p}} (\boldsymbol{v}+\hbar\#) f$$

Earlier Works

various derivations/covariance/collision/consistent anomaly/collective excitation/mass…

Stephanov, Yin (2012)
Chen, Son, Stephanov, Yee, Yin (2014)
Chen, Son, Stephanov (2015)
Mueller, Venugopalan (2017)
Son, Yamamoto (2012)
Son, Yamamoto (2013) Lin, Shukla (2019)
Chen, Pu, Wang, Wang (2013)
Hidaka, Pu, Yang (2017) Huang, et al. (2018)
Gorbar, Miransky, Shovkovy, Sukhachov (2017)
Weickgenannt, et al. (2019) Gao, Liang (2019) Hattori, Hidaka, Yang, (2019)

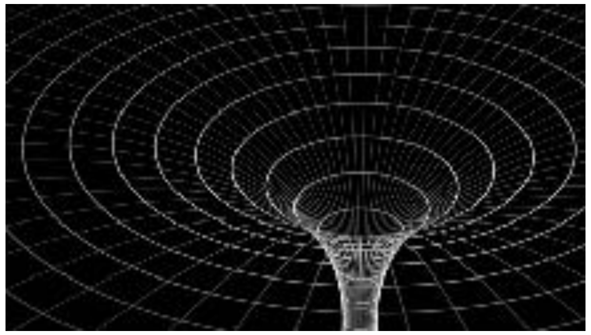
Relativistic spin transport theory under external E and B

Anything else coupled with spin ?

(inhomogeneous) vorticity

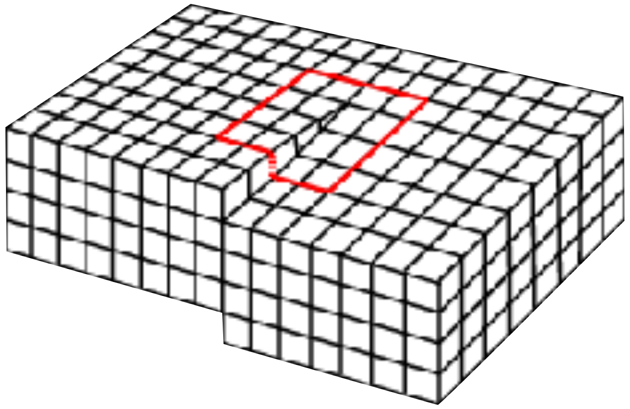
nuclear physics

gravity



astrophysics

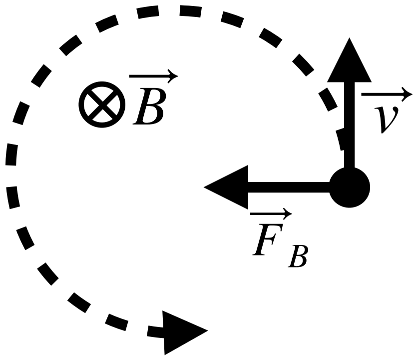
torsion



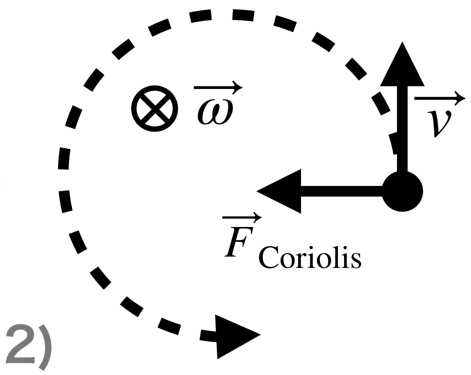
condensed matter

Chirality and Vorticity

$$\overrightarrow{J} = \sigma_{\mathrm{CME}}\,\overrightarrow{B} \quad \Longleftrightarrow \quad \overrightarrow{J} = \sigma_{\mathrm{CVE}}\,\overrightarrow{\omega}$$



$$\vec{F}_B = e\vec{v} \times \vec{B} \simeq \vec{F}_\omega = 2\varepsilon\vec{v} \times \vec{\omega}$$

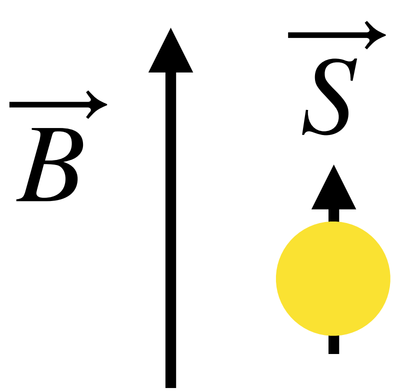


$$e\overrightarrow{B} \leftrightarrow 2\varepsilon\overrightarrow{\omega}$$

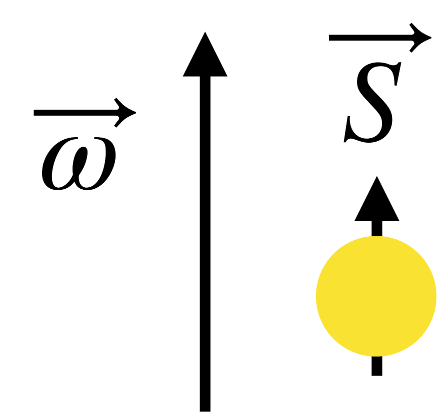
Stephanov, Yin (2012)

Why noninertial effect?

Why not spin part?

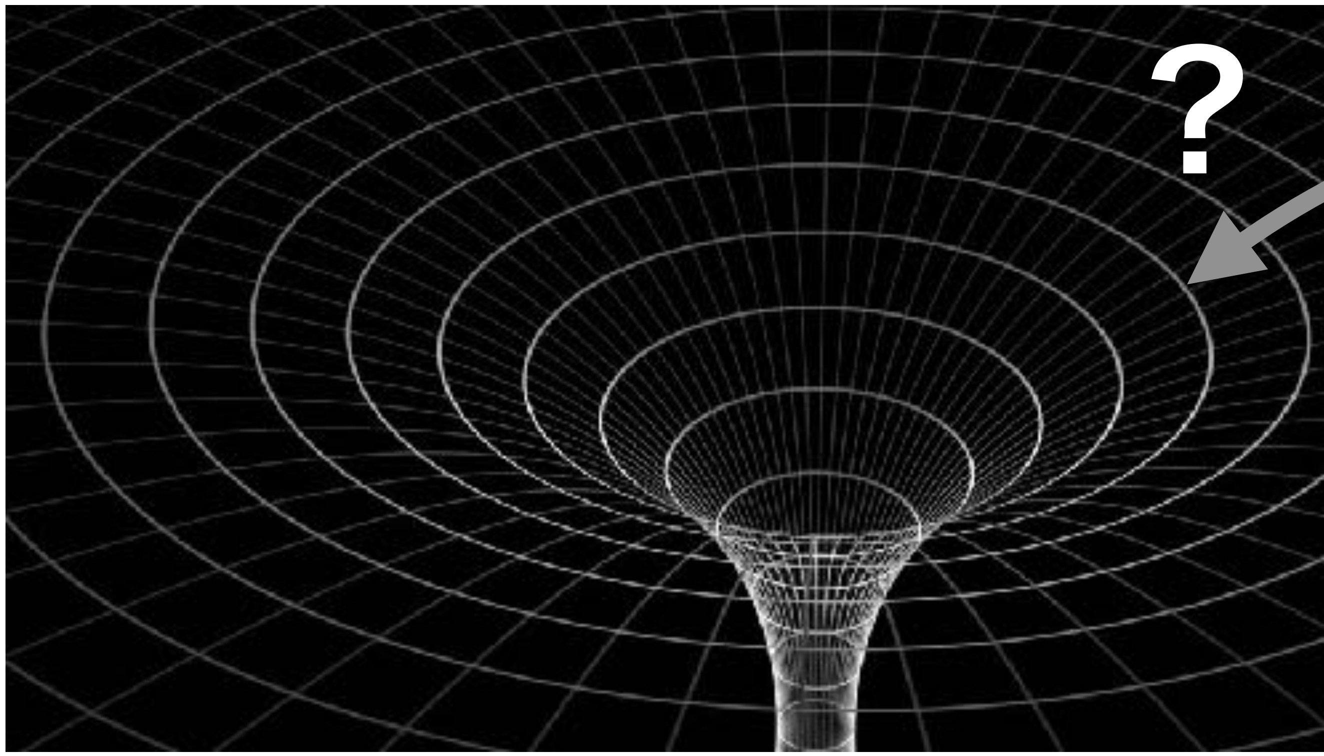


$$H_B = -\frac{e\vec{B}\cdot\vec{S}}{\varepsilon} \simeq H_\omega = -\vec{\omega}\cdot\vec{S}$$



$$e\overrightarrow{B} \leftrightarrow \varepsilon\overrightarrow{\omega}$$

Chirality and Gravity



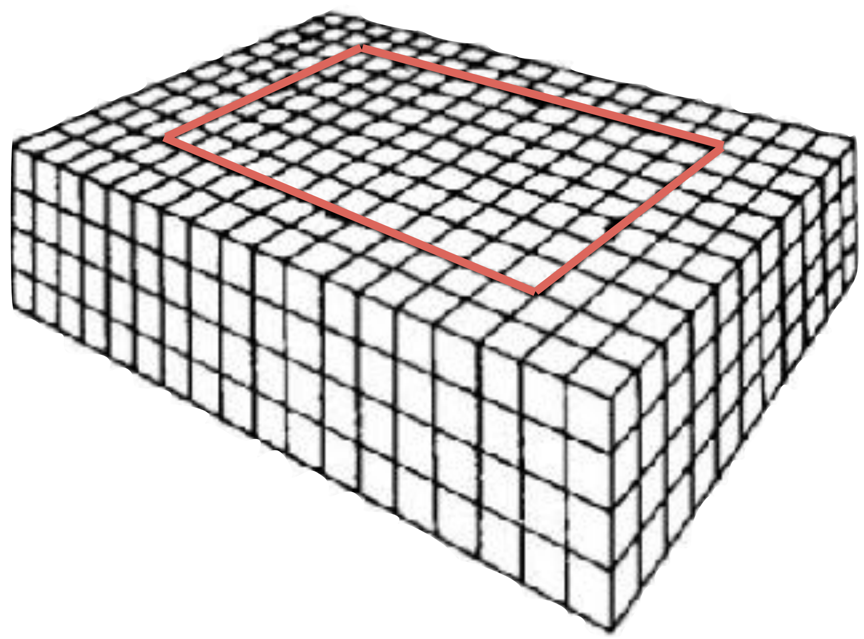
spinning classical particle
$S^{\mu\nu}$

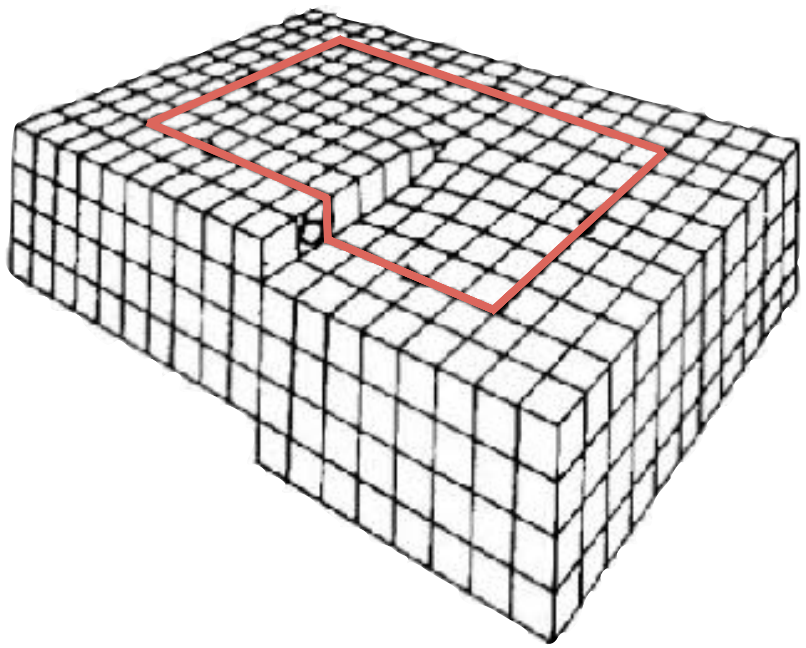
Mathisson-Papapetrou–Dixon (MPD) equation

$$\dot{p}_\mu = \left(-\Gamma^\rho_{\mu\nu} p_\rho + \frac{1}{2} S^{\alpha\beta} R_{\mu\nu\alpha\beta} \right) \dot{x}^\nu$$

spin-cavature coupling

Chirality and Torsion





$$e^a_\mu = \delta^a_\mu$$

$$e^a_\mu = \delta^a_\mu + \delta e^a_\mu$$

$$p_a \to p_a - \delta e^a_\mu p_\mu$$ **emergent gauge field**

torsional magnetic field

$$\mathcal{B}^\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho}T^a_{\nu\rho}p_a$$

$$T^a_{\mu\nu} = \partial_\mu e^a_\nu - \partial_\nu e^a_\mu$$

chirality dependent

spin-torsion coupling

Our Work

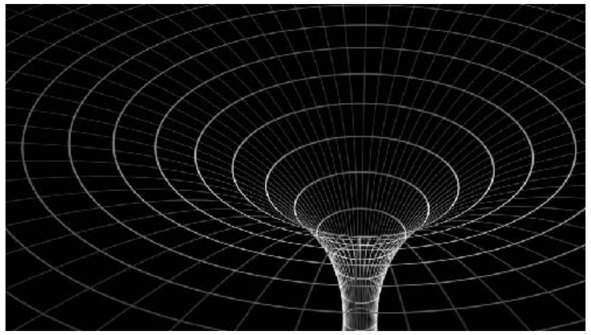
Spin coupled with Spacetime Geometry

CKT in curved spacetime

(inhomogeneous) vorticity

nuclear physics

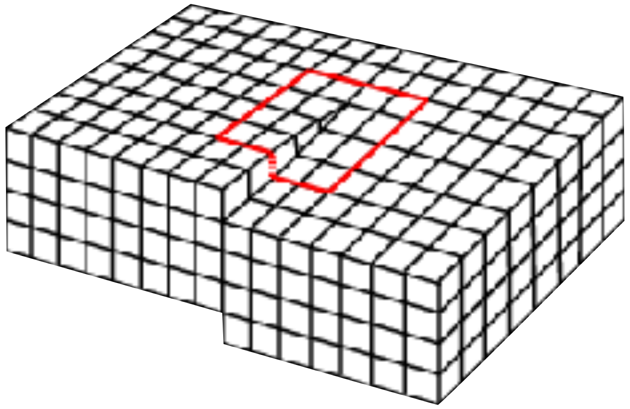
gravity



astrophysics

condensed matter

torsion



condensed matter

Contents

1. From QFT to CKT in curved spacetime

2. 1st order CKT in general coordinate

    ex.) rigidly rotating coordinate

3. 2nd order CKT in general coordinate

    ex.) gravito-electromagnetic field

1. From QFT to CKT in curved spacetime

From Quantum Field Theory to Chiral Kinetic Theory

Quantum Field Theory $S[\psi, \bar{\psi}]$

1. Construct Wigner function $W(x,p) = \int_y e^{-ip\cdot y/\hbar} \left\langle \bar{\psi}(x + \frac{1}{2}y)\psi(x - \frac{1}{2}y) \right\rangle$

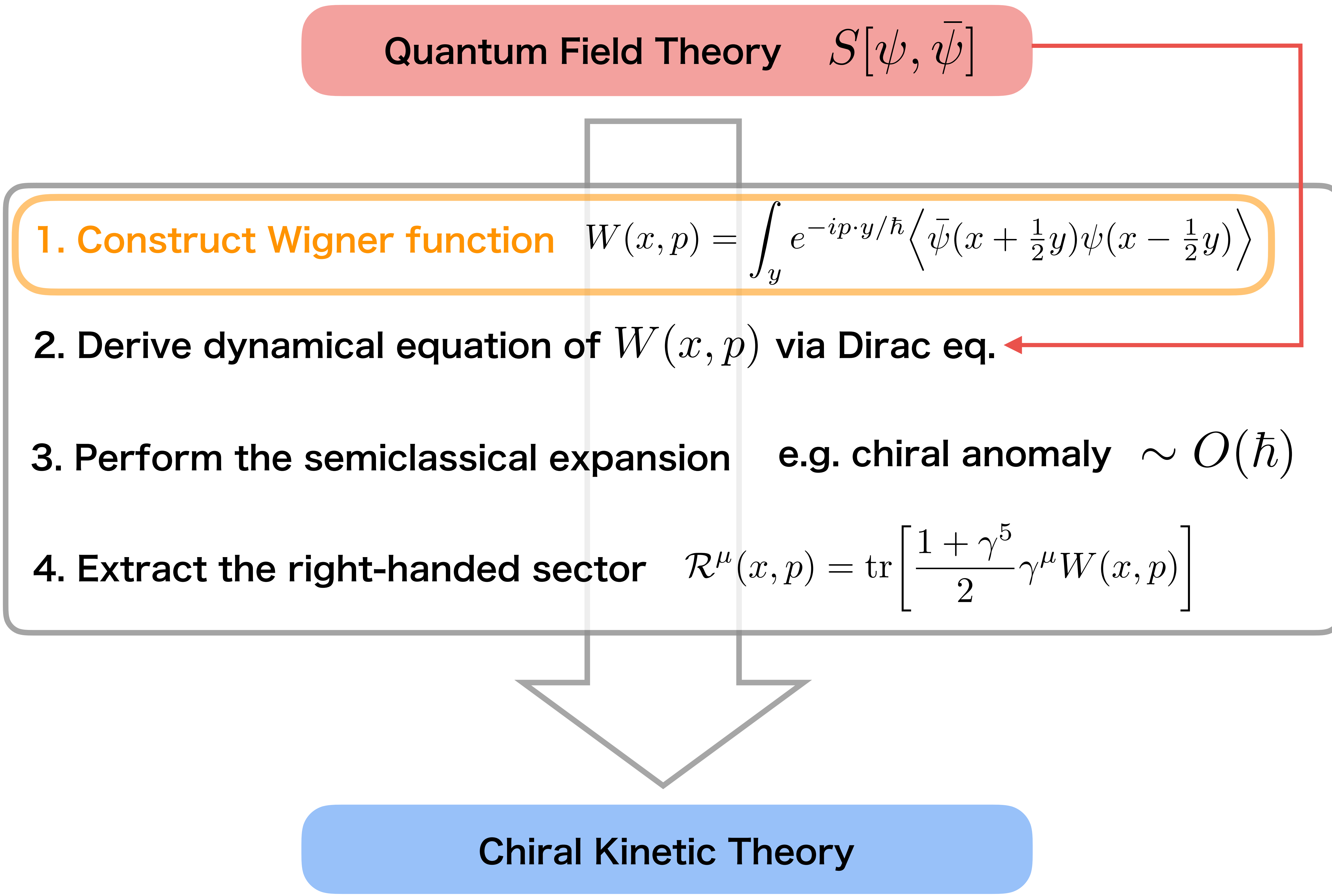
2. Derive dynamical equation of $W(x,p)$ via Dirac eq.

3. Perform the semiclassical expansion e.g. chiral anomaly $\sim O(\hbar)$

4. Extract the right-handed sector $\mathcal{R}^\mu(x,p) = \mathrm{tr}\left[\frac{1+\gamma^5}{2}\gamma^\mu W(x,p)\right]$

Chiral Kinetic Theory

From Quantum Field Theory to Chiral Kinetic Theory



Wigner Function

$$W(x,p) = \int_y e^{-ip\cdot y/\hbar} \Big\langle \bar{\psi}(x+\tfrac{1}{2}y)\psi(x-\tfrac{1}{2}y) \Big\rangle$$

$$\partial_\mu y^\nu = 0$$
$$\partial_\mu p_\nu = 0$$

Lorentz covariant translation

$$\psi(x+y) = \psi(x) + y^\mu \partial_\mu \psi(x) + \frac{1}{2} y^\mu y^\nu \partial_\mu \partial_\nu \psi(x) + \cdots$$
$$= \exp(y^\mu \partial_\mu)\psi(x) \qquad (\because [\partial_\mu, y^\nu] = 0)$$

ex.) Lorentz covariant vector current

$$J^\mu(x) = \int_p \mathrm{tr}\Big[\gamma^\mu W(x,p)\Big]$$

Wigner Function in EM-field

$$W(x,p) = \int_y e^{-ip\cdot y/\hbar}\Big\langle \bar{\psi}(x+\tfrac{1}{2}y)\psi(x-\tfrac{1}{2}y)\Big\rangle$$

$$\partial_\mu y^\nu = 0$$
$$\partial_\mu p_\nu = 0$$

covariant derivative

$$\nabla_\mu \psi(x) = (\partial_\mu + iA/\hbar)\psi(x)$$

U(1) gauge covariant translation

$$\psi(x+y) = \psi(x) + y^\mu \nabla_\mu \psi(x) + \frac{1}{2} y^\mu y^\nu \nabla_\mu \nabla_\nu \psi(x) + \cdots$$

$$= \exp(y^\mu \nabla_\mu)\psi(x) \qquad (\because [\nabla_\mu, y^\nu] = 0)$$

Wigner Function in Curved Spacetime

$$W(x,p) = \int_y e^{-ip\cdot y/\hbar} \left\langle \bar{\psi}(x+\tfrac{1}{2}y)\psi(x-\tfrac{1}{2}y) \right\rangle$$

$$\partial_\mu y^\nu = 0$$
$$\partial_\mu p_\nu = 0$$

covariant derivative

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\rho} V^\rho \qquad \nabla_\mu \psi(x) = (\partial_\mu + i\mathcal{A}_\mu)\psi(x)$$

General covariant & local Lorentz covariant translation

$$\psi(x+y) = \psi(x) + y^\mu \nabla_\mu \psi(x) + \frac{1}{2} y^\mu y^\nu \nabla_\mu \nabla_\nu \psi(x) + \cdots$$

$$\neq \exp(y^\mu \nabla_\mu)\psi(x) \qquad (\because [\nabla_\mu, y^\nu] = \Gamma^\nu_{\mu\rho} y^\rho \neq 0)$$

covariant derivative fixing y

$$D_\mu \equiv \nabla_\mu - \Gamma^\rho_{\mu\nu} y^\nu \partial^y_\rho \qquad [D_\mu, y^\nu] = 0$$

Wigner Function in Curved Spacetime

$$W(x,p) = \int_y e^{-ip\cdot y/\hbar} \left\langle \bar{\psi}(x+\tfrac{1}{2}y)\psi(x-\tfrac{1}{2}y) \right\rangle$$

$$\partial_\mu y^\nu = 0$$
$$\partial_\mu p_\nu = 0$$

covariant derivative

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\rho} V^\rho \qquad \nabla_\mu \psi(x) = (\partial_\mu + i\mathcal{A}_\mu)\psi(x)$$

General covariant & local Lorentz covariant translation

$$\psi(x+y) = \psi(x) + y^\mu \nabla_\mu \psi(x) + \frac{1}{2} y^\mu y^\nu \nabla_\mu \nabla_\nu \psi(x) + \cdots$$

$$= \exp(y^\mu D_\mu)\psi(x)$$

covariant derivative fixing y

$$D_\mu \equiv \nabla_\mu - \Gamma^\rho_{\mu\nu} y^\nu \partial^y_\rho \qquad [D_\mu, y^\nu] = 0$$

Wigner Function in Curved Spacetime

$$(x^\mu, y^\mu) \qquad\qquad (x^\mu, p_\mu)$$

$$D_\mu \equiv \nabla_\mu - \Gamma^\rho_{\mu\nu} y^\nu \partial^y_\rho \qquad\qquad D_\mu \equiv \nabla_\mu + \Gamma^\rho_{\mu\nu} p_\rho \partial^\nu_p$$

$$[D_\mu, y^\nu] = 0 \qquad\qquad [D_\mu, p_\nu] = 0$$

covariant Wigner function

$$\begin{aligned} W(x,p) &= \int_y e^{-ip\cdot y/\hbar} \left\langle \bar{\psi}(x + \tfrac{1}{2}y)\psi(x - \tfrac{1}{2}y) \right\rangle \\ &= \int_y e^{-ip\cdot y/\hbar} \left\langle \bar{\psi}(x) e^{y\cdot \overleftarrow{D}/2}\, e^{-y\cdot D/2} \psi(x) \right\rangle \end{aligned}$$

2. 1st order CKT in general coordinate

From Quantum Field Theory to Chiral Kinetic Theory

Quantum Field Theory $S[\psi, \bar{\psi}]$

1. **Construct Wigner function** $W(x,p) = \int_y e^{-ip\cdot y/\hbar} \left\langle \bar{\psi}(x + \frac{1}{2}y)\psi(x - \frac{1}{2}y) \right\rangle$

2. **Derive dynamical equation of** $W(x,p)$ **via Dirac eq.**

3. **Perform the semiclassical expansion** e.g. chiral anomaly $\sim O(\hbar)$

4. **Extract the right-handed sector** $\mathcal{R}^\mu(x,p) = \mathrm{tr}\left[\frac{1+\gamma^5}{2}\gamma^\mu W(x,p)\right]$

Chiral Kinetic Theory

1st order CKT

$$S = \frac{i}{2}\int \sqrt{-g}\, d^4x\, \bar{\psi}(\gamma^\mu \nabla_\mu \psi + \bar{\psi}\overleftarrow{\nabla}_\mu \gamma^\mu)\psi$$

kinetic eq.

$$D_\mu \mathcal{R}^\mu = 0$$

current cons.

$J^\mu \sim \mathcal{R}^\mu$

constraint

$$p_\mu \mathcal{R}^\mu = 0$$

conformal sym.

$T^{\mu\nu} \sim \mathcal{R}^\mu p^\nu$

constraint

$$\mathcal{R}^\mu p^\nu - \mathcal{R}^\nu p^\mu - \frac{\hbar}{2}\varepsilon^{\mu\nu\rho\sigma} D_\rho \mathcal{R}_\sigma = 0$$

AM cons.

$S^{\mu\nu\rho} \sim \frac{\hbar}{2}\varepsilon^{\mu\nu\rho\sigma}\mathcal{R}_\sigma$

$$\mathcal{R}^\mu = \mathrm{tr}\left[\gamma^\mu \frac{1+\gamma^5}{2} W\right] \qquad D_\mu \equiv \nabla_\mu + \Gamma^\rho_{\mu\nu} p_\rho \partial^\nu_p$$

Solution

$$p_\mu \mathcal{R}^\mu = 0 \qquad \mathcal{R}^\mu p^\nu - \mathcal{R}^\nu p^\mu - \frac{\hbar}{2}\varepsilon^{\mu\nu\rho\sigma} D_\rho \mathcal{R}_\sigma = 0$$

$$\Rightarrow \quad \mathcal{R}^\mu = 2\pi\delta(p^2)\left(p^\mu + \hbar\, \Sigma_n^{\mu\nu} D_\nu\right) f$$

Lorentz frame vector n^μ **spin tensor** $\Sigma_n^{\mu\nu} = \dfrac{\varepsilon^{\mu\nu\rho\sigma} p_\rho n_\sigma}{2p \cdot n}$

ex.) Minkowski spacetime

$$n^\mu = (1,0,0,0) \qquad \boldsymbol{\Sigma} = (\Sigma^{23}, \Sigma^{31}, \Sigma^{12}) = \frac{\boldsymbol{p}}{2p_0} \sim \frac{\hat{\boldsymbol{p}}}{2}$$

$$J^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu + \hbar \Sigma_n^{\mu\nu}$$

decomposed frame-dependently

Chen, Son, Stephanov (2015)

Kinetic Equation

$$D_\mu \mathcal{R}^\mu = 0 \qquad \mathcal{R}^\mu = 2\pi\delta(p^2)\Big(p^\mu + \hbar\,\Sigma_n^{\mu\nu} D_\nu\Big) f$$

$$\Rightarrow \quad \delta(p^2)\left[p^\mu D_\mu - \frac{\hbar}{2}\Sigma_n^{\alpha\beta} R_{\mu\nu\alpha\beta} p^\mu \partial_p^\nu + \hbar(D_\nu \Sigma_n^{\mu\nu}) D_\nu\right] f = 0$$

1. Vlasov-Einstein eq.

$$0 = \delta(p^2) p^\mu D_\mu f = \delta(p^2)(p^\mu \partial_\mu + \Gamma^\rho_{\mu\nu} p^\mu p_\rho \partial^p_\nu) f$$

2. MPD eq.

$$\dot{p}_\mu = \left(-\Gamma^\rho_{\mu\nu} p_\rho + \frac{\hbar}{2}\Sigma_n^{\alpha\beta} R_{\mu\nu\alpha\beta}\right)\dot{x}^\nu$$

$$S^{\alpha\beta} \leftrightarrow \hbar\Sigma_n^{\mu\nu}$$

Mathisson (1937)
Papapetrou (1951)
Dixon (1970)

Equilibrium Distribution

$$f = f_{(0)} + \hbar f_{(1)}$$

$f^{\rm eq}_{(0)} = f^{\rm eq}_{(0)}(-\beta\mu + \beta^\mu p_\mu)$ **but how about** $f^{\rm eq}_{(1)}$ **?**

Lorentz covariance under $n^\mu \to n'^\mu$

$$(\Lambda_n^{-1})_\mu{}^\nu \mathcal{R}'_\nu(x', p') - \mathcal{R}_\mu(x, p) = 0$$

$$\Rightarrow \quad \delta_n f_{(1)} = f'_{(0)} \frac{1}{2}\left(\Sigma^{\nu\rho}_{n'} - \Sigma^{\nu\rho}_n\right)\nabla_\nu \beta_\rho$$

(up to frame-independent part) $\quad f_{(1)} = f'_{(0)} \frac{1}{2} \Sigma^{\mu\nu}_n \nabla_\mu \beta_\nu$

$$f^{\rm eq} = f^{\rm eq}_{(0)}\left(-\beta\mu + \beta\cdot p + \frac{\hbar}{2}\Sigma^{\mu\nu}_n \nabla_\mu \beta_\nu\right)$$

This satisfies $\quad \delta(p^2)\left[p^\mu D_\mu - \frac{\hbar}{2}\Sigma^{\alpha\beta}_n R_{\mu\nu\alpha\beta} p^\mu \partial^\nu_p + \hbar (D_\nu \Sigma^{\mu\nu}_n) D_\nu\right] f = 0$

Ex.) Rotating Coordinate

$$g_{\mu\nu} = \eta_{\mu\nu}\,, \quad g_{0i} = -v_i = -(\boldsymbol{\omega} \times \boldsymbol{x})_i$$

$$n^\mu = (1, \boldsymbol{x} \times \boldsymbol{\omega})$$

$$f\Big(|\boldsymbol{p}| - \boldsymbol{\omega}\cdot(\boldsymbol{x}\times\boldsymbol{p}) - \hbar\boldsymbol{\omega}\cdot\hat{\boldsymbol{p}}/2\Big)$$

$$\boldsymbol{J}_{\mathrm{eq}} = \int_p \left(\hat{\boldsymbol{p}} - \hbar\frac{\hat{\boldsymbol{p}}}{2|\boldsymbol{p}|} \times \frac{\partial}{\partial \boldsymbol{x}}\right) f_{\mathrm{eq}} = \hbar\boldsymbol{\omega}\left(\frac{\mu^2}{4\pi^2} + \frac{T^2}{12}\right)$$ **CVE**

$$n^\mu = (1, 0, 0, 0)$$

Coriolis force

$$\boldsymbol{J}_{\mathrm{eq}} = \int_p \frac{\hbar}{2|\boldsymbol{p}|^2}\, 2|\boldsymbol{p}|\boldsymbol{\omega}\, f_{\mathrm{eq}} = \hbar\boldsymbol{\omega}\left(\frac{\mu^2}{4\pi^2} + \frac{T^2}{12}\right)$$ **CVE**

cf. chiral magnetic effect $2|\boldsymbol{p}|\boldsymbol{\omega} \leftrightarrow \boldsymbol{B}$ Stephanov, Yin (2012)

$$\boldsymbol{J}_{\mathrm{eq}} = \int_p \frac{\hbar}{2|\boldsymbol{p}|^2}\boldsymbol{B}\, f_{\mathrm{eq}} = \hbar\boldsymbol{B}\frac{\mu}{4\pi^2}$$

Ex.) Rotating Coordinate

$$g_{\mu\nu} = \eta_{\mu\nu}\,, \quad g_{0i} = -v_i = -(\boldsymbol{\omega} \times \boldsymbol{x})_i$$

$$n^\mu = (1, \boldsymbol{x} \times \boldsymbol{\omega})$$

$$f\Big(|\boldsymbol{p}| - \boldsymbol{\omega}\cdot(\boldsymbol{x}\times\boldsymbol{p}) - \hbar\boldsymbol{\omega}\cdot\hat{\boldsymbol{p}}/2\Big)$$

orbital spin

$$\boldsymbol{J}_{\rm eq} = \int_p \left(\hat{\boldsymbol{p}} - \hbar\frac{\hat{\boldsymbol{p}}}{2|\boldsymbol{p}|} \times \frac{\partial}{\partial \boldsymbol{x}}\right) f_{\rm eq} = \hbar\boldsymbol{\omega}\left(\frac{\mu^2}{4\pi^2} + \frac{T^2}{12}\right)$$ **CVE**

$$n^\mu = (1, 0, 0, 0)$$

Coriolis force

$$\boldsymbol{J}_{\rm eq} = \int_p \frac{\hbar}{2|\boldsymbol{p}|^2}\, 2|\boldsymbol{p}|\boldsymbol{\omega}\, f_{\rm eq} = \hbar\boldsymbol{\omega}\left(\frac{\mu^2}{4\pi^2} + \frac{T^2}{12}\right)$$ **CVE**

orbital

frame-dependently decomposed

$$J = L + S$$

but physics unchanged

Ex.) Rotating Coordinate

$$g_{\mu\nu} = \eta_{\mu\nu}\,, \quad g_{0i} = -v_i = -(\boldsymbol{\omega} \times \boldsymbol{x})_i$$

$$n^\mu = (1, \boldsymbol{x} \times \boldsymbol{\omega})$$

$$\left[\frac{\partial}{\partial t} + (\hat{\boldsymbol{p}} + \boldsymbol{x} \times \boldsymbol{\omega}) \cdot \frac{\partial}{\partial \boldsymbol{x}} + (\boldsymbol{p} \times \boldsymbol{\omega}) \cdot \frac{\partial}{\partial \boldsymbol{p}}\right] f = 0$$

$$\ddot{\boldsymbol{x}} = 2\dot{\boldsymbol{x}} \times \boldsymbol{\omega} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{x})$$

Coriolis centrifugal

$$n^\mu = (1, 0, 0, 0)$$

$$2\,|\boldsymbol{p}|\,\boldsymbol{\omega} \leftrightarrow \boldsymbol{B}$$

$$\left[(1 + \hbar\, 2|\boldsymbol{p}|\boldsymbol{\omega} \cdot \boldsymbol{\Omega}_{\boldsymbol{p}})\frac{\partial}{\partial t} + \left\{\boldsymbol{v}_{\boldsymbol{p}} + \hbar(\boldsymbol{v}_{\boldsymbol{p}} \cdot \boldsymbol{\Omega}_{\boldsymbol{p}})2|\boldsymbol{p}|\boldsymbol{\omega}\right\} \cdot \frac{\partial}{\partial \boldsymbol{x}} + (\boldsymbol{v}_{\boldsymbol{p}} \times 2|\boldsymbol{p}|\boldsymbol{\omega}) \cdot \frac{\partial}{\partial \boldsymbol{p}}\right] f = 0$$

$$\boldsymbol{v}_{\boldsymbol{p}} = \frac{\partial \varepsilon_{\boldsymbol{p}}}{\partial \boldsymbol{p}} \qquad \varepsilon_{\boldsymbol{p}} = |\boldsymbol{p}| - \frac{\hbar}{2}\boldsymbol{\omega} \cdot \hat{\boldsymbol{p}}$$

$$|\boldsymbol{p}|\,\boldsymbol{\omega} \leftrightarrow \boldsymbol{B}$$

3. 2nd order CKT in general coordinate

2nd Order CKT

QFT

$$S=\frac{i}{2}\int\sqrt{-g}\,d^4x\,\bar\psi(\gamma^\mu\nabla_\mu\psi+\bar\psi\overleftarrow{\nabla}_\mu\gamma^\mu)\psi$$

CKT

kinetic eq.	$\Delta\cdot\mathcal{R}=\frac{\hbar^2}{24}(\nabla_\rho R_{\mu\nu})\partial_p^\rho\partial_p^\mu\mathcal{R}^\nu$	**current cons.**
constraint	$\Pi\cdot\mathcal{R}=\frac{\hbar^2}{8}R_{\mu\nu}\partial_p^\mu\mathcal{R}^\nu$	**conformal sym.**
constraint	$\frac{\hbar}{2}\varepsilon^{\mu\nu\rho\sigma}\Delta_\rho\mathcal{R}_\sigma+\Pi^\mu\mathcal{R}^\nu-\Pi^\nu\mathcal{R}^\mu=\frac{\hbar^2}{16}R^{\mu\nu\rho\sigma}\partial_\rho^p\mathcal{R}_\sigma$	**AM cons.**

$$\Pi_\mu=p_\mu+\frac{\hbar^2}{24}R^\rho{}_{\sigma\mu\nu}\partial_p^\sigma\partial_p^\nu p_\rho+\frac{\hbar^2}{4}R_{\mu\nu}\partial_p^\nu$$

$$\Delta_\mu=D_\mu-\frac{\hbar^2}{12}(\nabla_\rho R_{\mu\nu})\partial_p^\rho\partial_p^\nu-\frac{\hbar^2}{24}(\nabla_\lambda R^\rho{}_{\sigma\mu\nu})\partial_p^\nu\partial_p^\sigma\partial_p^\lambda p_\rho+\frac{\hbar^2}{8}R^\rho{}_{\sigma\mu\nu}\partial_p^\nu\partial_p^\sigma D_\rho$$

$$D_\mu=\nabla_\mu+\Gamma^\rho_{\mu\nu}p_\rho\partial_p^\nu$$

Solution

$$\mathcal{R}_\mu = 2\pi\delta(p^2)\Big[p_\mu(f_{(0)} + \hbar f_{(1)} + \hbar^2 f_{(2)}) + \hbar\underline{\Sigma^n_{\mu\nu}}D^\nu f_{(0)} + \hbar^2\underline{\Sigma^u_{\mu\nu}}D^\nu f_{(1)}\Big]$$
$$+ 2\pi\hbar^2\frac{1}{p^2}\left[-p_\mu X^\lambda p_\lambda + 2p^\nu\Big(Y_{[\mu}p_{\nu]} + Z_{\alpha\mu\nu}p^\alpha\Big)\right]\delta(p^2)f_{(0)}$$
$$+ 2\pi\hbar^2\frac{\delta(p^2)}{p^2}\left[\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}p^\nu D^\rho\underline{\Sigma_n^{\sigma\lambda}}D_\lambda - \underline{\Sigma^u_{\mu\nu}}\left(\frac{1}{2}\tilde{R}^{\alpha\beta\nu\rho}p_\rho p_\alpha\partial^p_\beta + p\cdot D\underline{\Sigma_n^{\nu\rho}}D_\rho\right)\right]f_{(0)}$$

$$D_\mu = \nabla_\mu + \Gamma^\rho_{\mu\nu}p_\rho\partial^\nu_p\,, \quad \tilde{R}_{\alpha\beta\mu\nu} = \frac{1}{2}R_{\alpha\beta}{}^{\rho\sigma}\varepsilon_{\rho\sigma\mu\nu}\,,$$
$$X_\mu = \frac{1}{8}R_{\mu\nu}\partial^\nu_p + \frac{1}{24}R^\rho{}_{\sigma\mu\nu}\partial^\nu_p\partial^\sigma_p p_\rho\,, \quad Y_\mu = \frac{1}{4}R_{\mu\nu}\partial^\nu_p + \frac{1}{24}R^\rho{}_{\sigma\mu\nu}\partial^\nu_p\partial^\sigma_p p_\rho\,,$$
$$Z_{\alpha\mu\nu} = -\frac{1}{16}R_{\lambda\alpha\mu\nu}\partial^\lambda_p$$

two ambiguities (even in flat spacetime)

$$n^\mu \qquad\qquad u^\mu$$

Lorentz frame **Lorentz frame?**

Side-Jump term at 2nd Order

ex) Minkowski spacetime

$$n^\mu = u^\mu = (1,0,0,0) \qquad \boldsymbol{\Sigma} = (\Sigma^{23}, \Sigma^{31}, \Sigma^{12}) = \frac{\boldsymbol{p}}{2p_0} \sim \frac{\hat{\boldsymbol{p}}}{2}$$

$$\boldsymbol{\mathcal{R}} \sim \hat{\boldsymbol{p}} f + \frac{1}{|\boldsymbol{p}|}\boldsymbol{\nabla} \times \left(\frac{\hbar}{2}\hat{\boldsymbol{p}} f\right) + \frac{\hbar}{2|\boldsymbol{p}|}\boldsymbol{\nabla} \times \left[\frac{1}{|\boldsymbol{p}|}\boldsymbol{\nabla} \times \left(\frac{\hbar}{2}\hat{\boldsymbol{p}} f\right)\right]$$

$$\boldsymbol{j} = \frac{\hbar}{2i|\boldsymbol{p}|}\left[\psi^\dagger \boldsymbol{\nabla} \times \psi - (\nabla \psi^\dagger)\psi\right] - \frac{\hbar}{2|\boldsymbol{p}|}\boldsymbol{\nabla} \times \left(\psi^\dagger \boldsymbol{\sigma} \psi\right)$$

quantum mechanics

$(\hbar/i)\overset{\leftrightarrow}{\boldsymbol{\nabla}} \to \boldsymbol{p}$

$\boldsymbol{\sigma} \to \boldsymbol{S}$

$$\sim \hat{\boldsymbol{p}} f + \frac{1}{|\boldsymbol{p}|}\boldsymbol{\nabla} \times (\boldsymbol{S} f)$$

kinetic theory

Stone (2015)
Fukushima, Pu, Zebin (2018)

$$\boldsymbol{j} \propto \boldsymbol{S} f \qquad \sim \hat{\boldsymbol{p}} f + \frac{1}{|\boldsymbol{p}|}\boldsymbol{\nabla} \times \left(\frac{\hbar}{2}\hat{\boldsymbol{p}} f\right) + \frac{\hbar}{2|\boldsymbol{p}|}\boldsymbol{\nabla} \times \left[\frac{1}{|\boldsymbol{p}|}\boldsymbol{\nabla} \times \left(\frac{\hbar}{2}\hat{\boldsymbol{p}} f\right)\right] + O(\hbar^3)$$

Equilibrium Distribution

Lorentz covariance under $n^\mu \to n'^\mu$ and $u^\mu \to u'^\mu$

$$(\Lambda_n^{-1})_\mu{}^\nu \mathcal{R}'_\nu(x',p') - \mathcal{R}_\mu(x,p) = 0$$

⇨

$$\delta_n f_{(2)}(x',p') = \Sigma^u_{\mu\nu} D^\mu \left[\frac{1}{4} f'_{(0)} \varepsilon^{\rho\sigma\nu\lambda} \left(\frac{n'_\lambda}{p\cdot n'} - \frac{n_\lambda}{p\cdot n} \right) \nabla_\rho \beta_\sigma \right]$$

$$\delta_u f_{(2)}(x',p') = \left(\Sigma^{u'}_{\mu\nu} - \Sigma^u_{\mu\nu} \right) D^\mu \left(f'_{(0)} \frac{\varepsilon^{\rho\sigma\nu\lambda} n_\lambda}{4p\cdot n} \nabla_\rho \beta_\sigma \right)$$

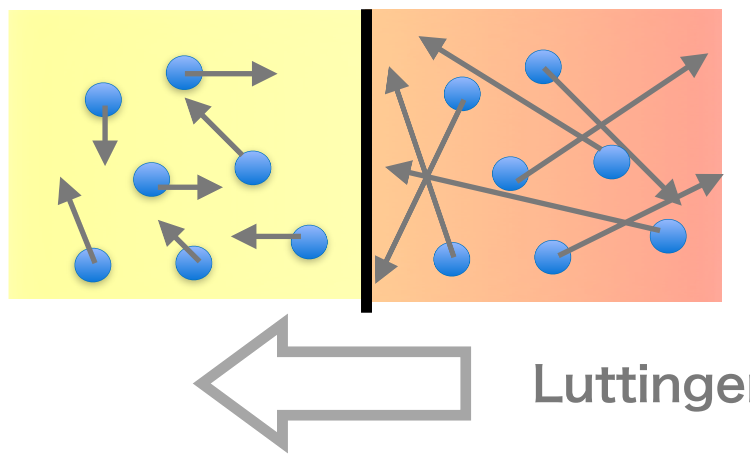
(up to frame-independent part)

$$f_{(2)} = \Sigma^u_{\mu\nu} D^\mu \left(f'_{(0)} \frac{\varepsilon^{\nu\rho\sigma\lambda}}{4\,p\cdot n} n_\rho \nabla_\sigma \beta_\lambda \right)$$

Ex.) Gravito-Electromagnetic Field

$$g_{00} = 1 - 2\varphi\,, \quad g_{ij} = \delta_{ij}\,, \quad g_{0i} = -v_i(\boldsymbol{x})$$

thermal gradient



Luttinger (1964)

gravity

(inhomogeneous) vorticity

$$\boldsymbol{E}_g = -\boldsymbol{\nabla}\varphi = -\frac{\boldsymbol{\nabla}T}{T}$$

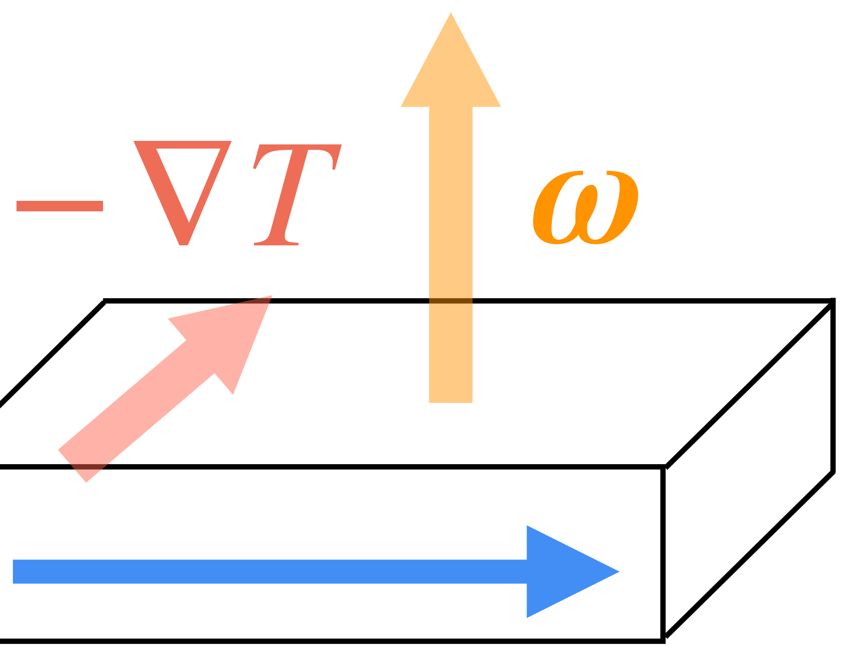
$$\boldsymbol{B}_g = \frac{1}{2}\boldsymbol{\nabla}\times\boldsymbol{v} = \boldsymbol{\omega}$$

Ricci tensor

$$\boldsymbol{R} = (R^1{}_0, R^2{}_0, R^3{}_0) = 3\boldsymbol{E}_g\times\boldsymbol{B}_g + \boldsymbol{\nabla}\times\boldsymbol{B}_g$$

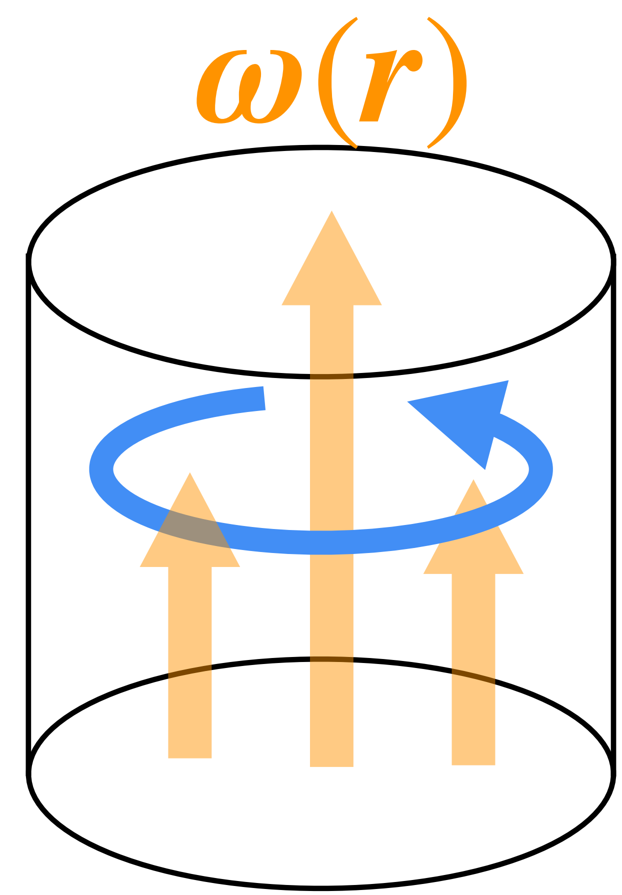
Current

$$\boldsymbol{J}_R = \int_p \boldsymbol{\mathcal{R}} = \hbar^2 \frac{\mu \boldsymbol{R}}{24\pi^2} = \hbar^2 \frac{\mu}{24\pi^2}\left(3\underbrace{\boldsymbol{E}_g}_{-\nabla T} \times \underbrace{\boldsymbol{B}_g}_{\boldsymbol{\omega}} + \boldsymbol{\nabla} \times \underbrace{\boldsymbol{B}_g}_{\boldsymbol{\omega}}\right)$$



$$J_{\mathrm{GR}} \sim -\nabla T \times \boldsymbol{\omega}$$

Gravitational Nernst Effect



$$J_{\mathrm{GR}} \sim \nabla \times \boldsymbol{\omega}$$

Gravitational Ampere's law

Summary & Outlook

[1] The CKT has been extended to general coordinate

The CKT in rot. coord. explains the B-ω duality

The 2nd order CKT in curved spacetime can be solved
(unlike the one in EM backgrounds)

[2] Phenomenological outlook

To describe the dynamics with GR effects
(collisions necessary)

Gravity-induced current can be generated
(practical estimation)

[3] Theoretical outlook

Why two frame vectors emerges

How anomaly is reproduced (Nieh-Yan anomaly?)