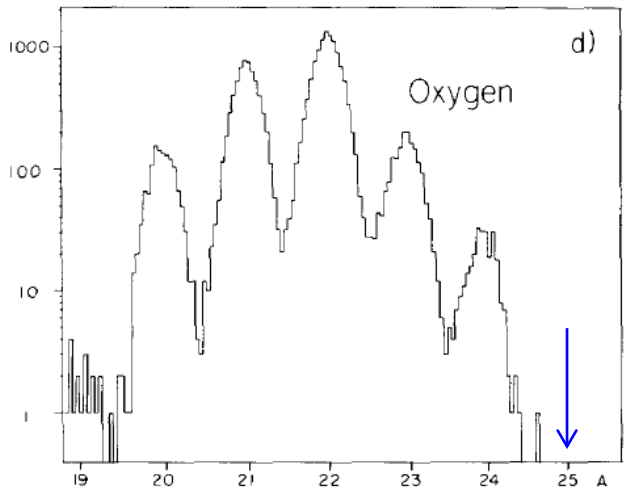
Nuclei outside the drip lines
: properties of one-particle resonances

- Nuclei outside the drip lines
- resonance states
- one proton emission

Drip-line of O isotopes

Oxygen isotopes (Z=8)

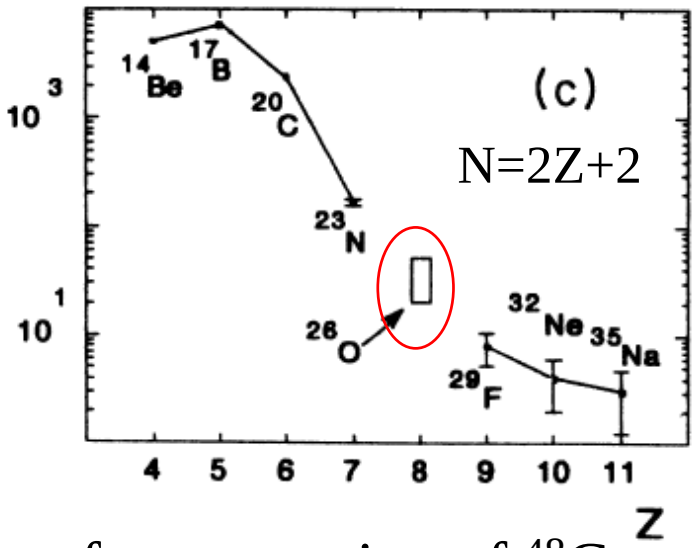
- ✓ stable isotopes: ^{16}O (99.757%), ^{17}O (0.038%), ^{18}O (0.205%)
- ✓ discovery of ^{24}O : A.G. Artukh et al., PL32B (1970) 43



fragmentation of ^{40}Ar

M. Langevin et al.,
PL150B (‘85) 71

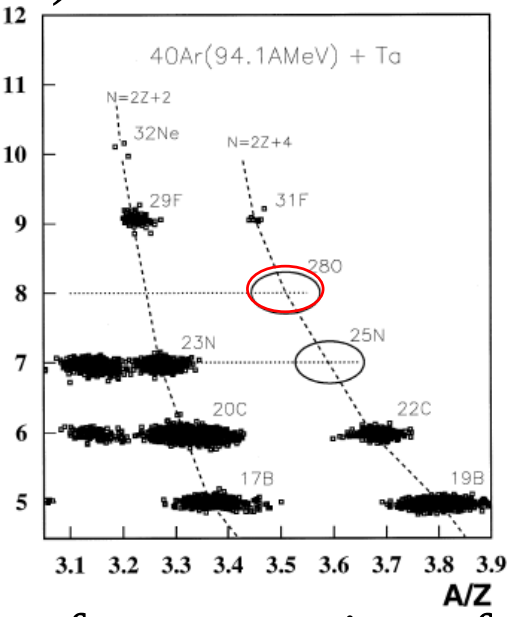
^{25}O : undetected



fragmentation of ^{48}Ca

D. Guillemaud-Mueller et al.,
PRC41 (‘90) 937

^{26}O : undetected



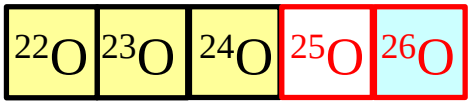
fragmentation of ^{40}Ar

H. Sakurai et al.,
PLB448 (‘99) 180

^{28}O : undetected

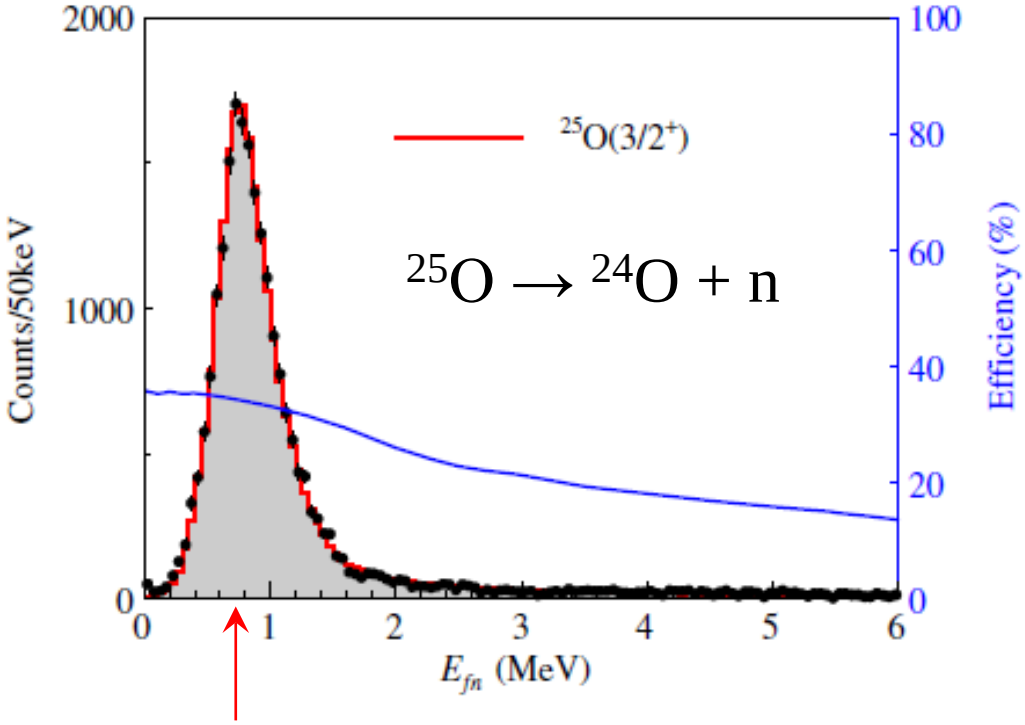
➡ n-drip line of O: ^{24}O ; ^{25,26,28}O → unbound

how can ^{25}O be observed?

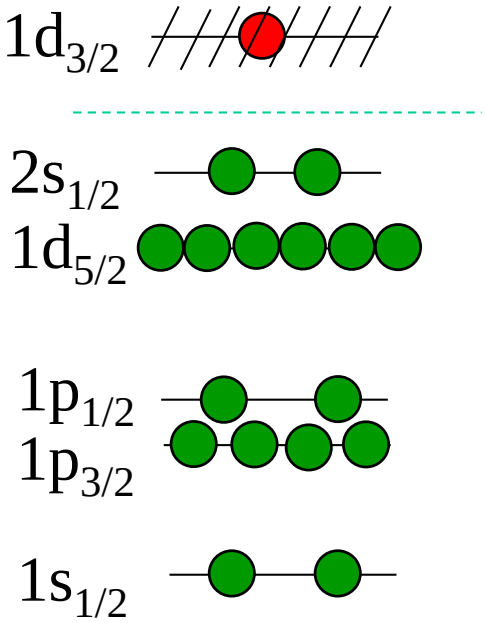


1 proton removal from $^{26}_{9}F_{17} \rightarrow {}^{25}_{8}O_{17}$

→ decay by emitting one neutron



peak

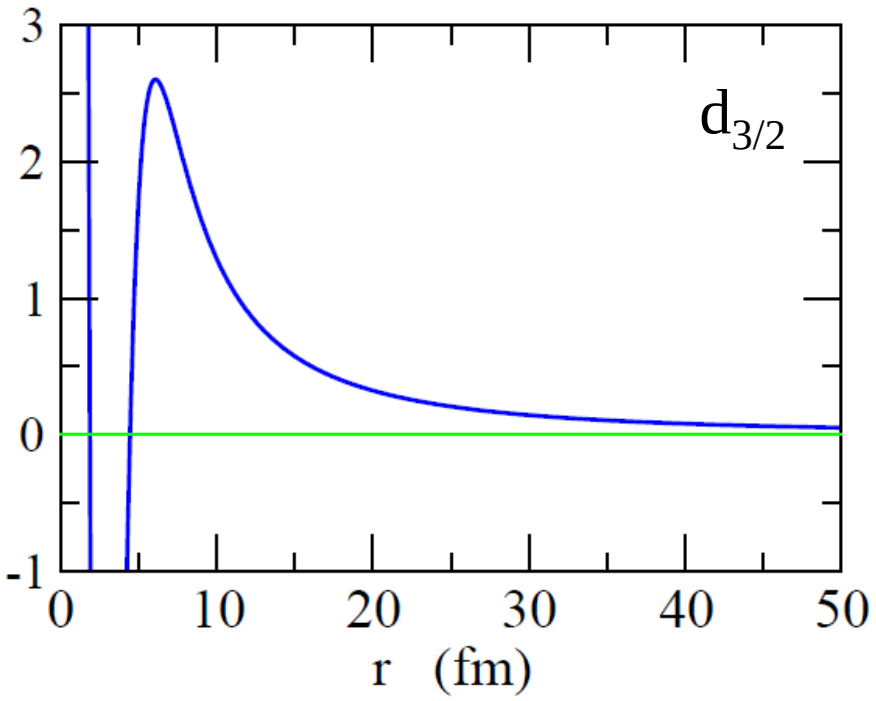


one-particle state of neutrons

Y. Kondo et al., PRL116 (‘16) 102503

this state can be interpreted as a “quasi-bound” state with $1d_{3/2}$

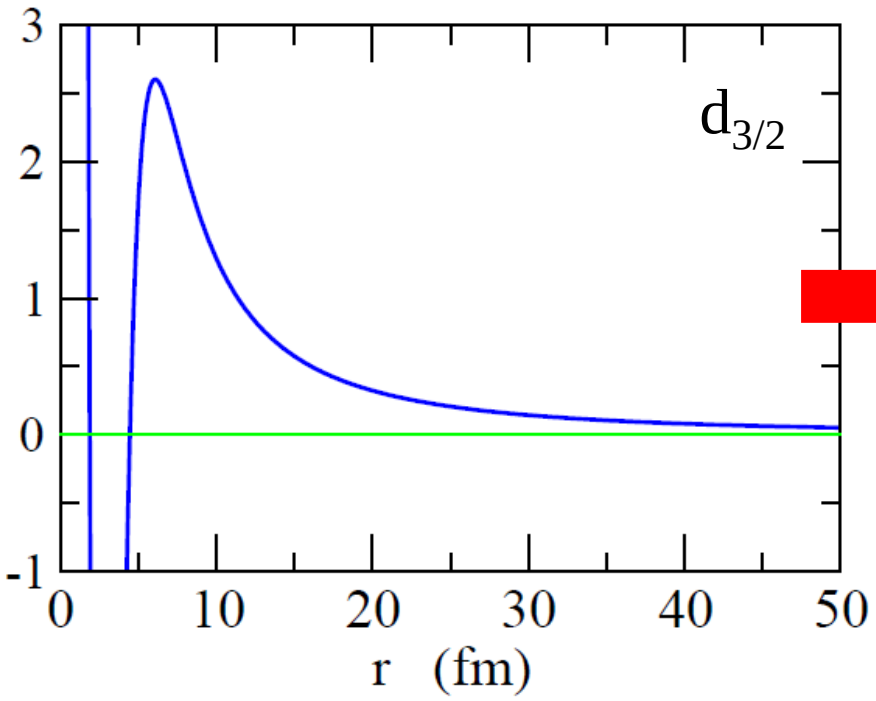
what is a quasi-bound state?

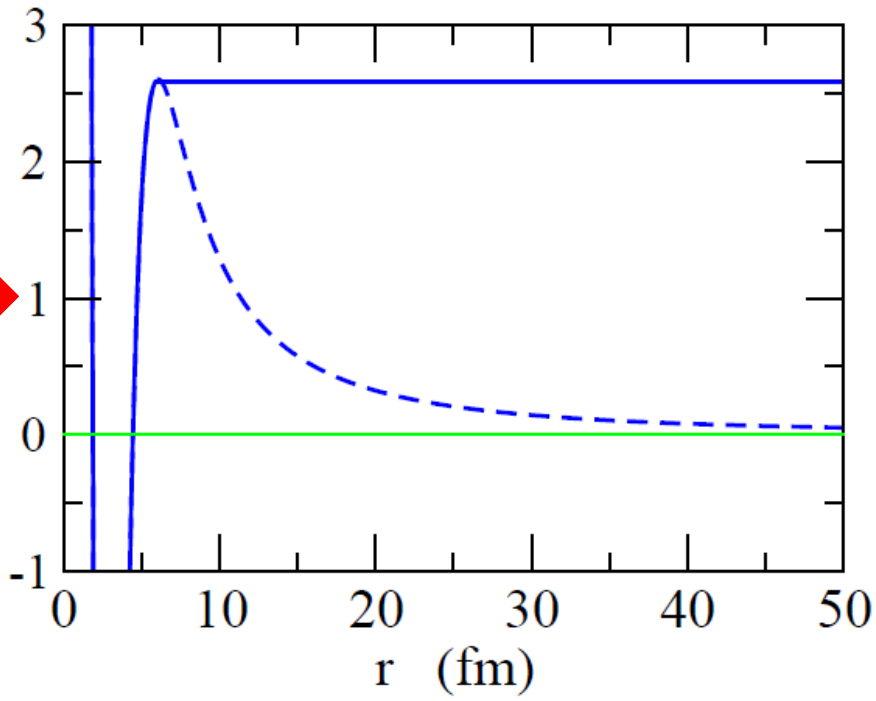


potential energy

a bound state: may exist only in the region of $E < 0$

what is a quasi-bound state?





potential energy

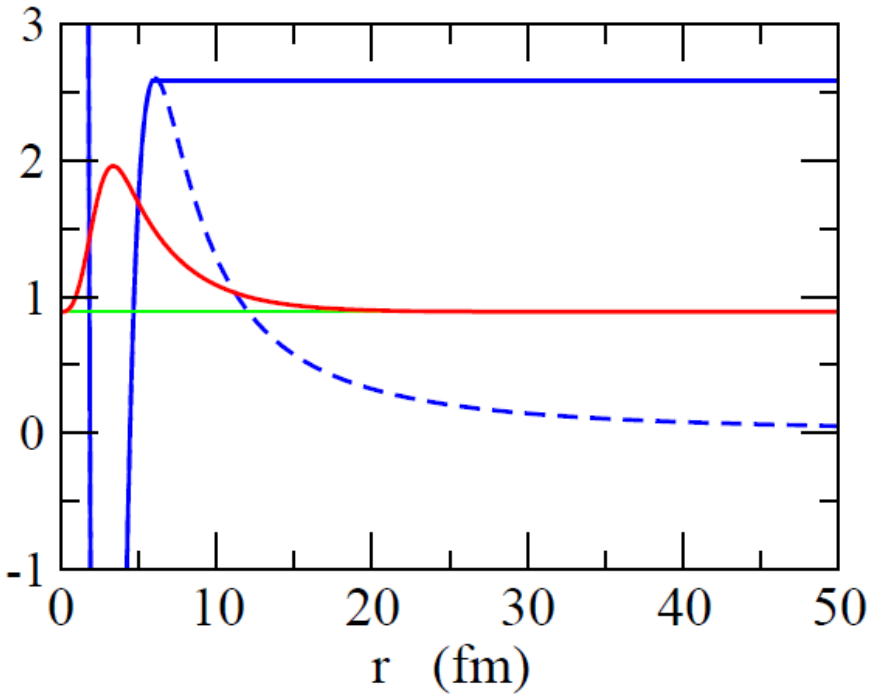
a bound state: may exist only in the region of $E < 0$

if the potential is modified in this way....

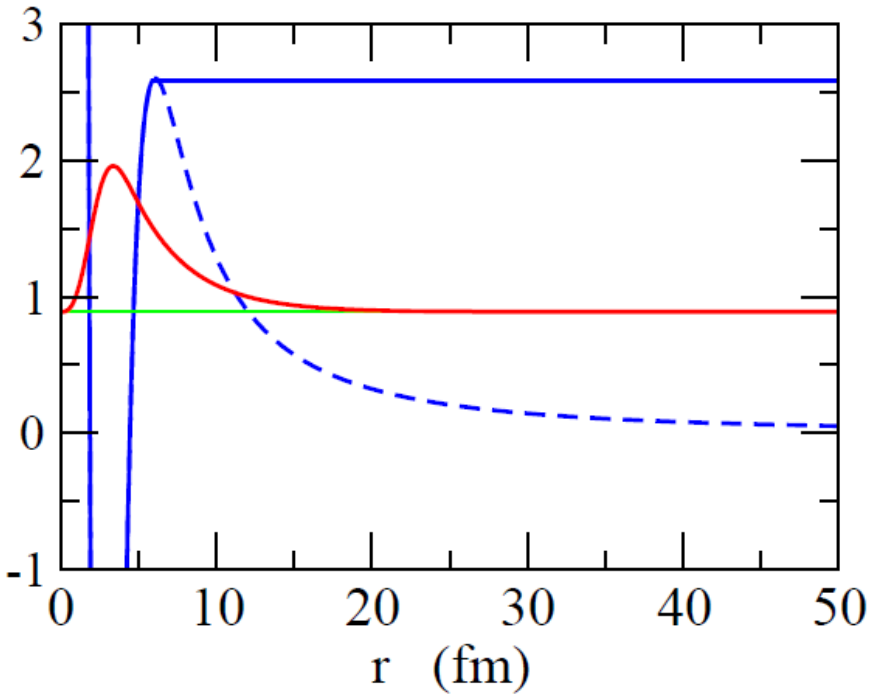
$\longrightarrow$ a bound state with $E > 0$

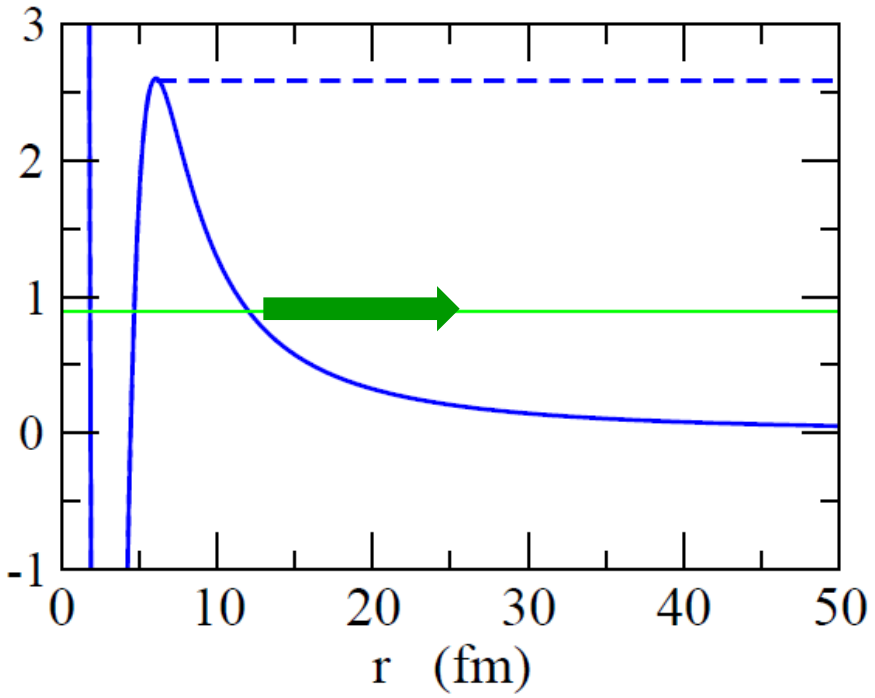
= quasi-bound (quasi-stable) state

what is a quasi-bound state?



what is a quasi-bound state?



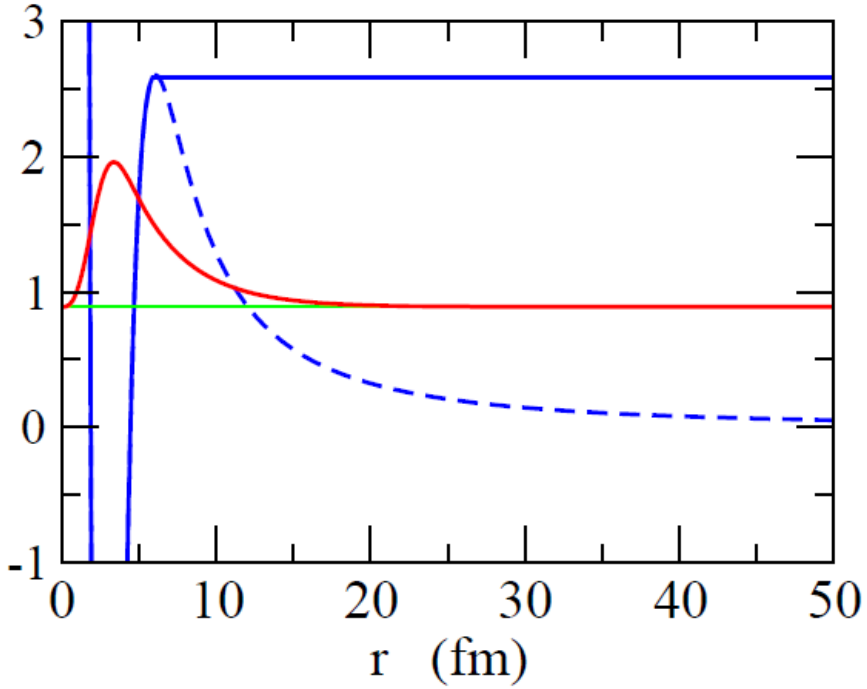


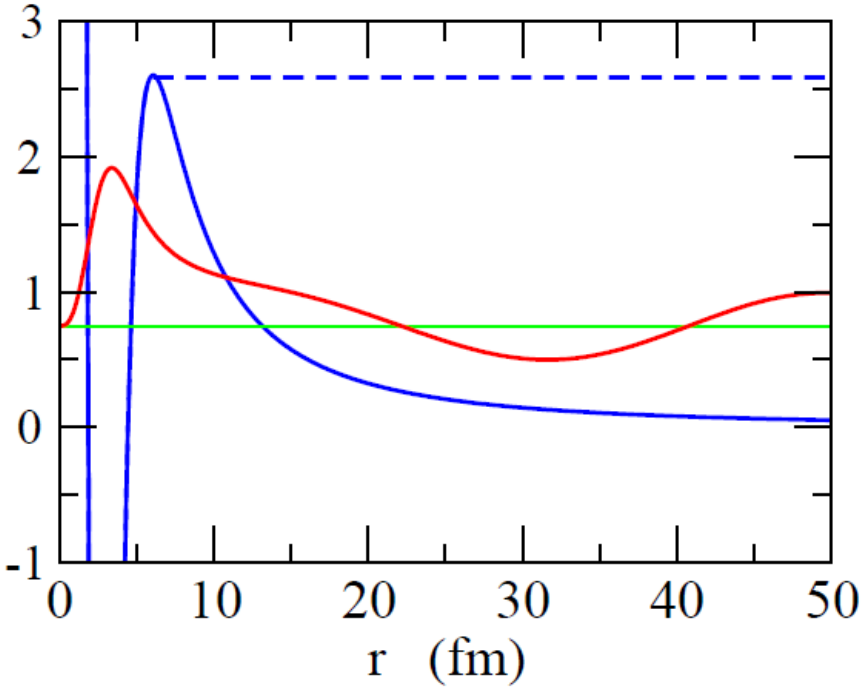
bound state = infinite life-time

In reality, it decays within a finite time due to tunneling

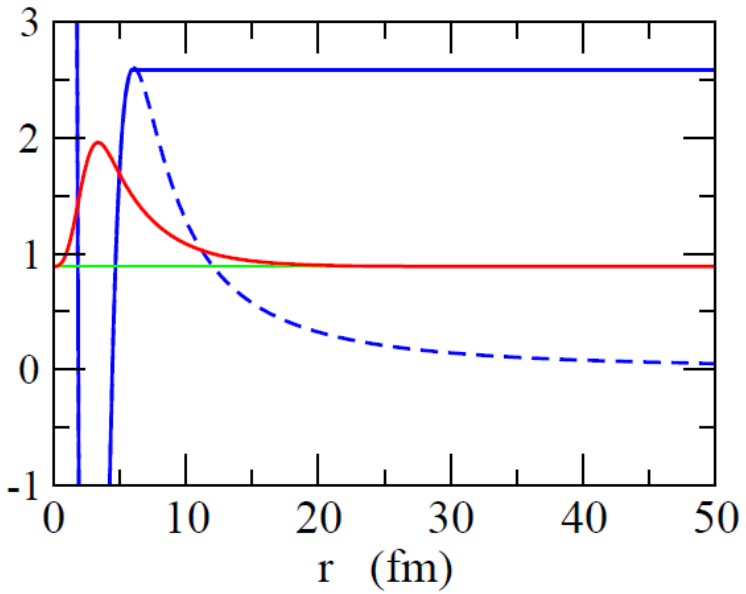
"quasi-bound" state

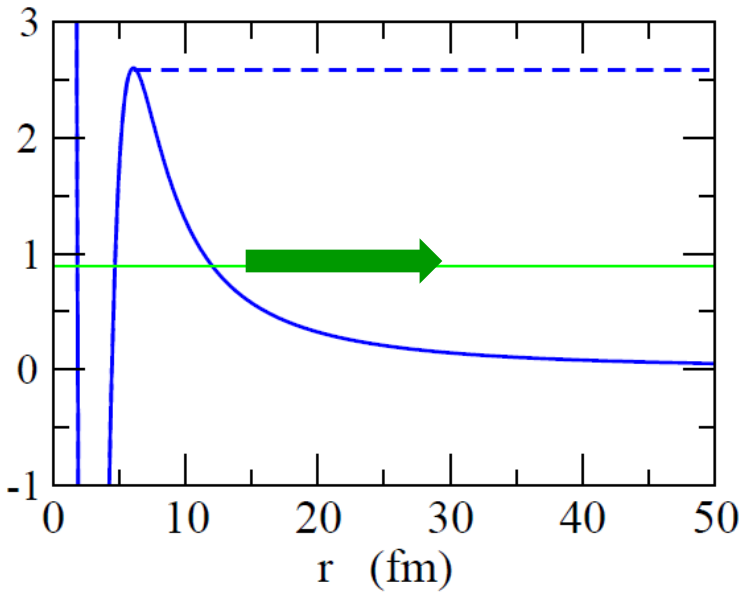
what is a quasi-bound state?





Gamow state





tunnel effect

→decay as an outgoing wave

$$u(r) \sim r^{l+1} \quad (r \to 0)$$
$$\to \mathcal{N} e^{i(kr - l\pi/2)} \quad (r \to \infty)$$

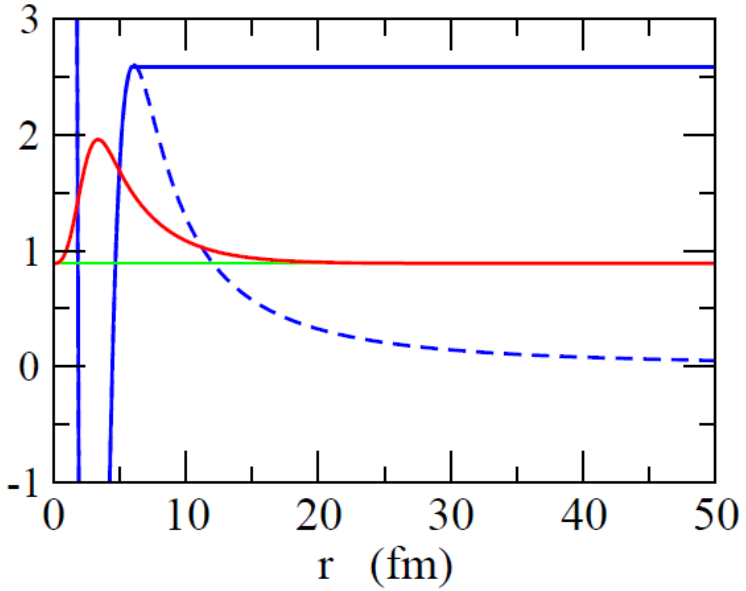
⇨ a complex energy

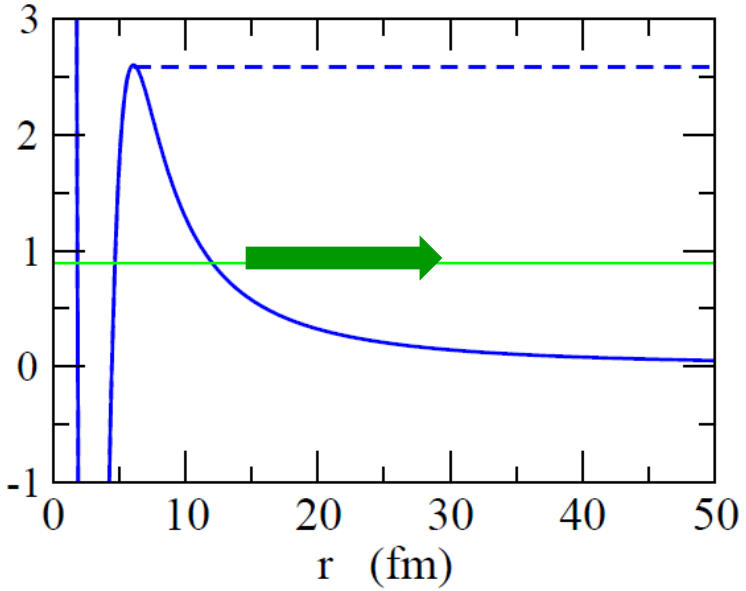
resonance width

$$E \to E_R - i\frac{\Gamma}{2}$$

resonance energy

Gamow state





tunnel effect

→decay as an outgoing wave

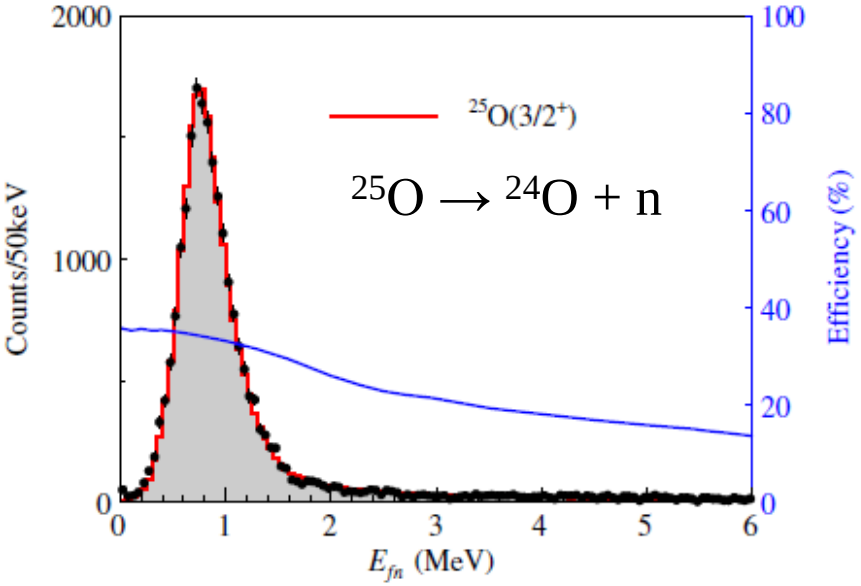
$$E \to E_R - i\frac{\Gamma}{2}$$

$$\begin{aligned} P_{\mathrm{sur}}(t) &\equiv |\langle\psi(0)|\psi(t)\rangle|^2 \\ &= |\langle\psi(0)|e^{-iHt/\hbar}|\psi(0)\rangle|^2 \\ &= |\langle\psi(0)|e^{-i(E_R - i\Gamma/2)t/\hbar}|\psi(0)\rangle|^2 \\ &= e^{-\Gamma t/\hbar} \end{aligned}$$

$\longrightarrow$ $\hbar/\Gamma$: a life time

invariant mass spectroscopy

a proton removal from $^{26}_{9}F_{17}$
$\rightarrow$ $^{25}_{8}O_{17}$ $\rightarrow$ decay by emitting a neutron



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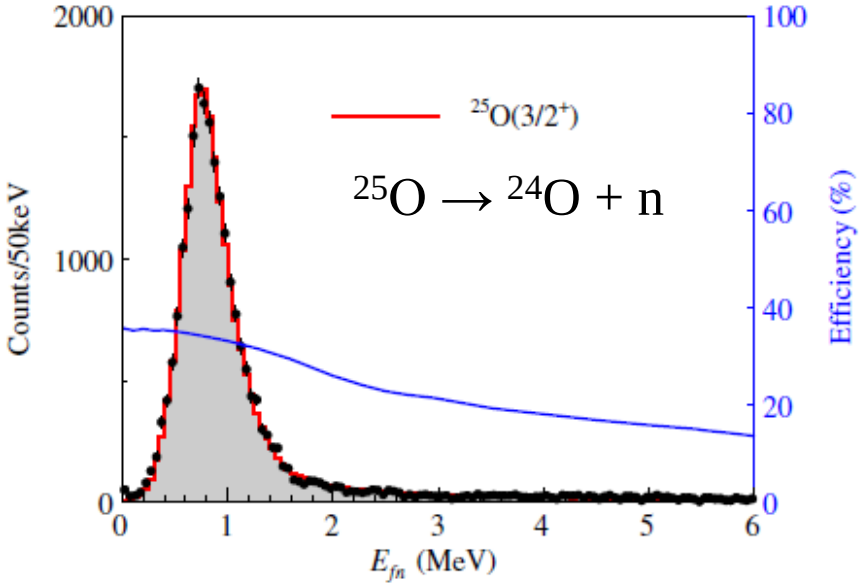

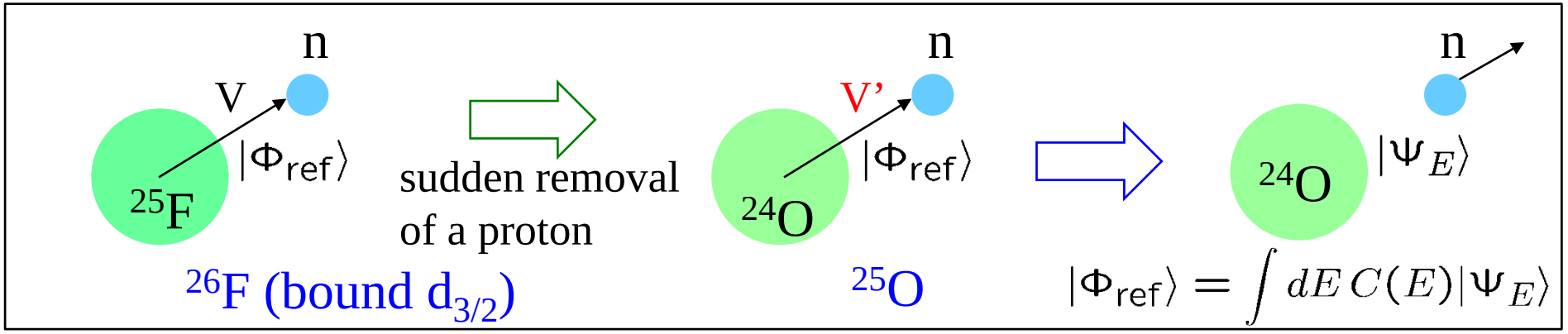

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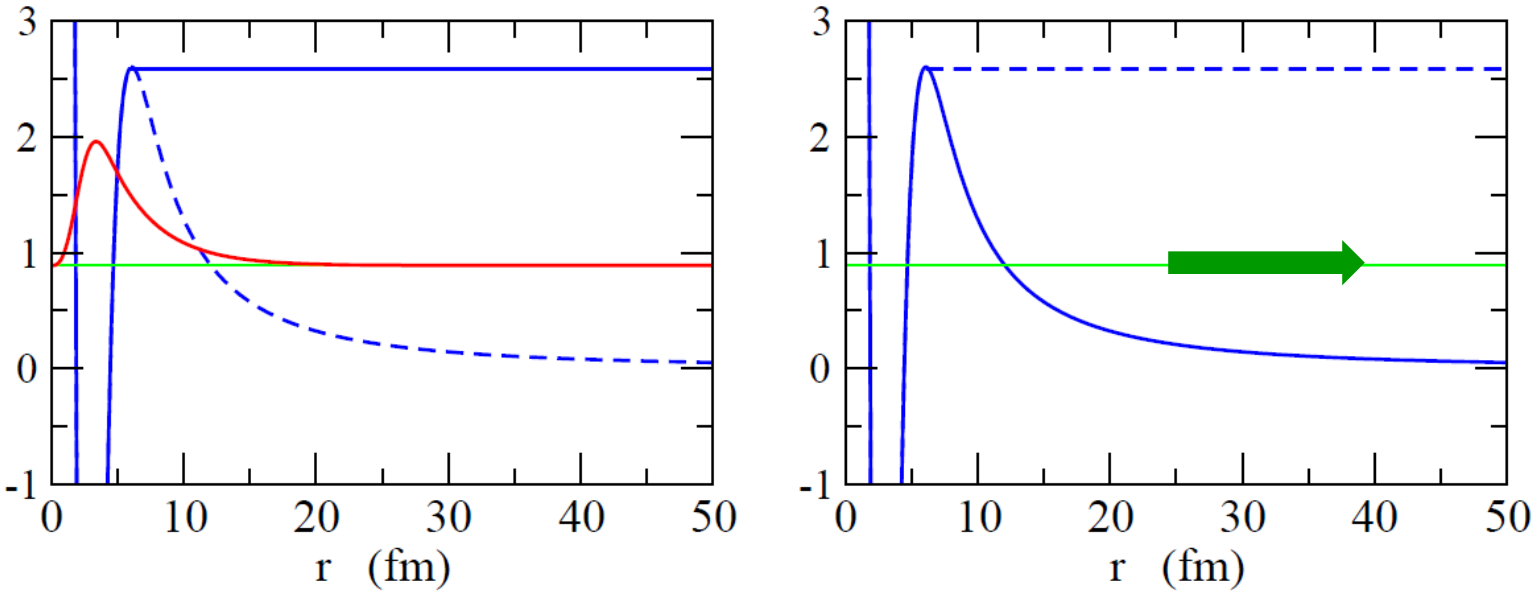

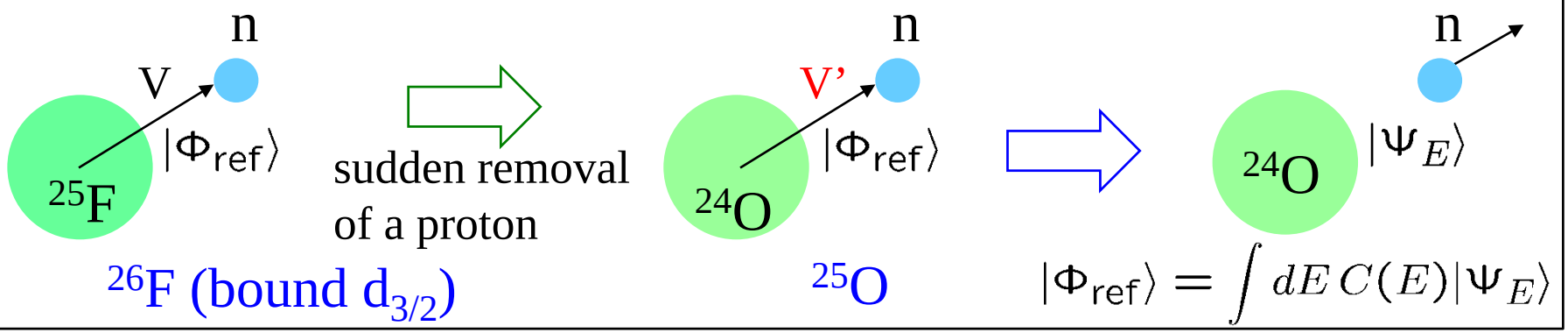

invariant mass spectroscopy

a proton removal from $^{26}_{9}F_{17}$
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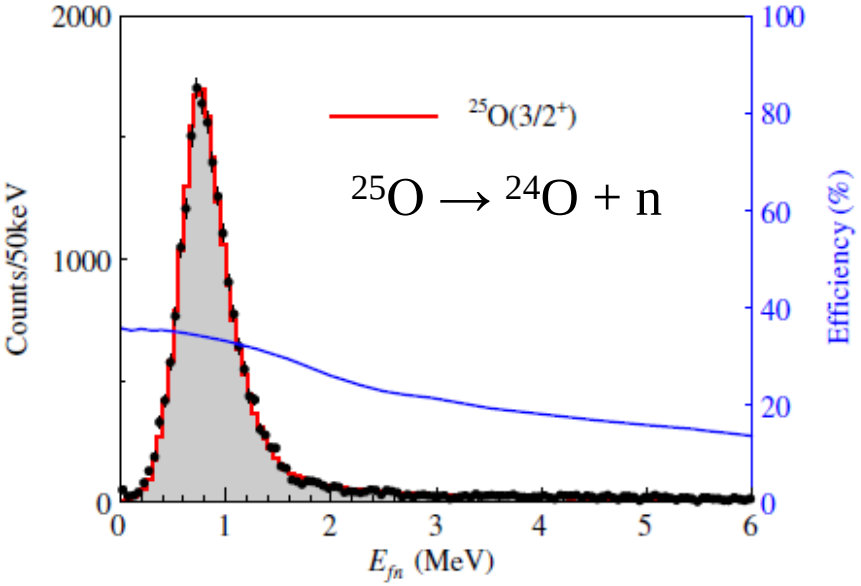
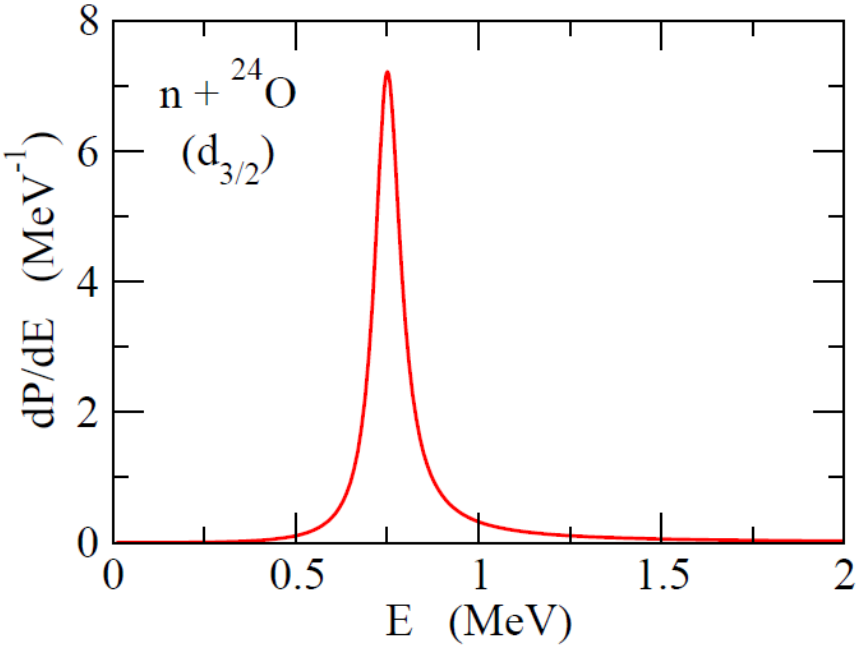
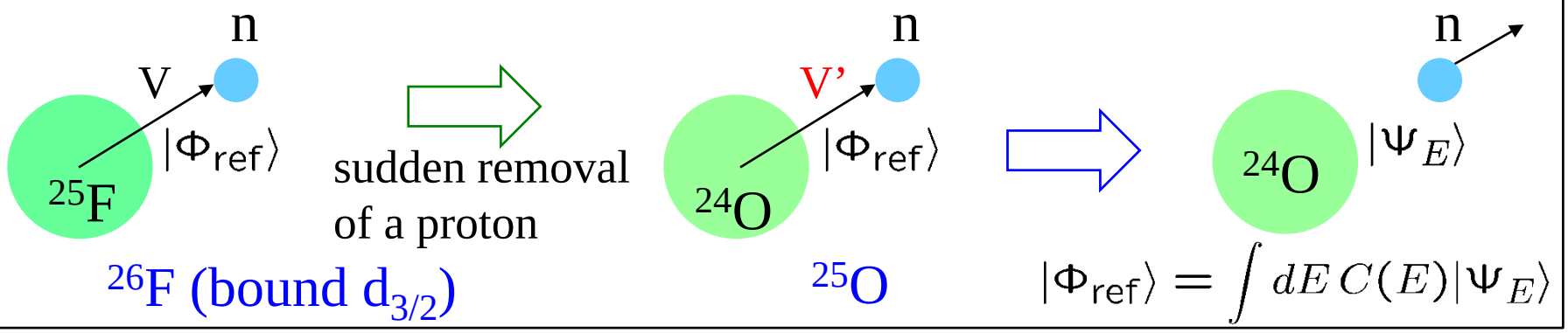
$$\frac{dP}{dE} = |\langle \Phi_{\text{ref}} | \Psi_E \rangle|^2$$







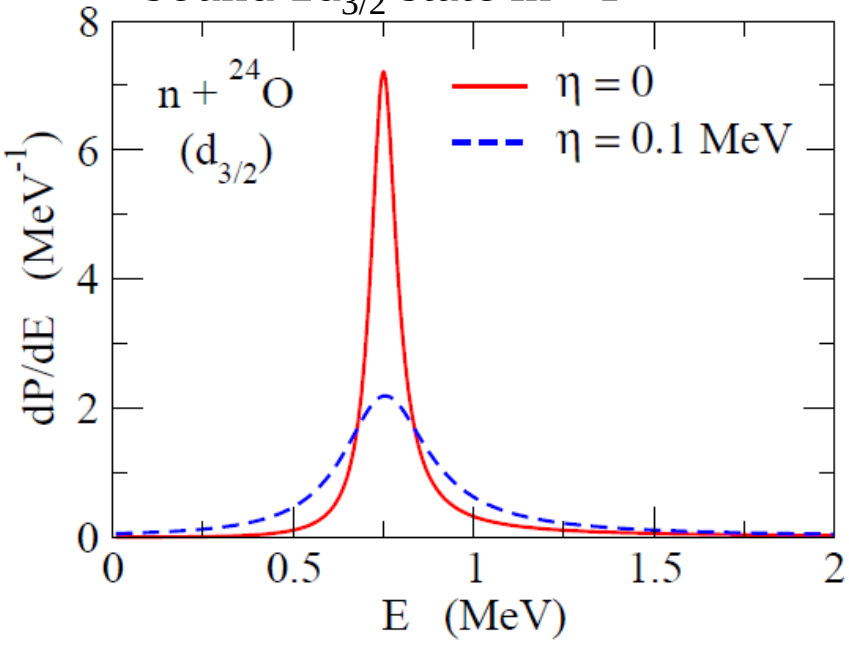
invariant mass spectroscopy

$$\frac{dP}{dE} = |\langle \Phi_{\text{ref}} | \Psi_E \rangle|^2 = \int dE' \, |\langle \Phi_{\text{ref}} | \Psi_{E'} \rangle|^2 \delta(E - E')$$

$$\rightarrow \frac{1}{\pi} Im \int dE' \, |\langle \Phi_{\text{ref}} | \Psi_{E'} \rangle|^2 \frac{1}{E' - E - i\eta}$$

$$= 1 / (H - E - i\eta) = G(E)$$

Reference state:
bound $1d_{3/2}$ state in ^{26}F



$$\lim_{\eta \to 0} \frac{1}{x - i\eta} = P\frac{1}{x} + i\pi\delta(x)$$

calculation with a finite η
(convenient for a numerical calculation)

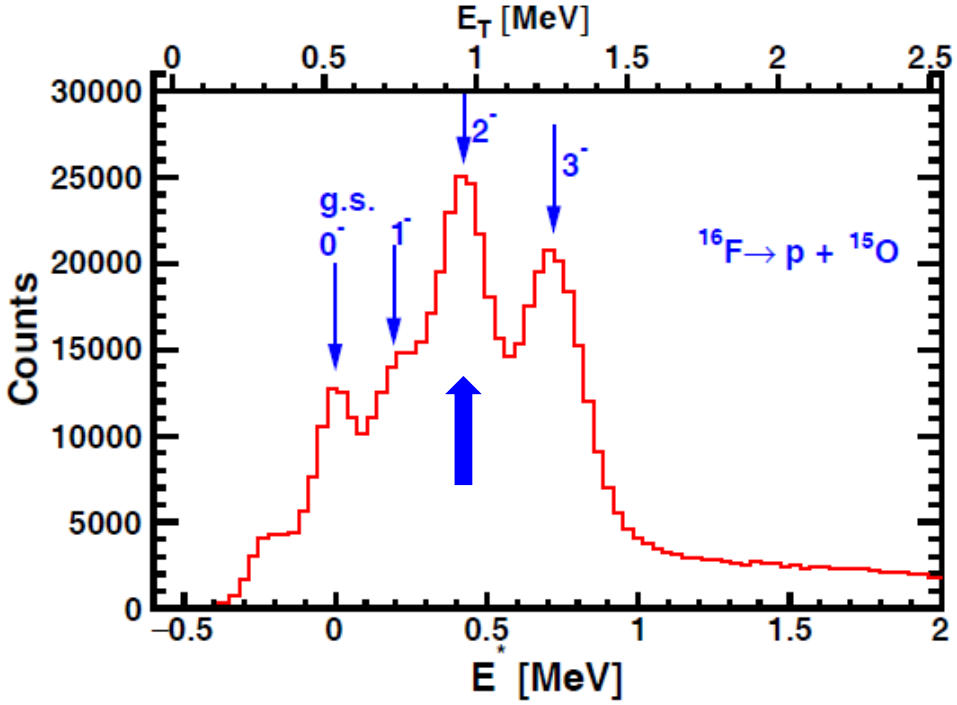
resonance and scattering

let us look at an actual data

proton unbound nucleus $^{16}_{9}F_{7}$

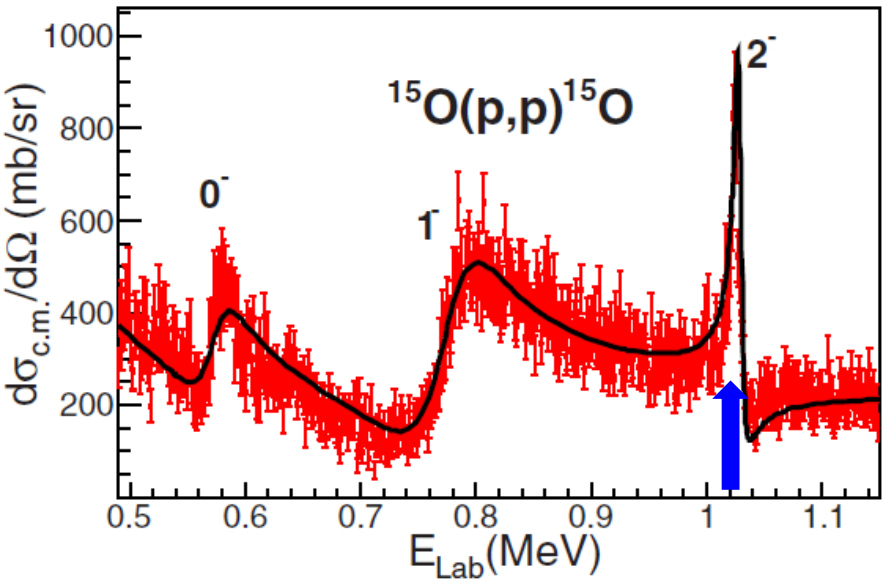
one proton removal from $^{17}_{10}Ne$

$\rightarrow$ $^{16}_{9}F$ $\rightarrow$ decay spectrum



R.J. Charity,
Eur. Phys. J. Plus 131 (‘16) 63

^{15}O + p elastic scattering



θ_{cm} = 180 deg.

I. Stefan et al., PRC90(‘14) 014307

enhancement of el. scat. cross section at res. energies (resonant scattering)

resonance and scattering

next, scattering theory

a free particles :

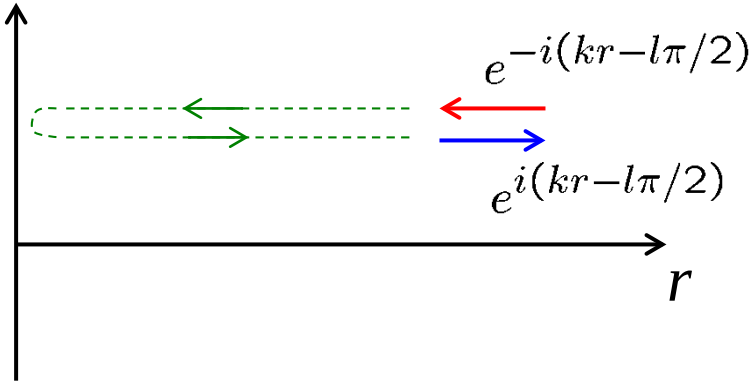
$$\left(-\frac{\hbar^2}{2m}\nabla^2 - E\right)\psi(\boldsymbol{r}) = 0$$

✓solution :

$$\psi(\boldsymbol{r}) \propto j_l(kr)Y_{lm}(\hat{\boldsymbol{r}})\chi_{m_s}$$

✓asymptotic behavior :

$$\begin{aligned} j_l(kr) &\rightarrow \frac{1}{kr}\sin(kr - \frac{l\pi}{2}) \\ &= \frac{-1}{2ikr}\left(e^{-i(kr-l\pi/2)} \right. \\ &\qquad \left. -e^{i(kr-l\pi/2)}\right) \end{aligned}$$



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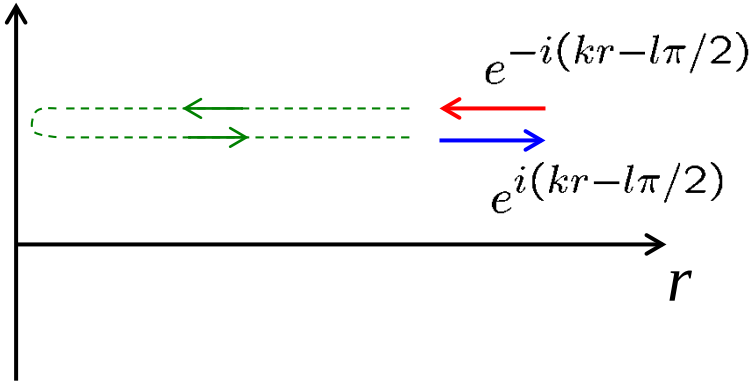

a particle in a potential :

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V(r) - E\right)\psi(\boldsymbol{r}) = 0$$

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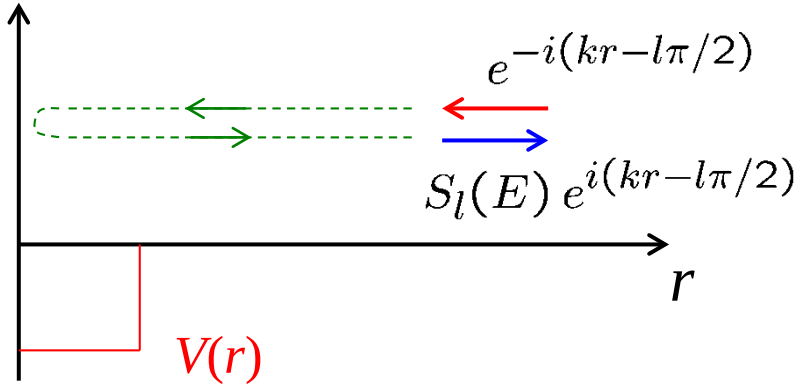
* $|S_l(E)| = 1$ if no absorption



resonance and scattering

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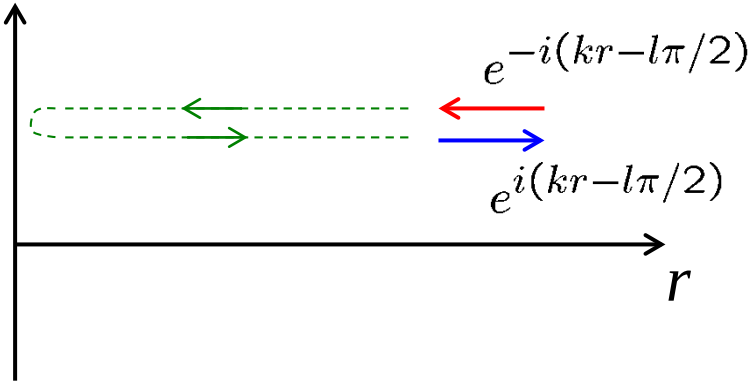

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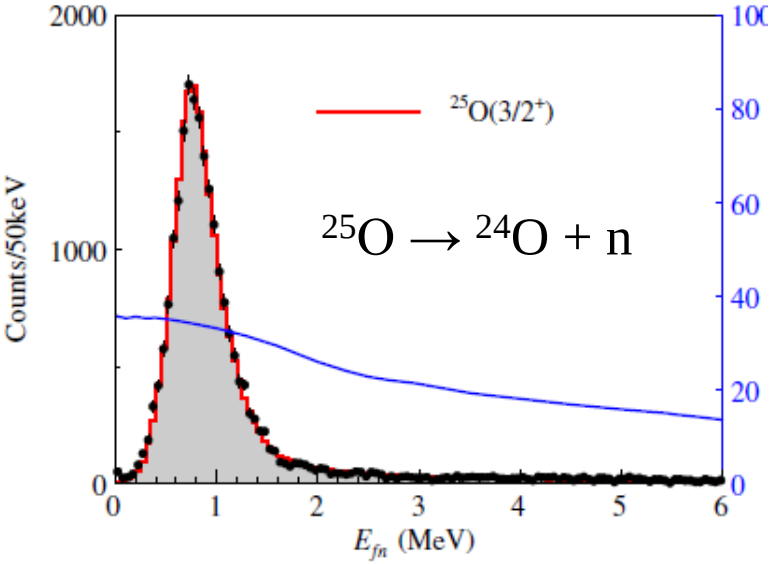
phase shift

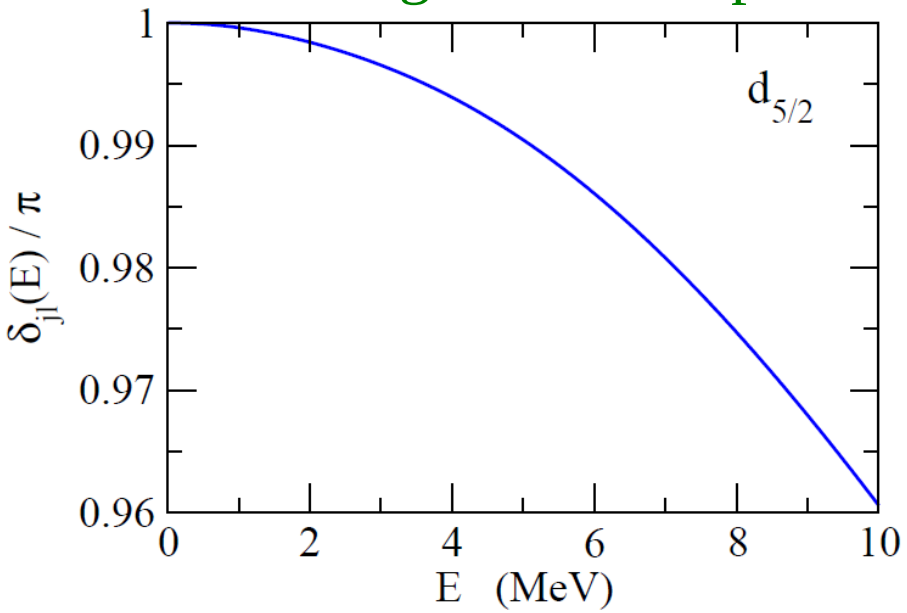
$$S_l(E) = e^{2i\delta_l(E)}$$

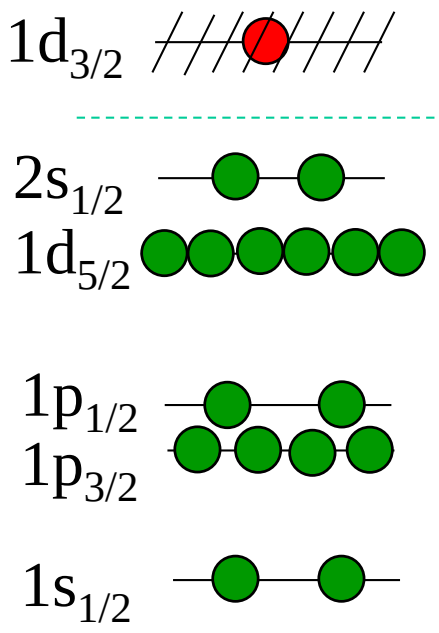
$$R_l(r) \rightarrow -\frac{e^{i\delta_l(E)}}{kr}\sin(kr - l\pi/2 + \delta_l(E))$$

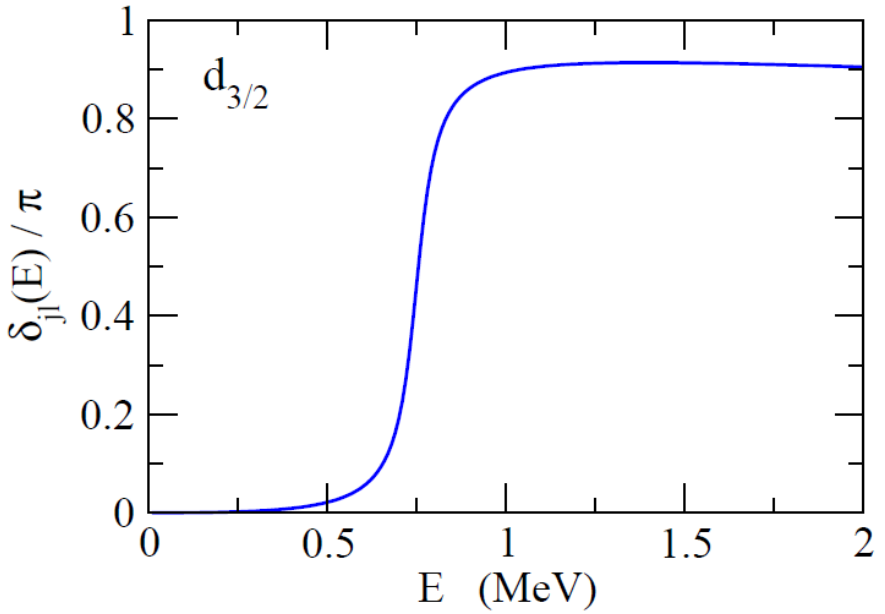
how does the phase shift behave when a resonance exists?



n-^{24}O scattering with a WS potential







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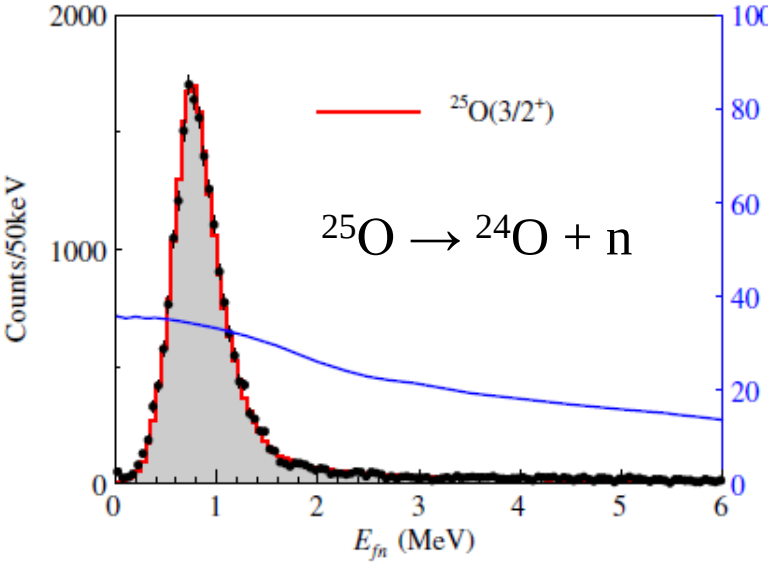
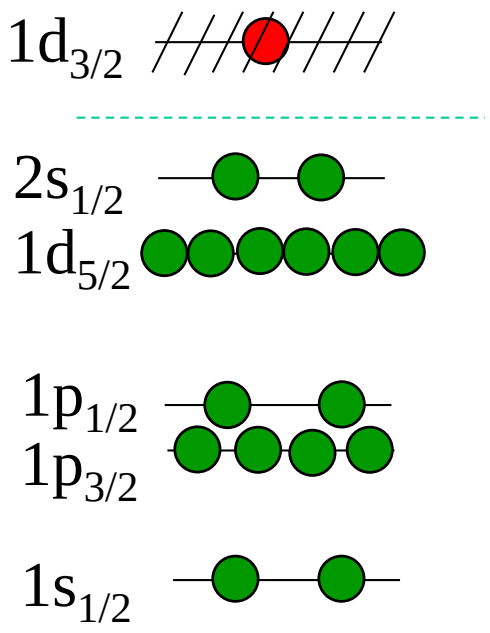
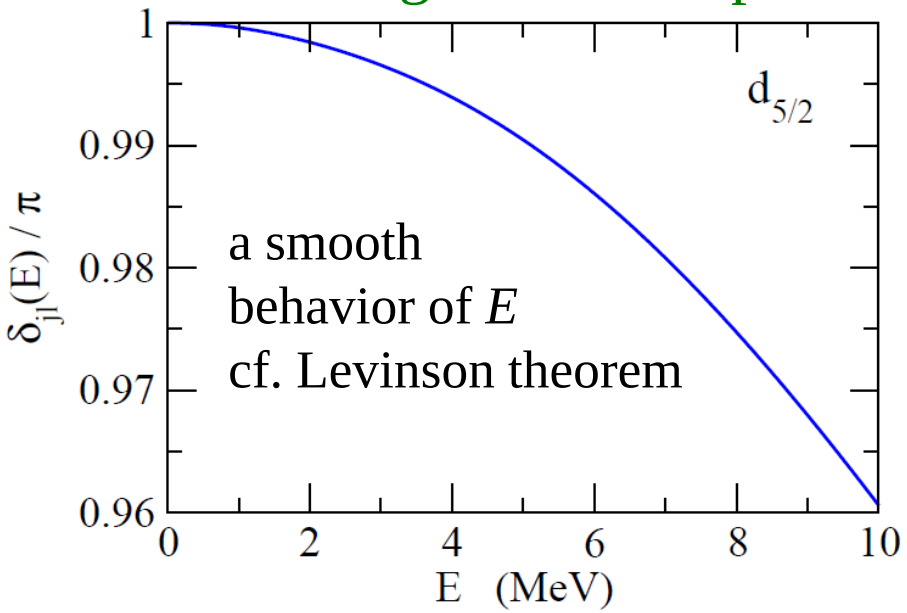
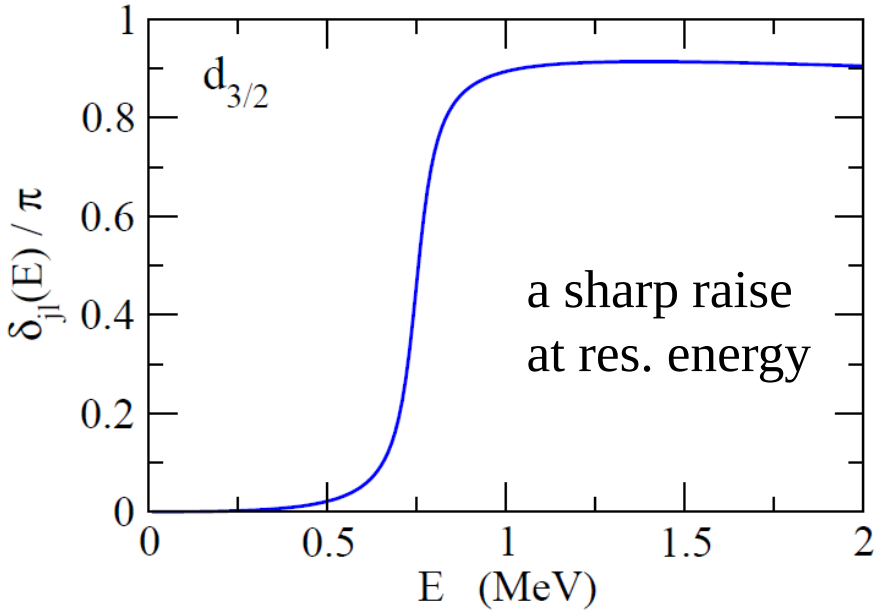


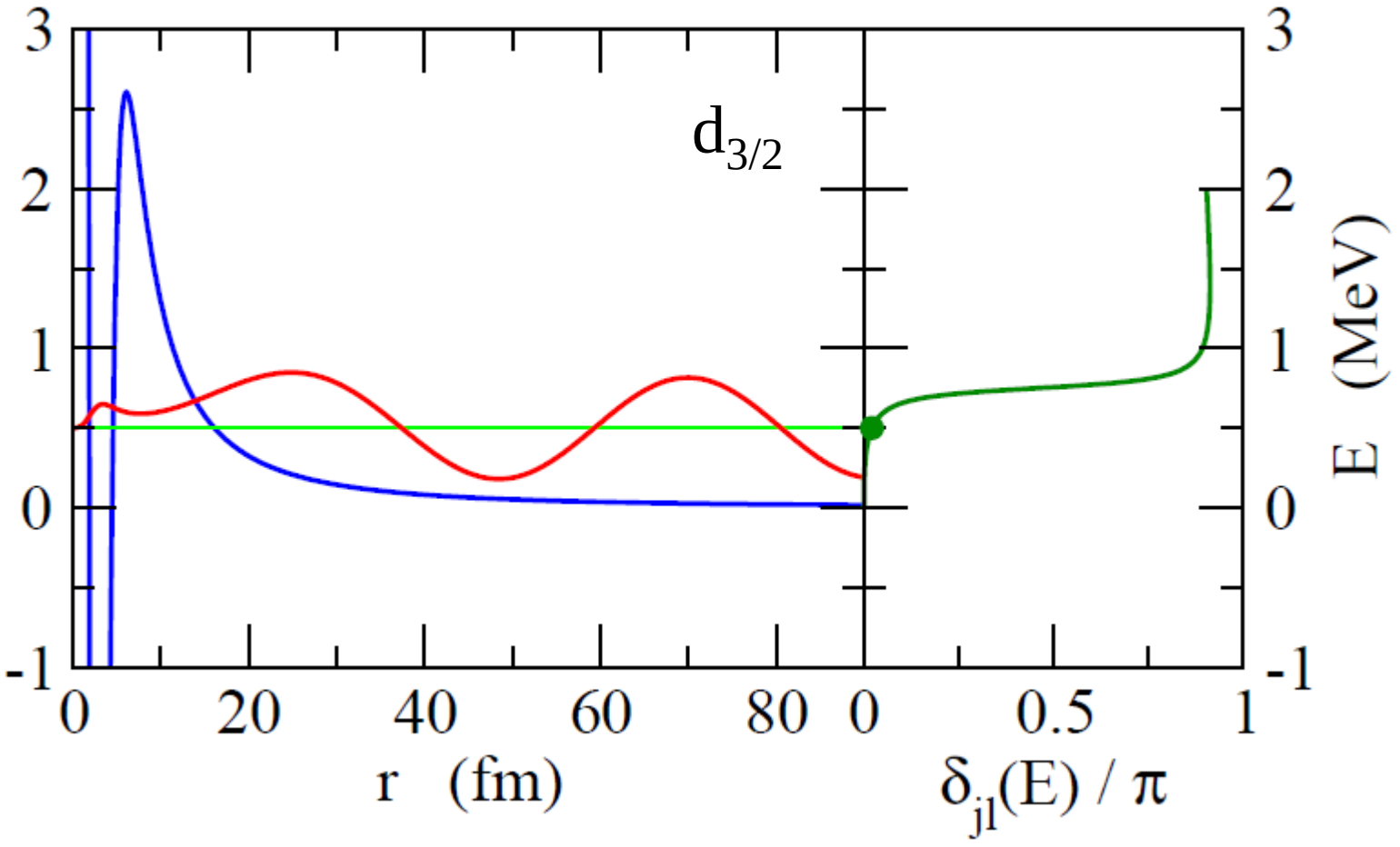


n-^{24}O scattering with a WS potential



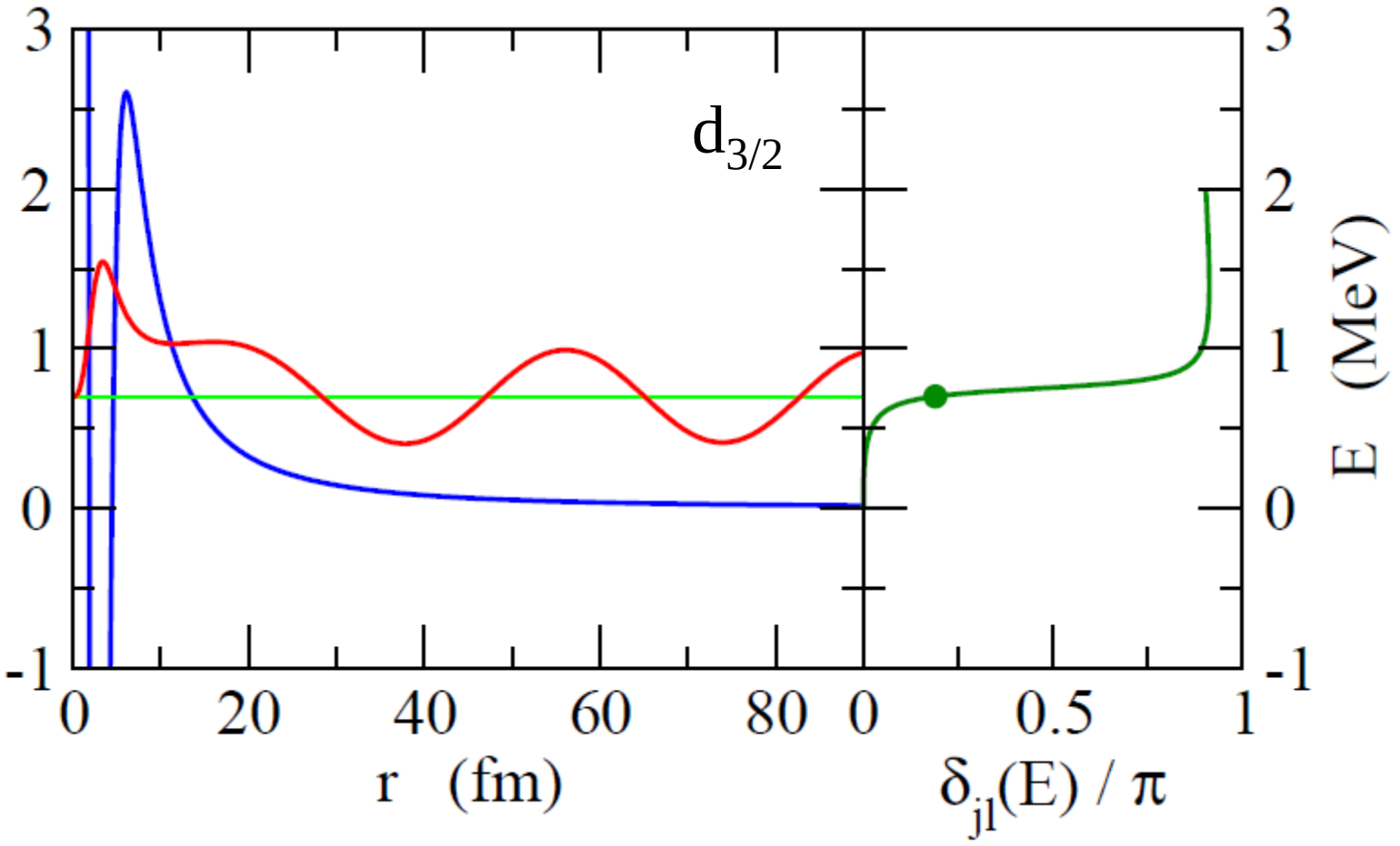


how about the wave function?



$$rR_{jl}(r) \to \sqrt{\frac{2m}{\pi k\hbar^2}} \sin\left(kr - \frac{l\pi}{2} + \delta_{jl}(E)\right); \qquad \int dr\psi_E(\boldsymbol{r})\psi^*_{E'}(\boldsymbol{r}) = \delta(E - E')$$

how about the wave function?



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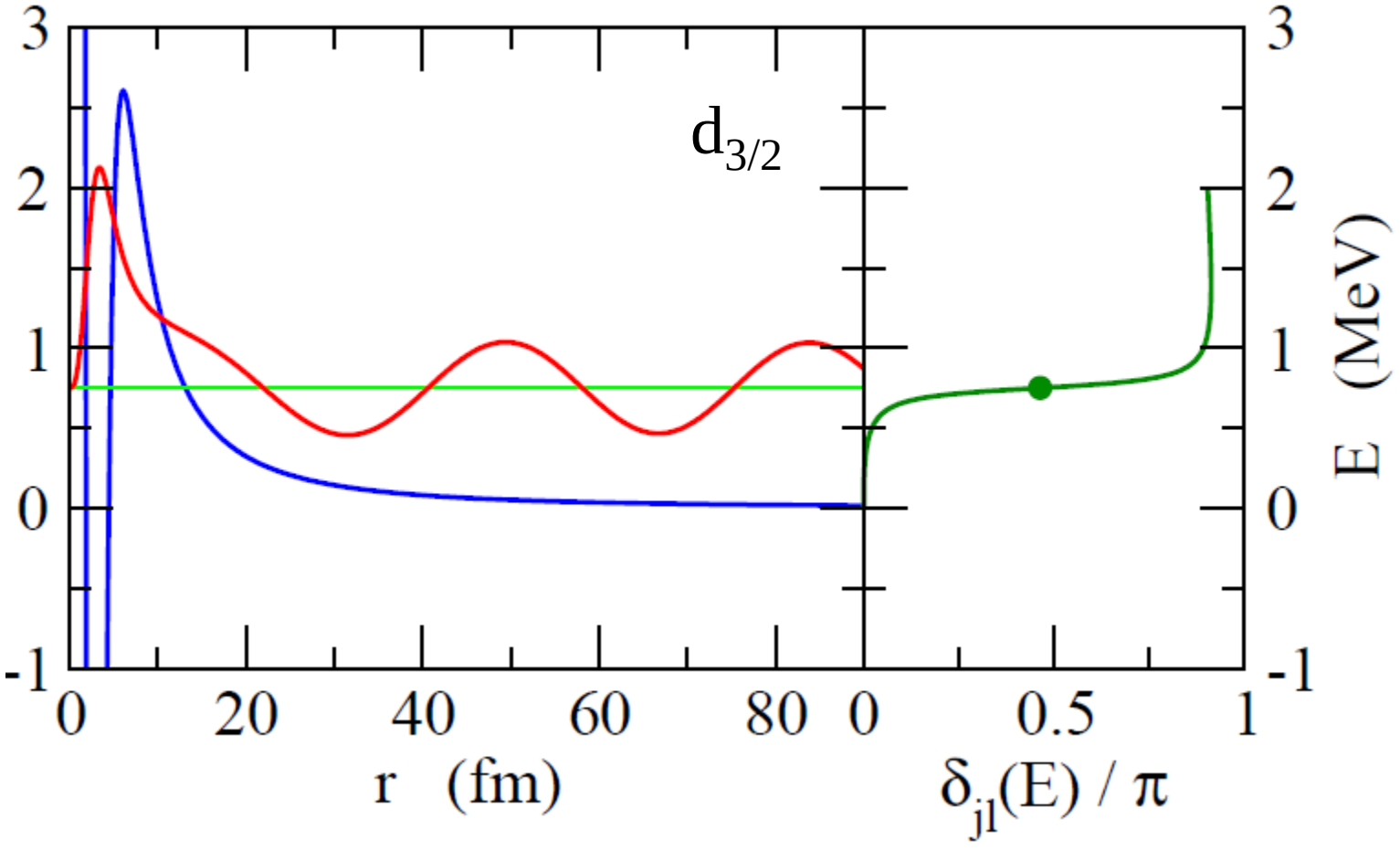

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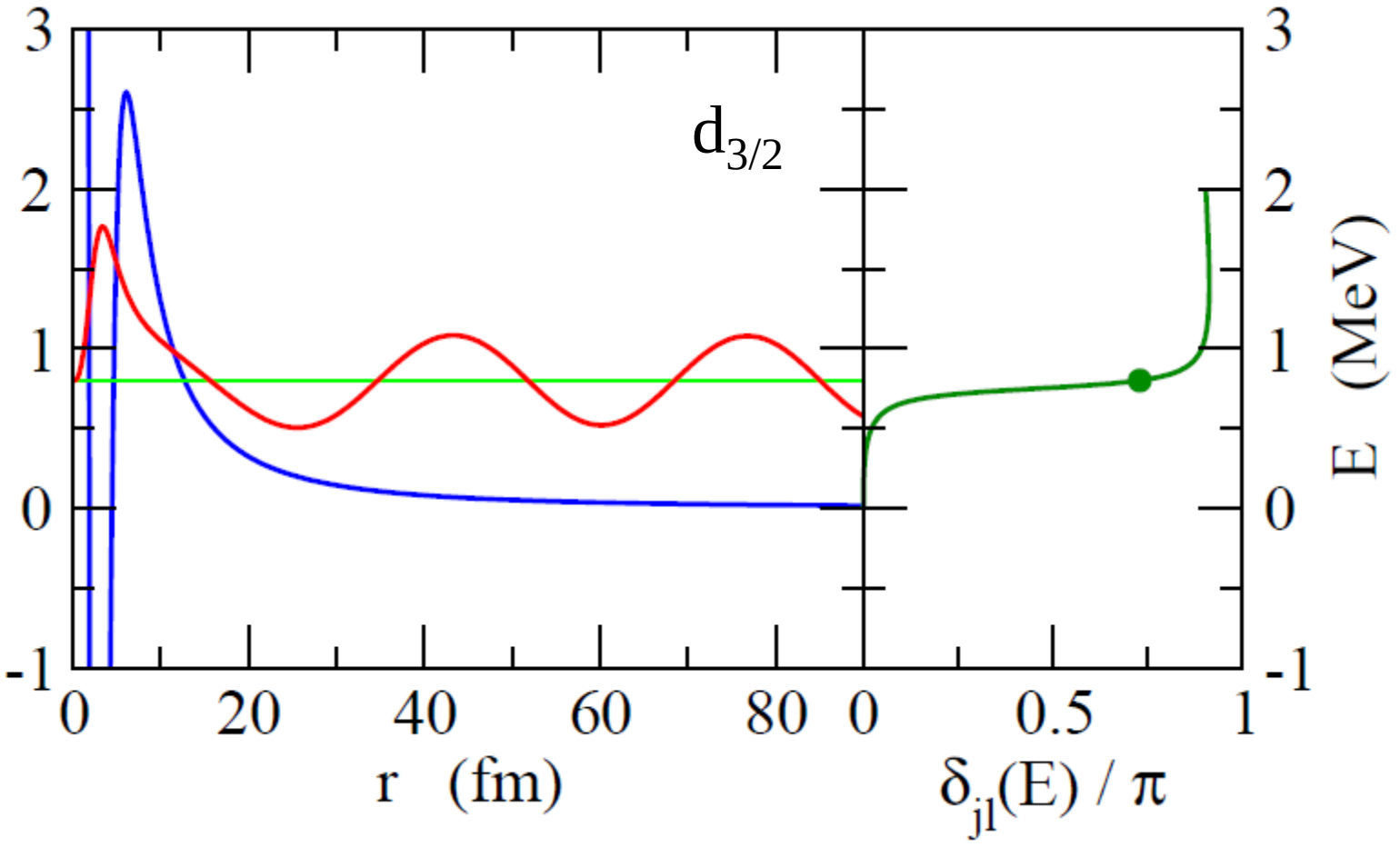

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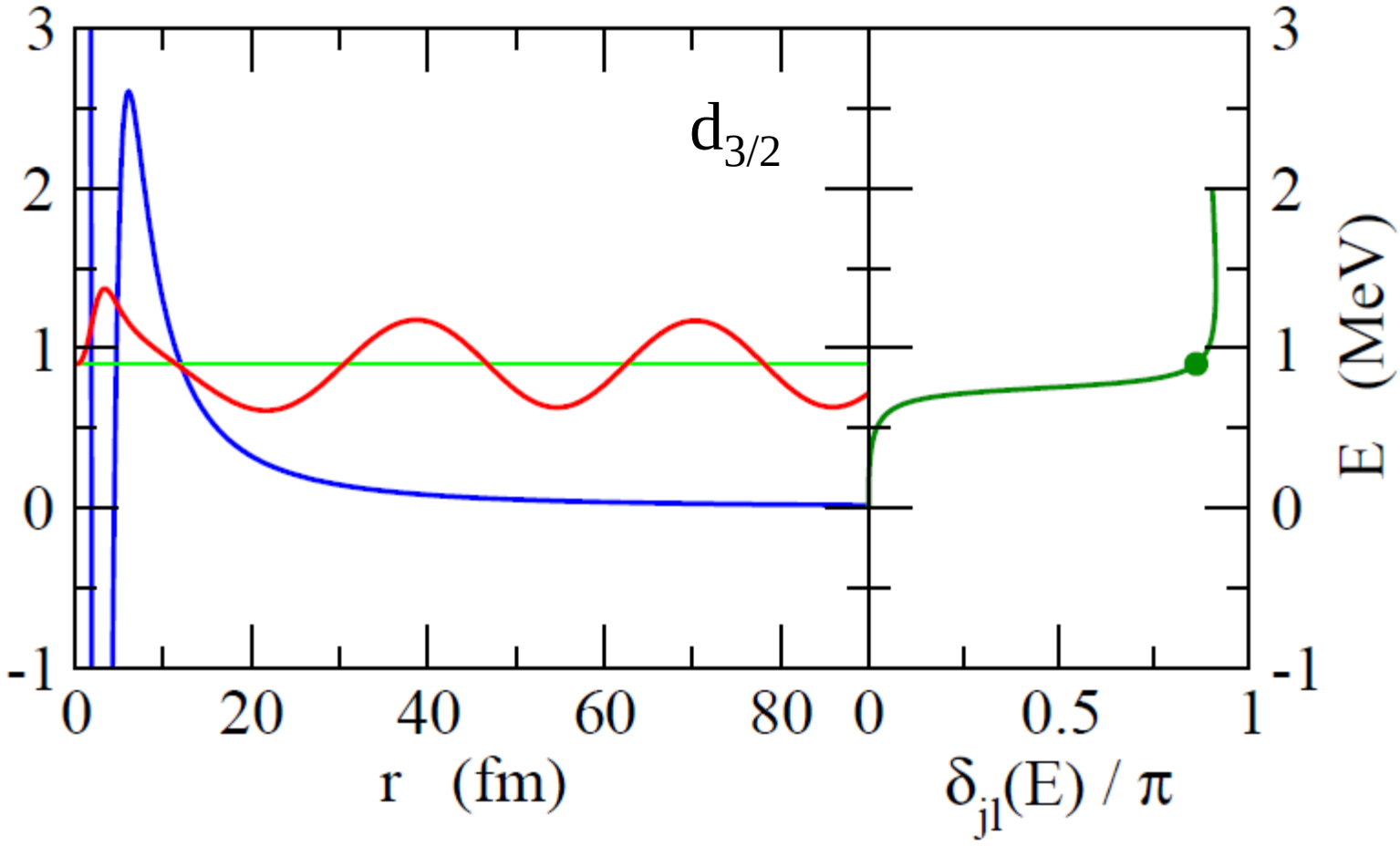

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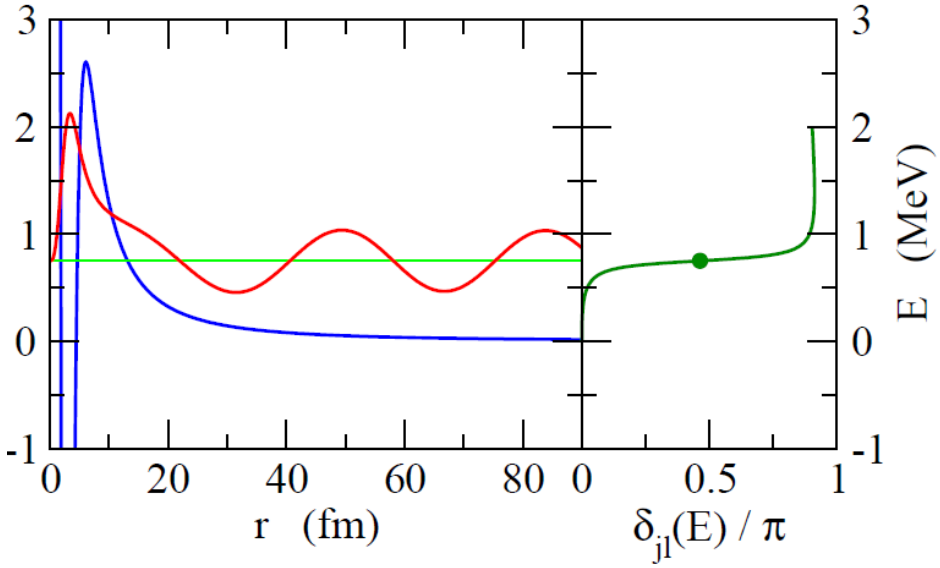
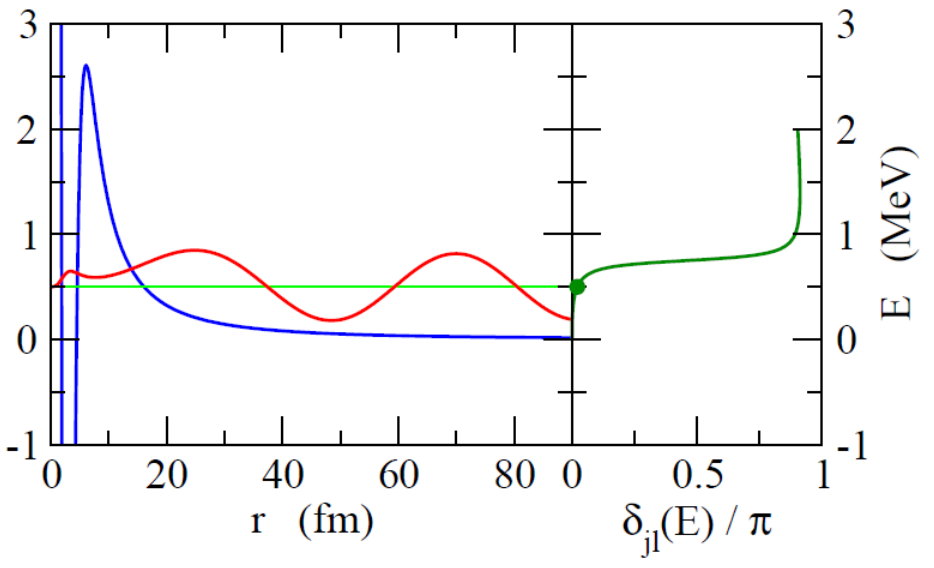
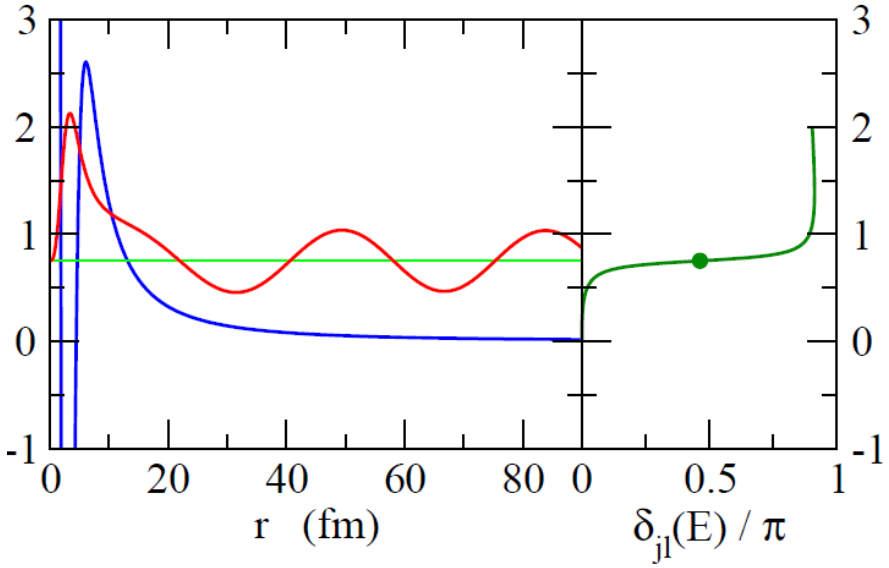




on-resonance:
a large amplitude inside the barrier

off-resonance:
a small amplitude inside the barrier

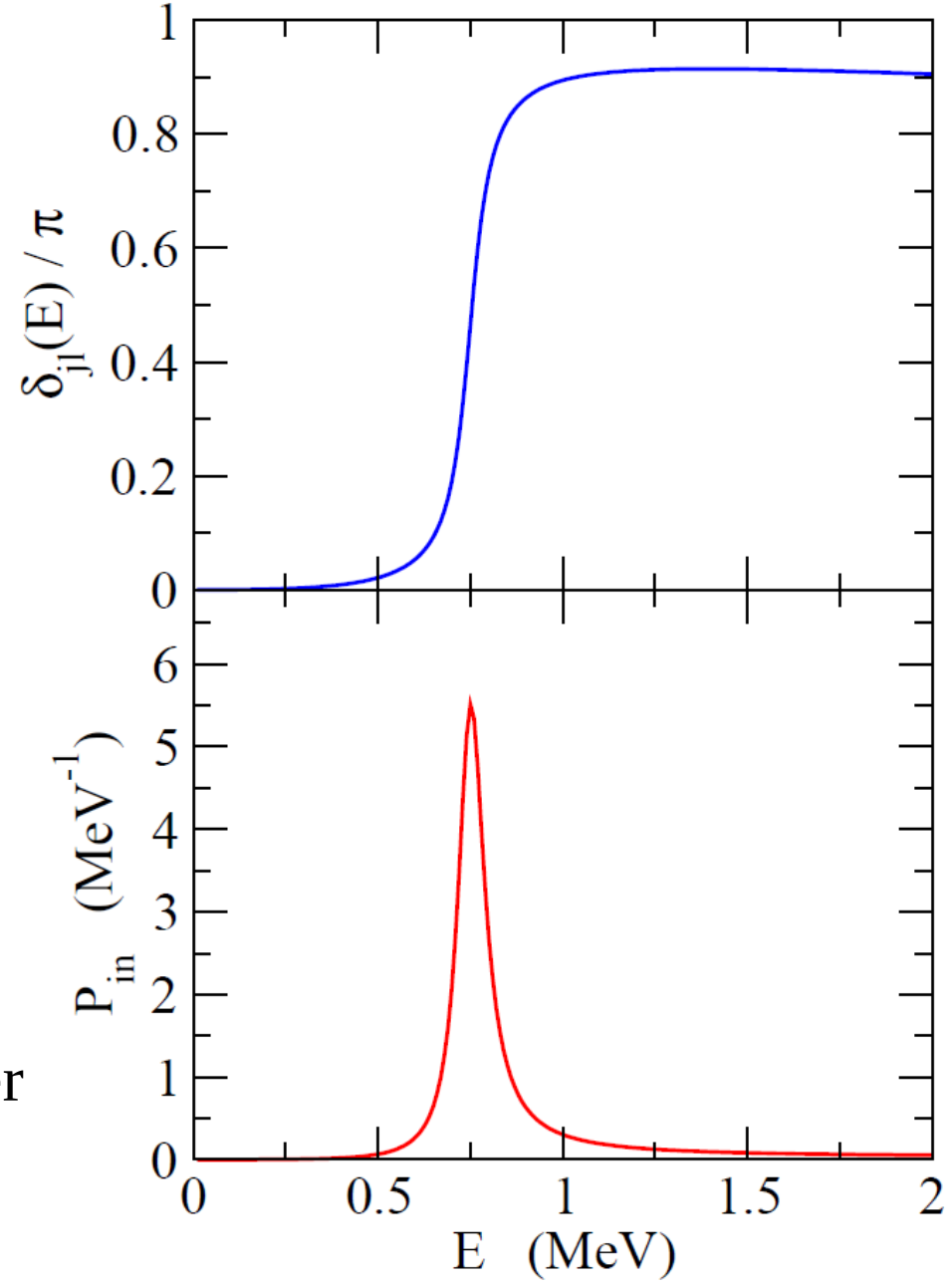
how about the wave function?



on-resonance:
a large amplitude inside the barrier

probability inside the barrier

$$P_{\text{in}} \equiv \int_0^{r_b} r^2 dr \, |R_{jl}(r)|^2$$

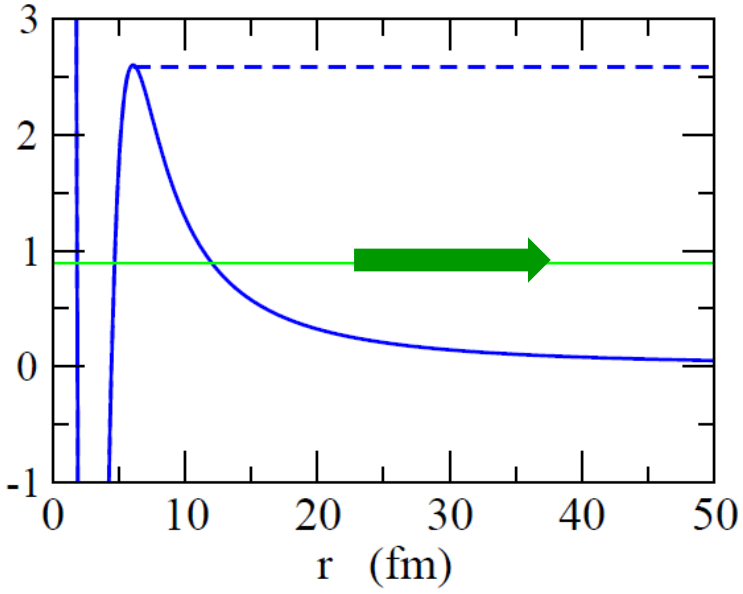


relation between a Gamow state and resonance scattering

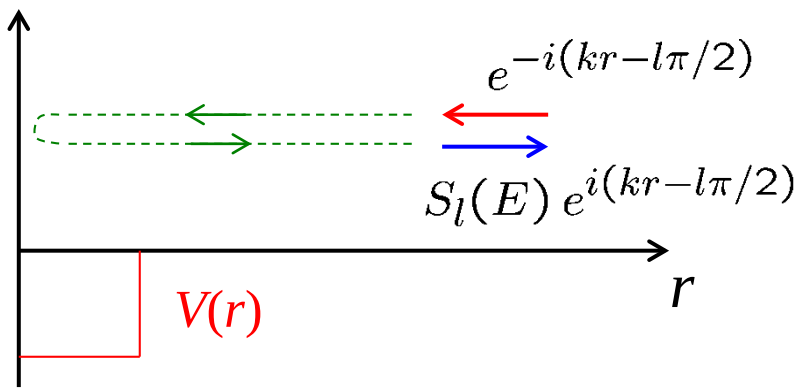
- What is the meaning of “a resonance = a pole of S-matrix”?
- Why is a pole of S-matrix related to a resonance?

relation between a Gamow state and resonance scattering

Gamow state



Scattering state



Outgoing wave boundary cond.

$$u_l(r) \to \mathcal{N}\, e^{i(kr-l\pi/2)}$$

$$E \to E_R - i\frac{\Gamma}{2}$$

scattering boundary cond.

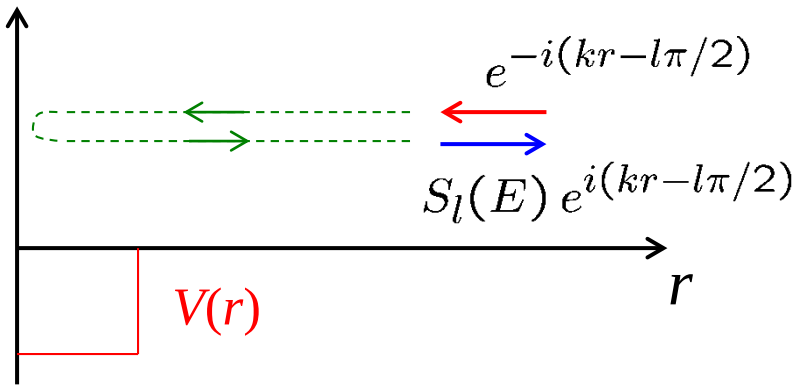
$$u_l(r) \to \mathcal{N}\left(e^{-i(kr-l\pi/2)} - S_l(E)\, e^{i(kr-l\pi/2)}\right)$$

$$S_l(E) = e^{2i\delta_l(E)}$$

E : real

relation between a Gamow state and resonance scattering

Scattering state



Scattering boundary cond.

$$u_l(r) \;\rightarrow\; \mathcal{N}\left(e^{-i(kr-l\pi/2)} - S_l(E)\, e^{i(kr-l\pi/2)}\right)$$

$$S_l(E) = e^{2i\delta_l(E)}$$

E : real

If there is any E at which $S_l(E)$ diverges:

$$u_l(r) \sim \widetilde{\mathcal{N}}\, e^{i(kr-l\pi/2)}$$ (outgoing wave)

To this end, E has to be extended to a complex plane:

$$E \rightarrow E_R - i\frac{\Gamma}{2}$$

➡ Gamow state $\longleftrightarrow$ a pole of S-matrix

Breit-Wigner formula

If S-matrix has a ploe at $\epsilon = E_R - i\frac{\Gamma}{2}$

$$S(E) = e^{2i\delta_0(E)} \cdot \frac{E - \epsilon^*}{E - \epsilon} \quad \longleftarrow \quad |S(E)| = 1$$

$\delta_0(E)$: a smoth function of E (background phase shift)

In this case,

$$\begin{aligned} S(E) &= e^{2i\delta_0(E)} \cdot \frac{E - E_R - i\Gamma/2}{E - E_R + i\Gamma/2} \\ &= e^{2i\delta_0(E)} \left(1 - \frac{i\Gamma}{E - E_R + i\Gamma/2}\right) \end{aligned}$$

Breit-Wigner formula

If S-matrix has a ploe at $\epsilon = E_R - i\dfrac{\Gamma}{2}$

$$S(E) = e^{2i\delta_0(E)} \cdot \frac{E - \epsilon^*}{E - \epsilon} \quad \longleftarrow \quad |S(E)| = 1$$

$\delta_0(E)$: a smoth function of E (background phase shift)

Take $E - \epsilon^* = c\, e^{i\delta_r(E)}$

$$S(E) = e^{2i\delta_0(E)} \cdot \frac{c\, e^{i\delta_r(E)}}{c\, e^{-i\delta_r(E)}} = e^{2i\delta_0(E)} e^{2i\delta_r(E)}$$

$$E - \epsilon^* = E - E_R - i\Gamma/2 = c\, e^{i\delta_r} = c(\cos\delta_r + i \sin\delta_r)$$

$$\longrightarrow \quad \tan\delta_r = \frac{\sin\delta_r}{\cos\delta_r} = \frac{-\Gamma/2}{E - E_R}$$

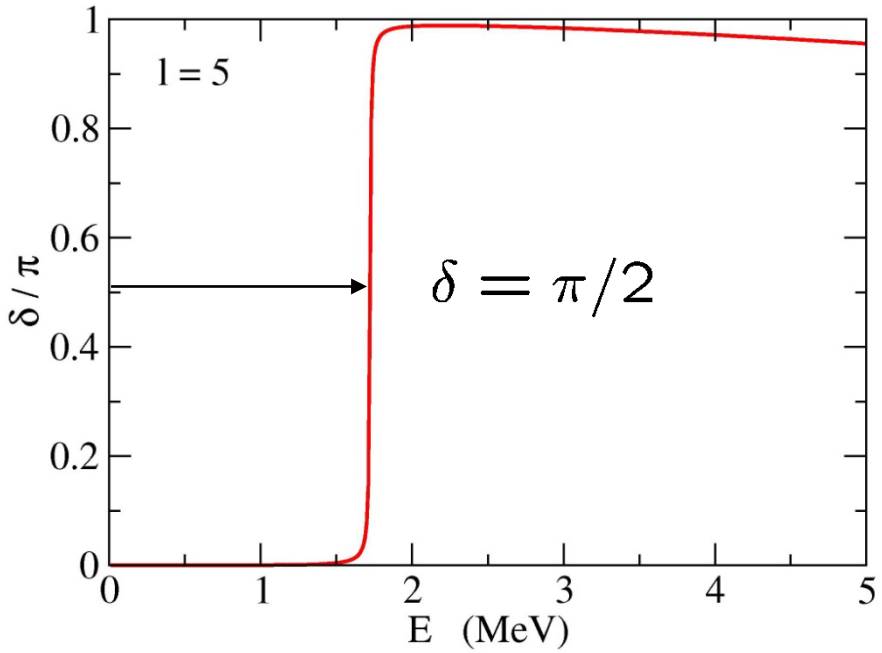
$$\delta(E) = \delta_r(E) + \delta_0(E) = \tan^{-1}\left(\frac{\Gamma}{2(E_R - E)}\right) + \delta_0(E)$$

$$\delta_l(E) \sim \delta_r(E) = \tan^{-1}\left(\frac{\Gamma}{2(E_R - E)}\right)$$

$$\Longrightarrow \qquad \begin{aligned} \sigma_l &= \frac{4\pi}{k^2}(2l+1)\sin^2\delta_l \\ &= \frac{4\pi}{k^2}(2l+1)\frac{1}{1+\cot^2\delta_l} \\ &= \frac{4\pi}{k^2}(2l+1)\frac{\Gamma^2/4}{\Gamma^2/4+(E-E_R)^2} \end{aligned}$$

Breit-Wigner formula

For a narrow resonance, a resonance = the energy at which the phase shift crosses $\pi/2$



Breit-Wigner formula:

$$\delta(E) = \tan^{-1}\frac{\Gamma}{2(E_R - E)} + \delta_0(E)$$

$$\Rightarrow \quad \delta(E_R) \sim \pi/2$$

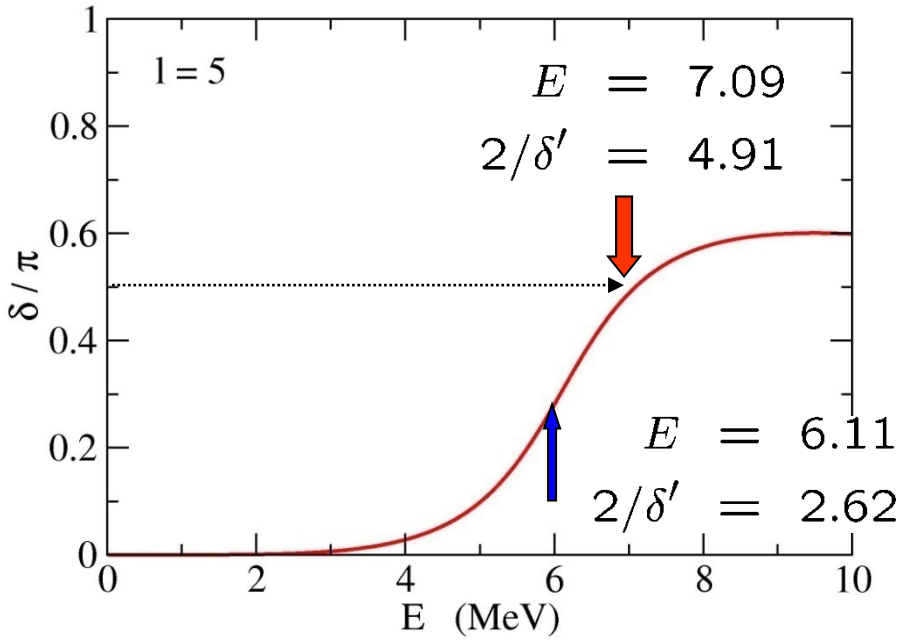
$$\Gamma = \frac{2}{\delta'(E_R)}$$

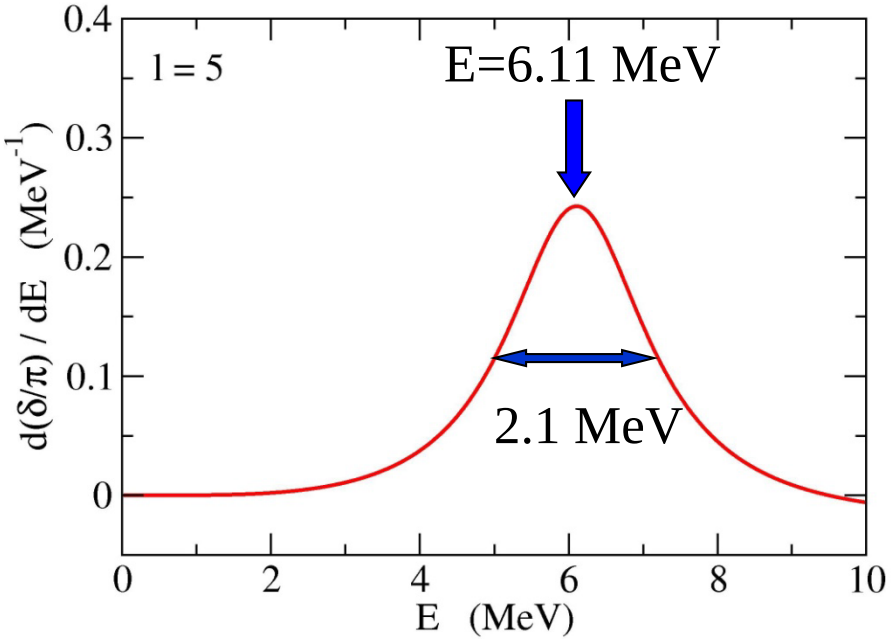
V_0 = -50 MeV
R_0 = 1.27 * $200^{1/3}$ fm
a = 0.67 fm
μ = 200 m_N / 201

(note) but this is not the case for a broad resonance

$$\delta(E) = \tan^{-1}\frac{\Gamma}{2(E_R - E)} + \delta_0(E)$$

background phase shift





Gamow state: $E = 6.01$ MeV

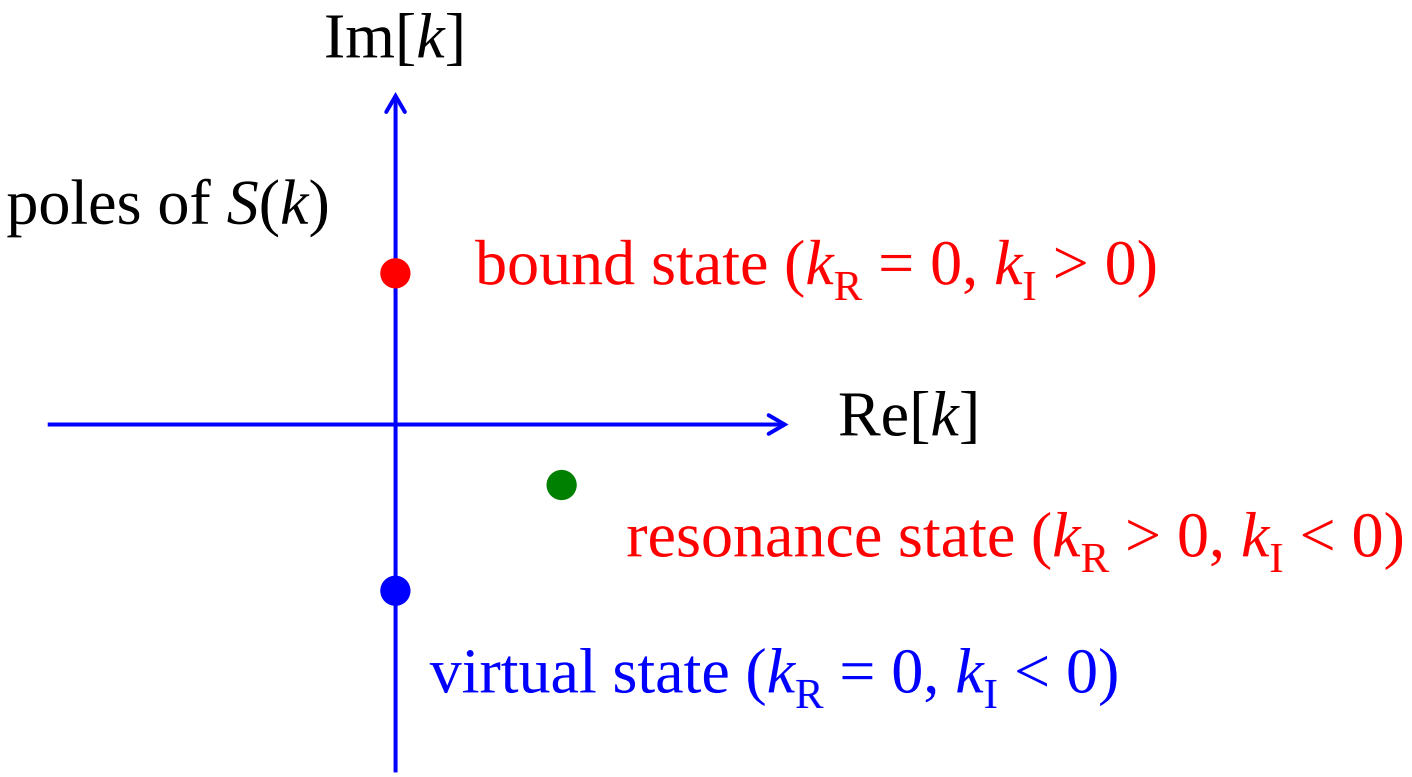
$\Gamma = 2.22$ MeV

On virtual state

Scattering problem: real positive energy (physical observable)

⇧ influence

a pole of S-matrix close to the real axis in a complex E plance
(or in a complex momentum plane)



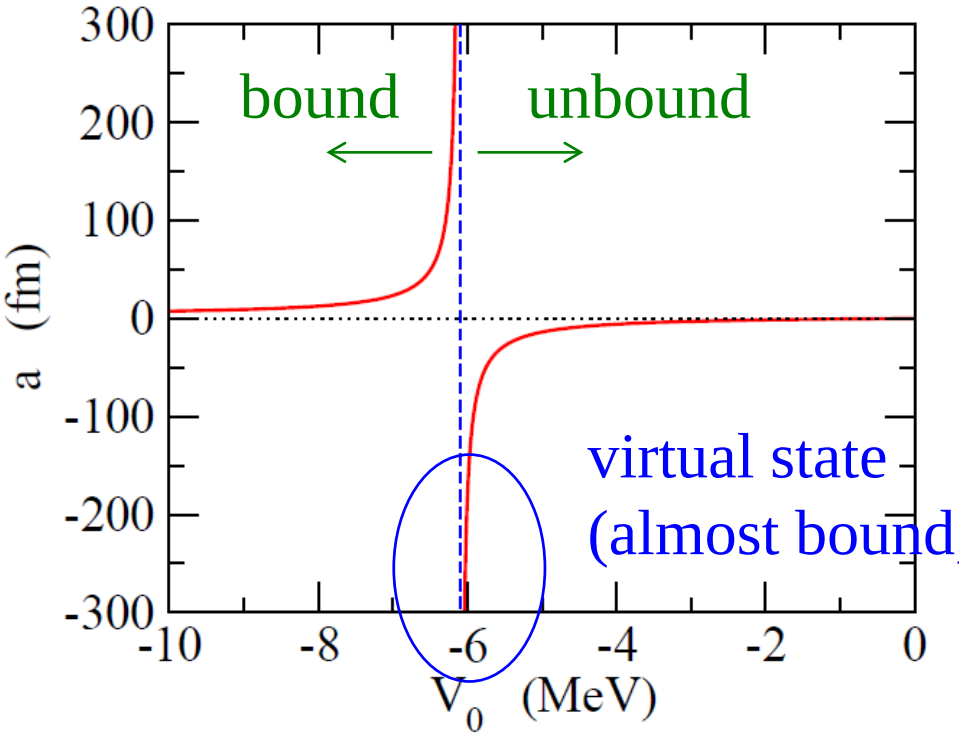
On virtual state

let us consider scattering at very low energies (thus, only s-wave)

effective range expansion: $$k \cot \delta(k) \sim -\frac{1}{a} + \frac{1}{2} r_e k^2 + \cdots$$

On virtual state

scattering length: diverges at zero binding



virtual state
(almost bound, but not really)

vary the depth of Woods-Saxon potential

(R = 2.736 fm, a_0 = 0.67 fm)

the meaning of scattering length

a square-well potential with radius R

$$a = R\left(1 - \frac{\tan \kappa R}{\kappa R}\right)$$

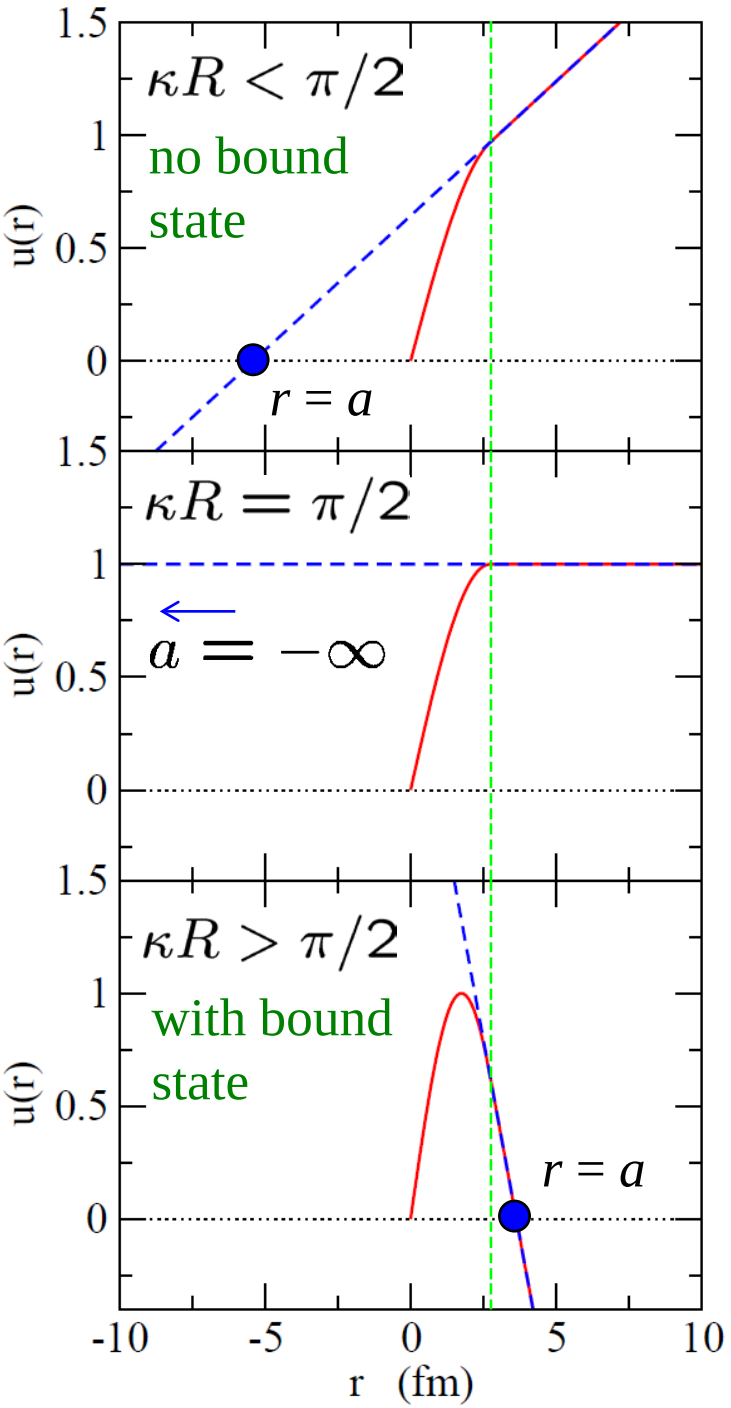
$$\kappa = \sqrt{\frac{2mV_0}{\hbar^2}}$$

$$u(r) = A \sin(\kappa r) \quad (r < R)$$

$$f(r) \equiv u(R) + u'(R)(r - R)$$

is $f(a)$=0 at $r = a$

scattering length
= a crossing point of
a linear approximation of wf at $r = R$



On virtual state

let us consider scattering at very low energies (thus, only s-wave)

effective range expansion: $$k \cot \delta(k) \sim -\frac{1}{a} + \frac{1}{2} r_e k^2 + \cdots$$

$$\begin{aligned} S(k) &= \frac{e^{i\delta}}{e^{-i\delta}} = \frac{\cos\delta + i \sin\delta}{\cos\delta - i \sin\delta} \\ &= \frac{\cot\delta + i}{\cot\delta - i} \\ &\sim \frac{-1/a + ik}{-1/a - ik} \end{aligned}$$

a pole at: $k = i \dfrac{1}{a}$

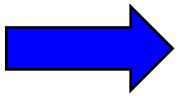
$a < 0$: virtual state

$a > 0$: a weakly bound state

On virtual state

let us consider scattering at very low energies (thus, only s-wave)

effective range expansion: $$k\cot\delta(k) \sim -\frac{1}{a} + \frac{1}{2}r_e k^2 + \cdots$$



$$S(k) \sim \frac{-1/a + ik}{-1/a - ik}$$

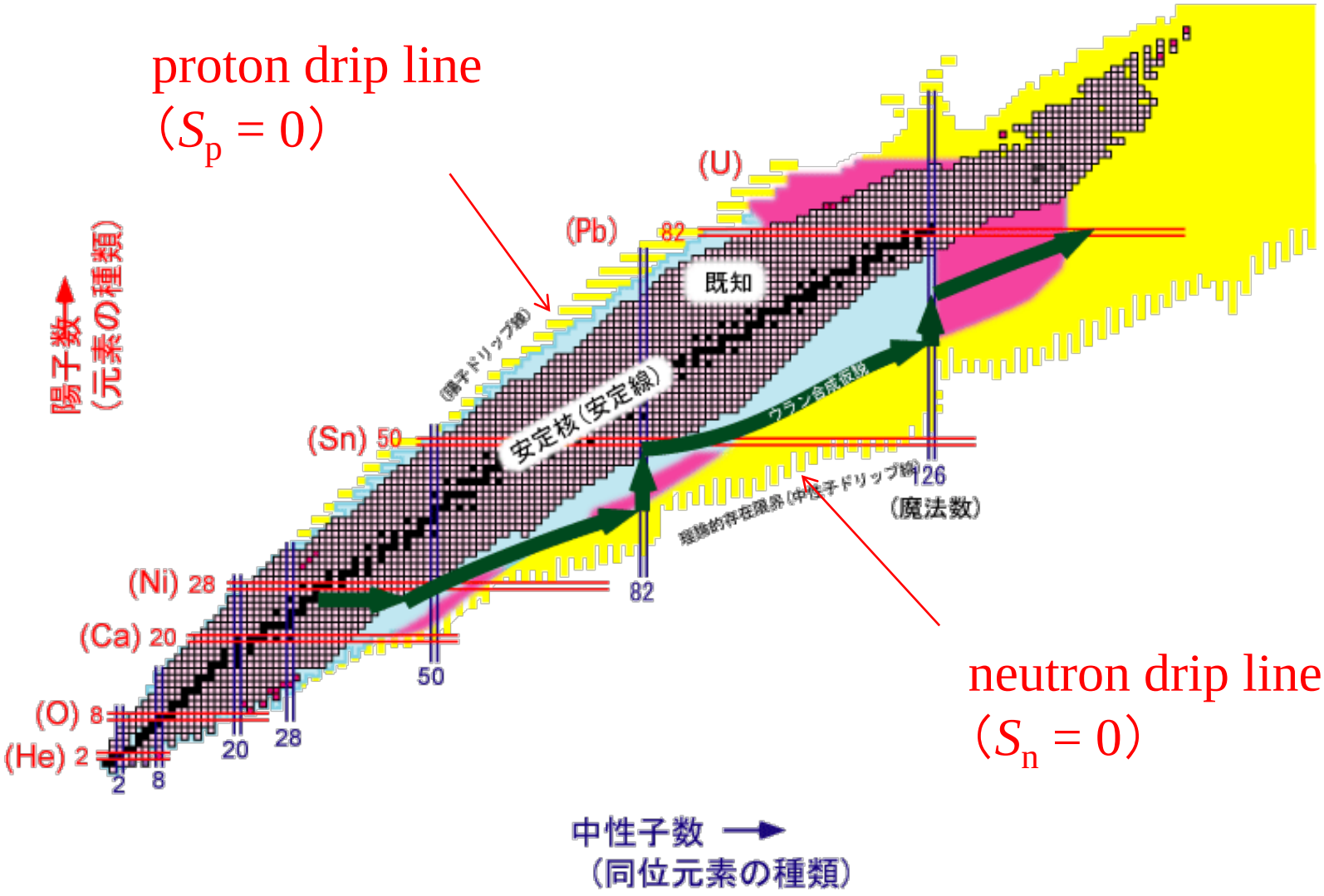
a pole at: $k = i\dfrac{1}{a}$

if the pole is close to the real axis → a large $|a|$

the total cross section of elastic stattering:

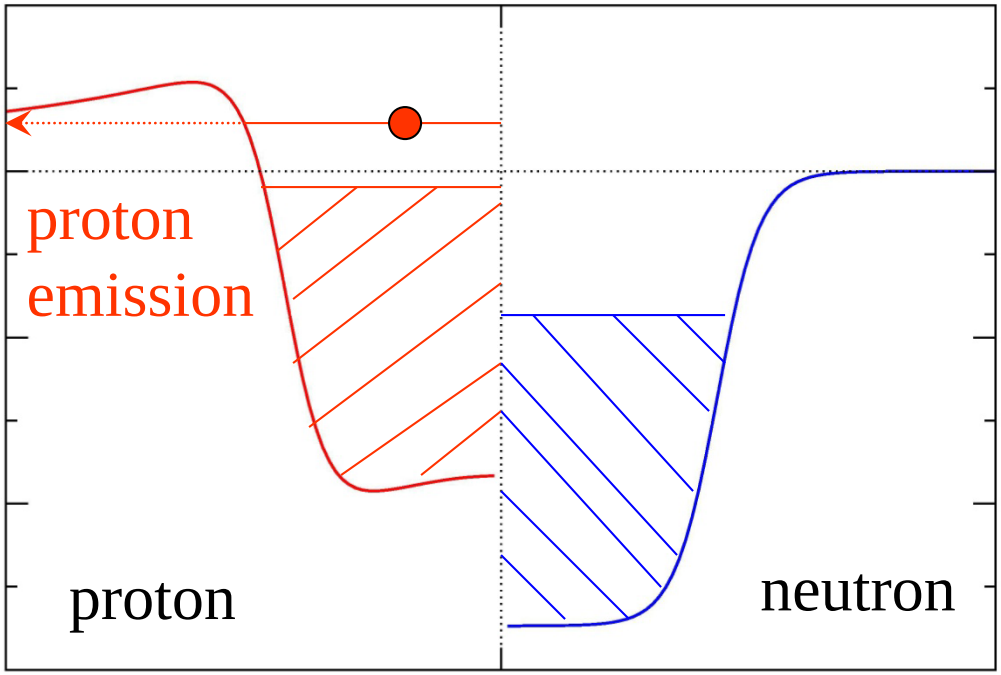
$$\sigma = \frac{4\pi}{k^2}\sin^2\delta = \frac{4\pi}{(k\cot\delta)^2 + k^2} \sim 4\pi a^2$$: large

proton emission of nuclei beyond the p-drip line



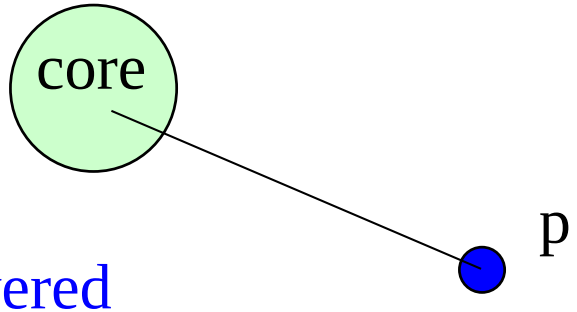
proton emission decay

nuclei beyond the proton dripline



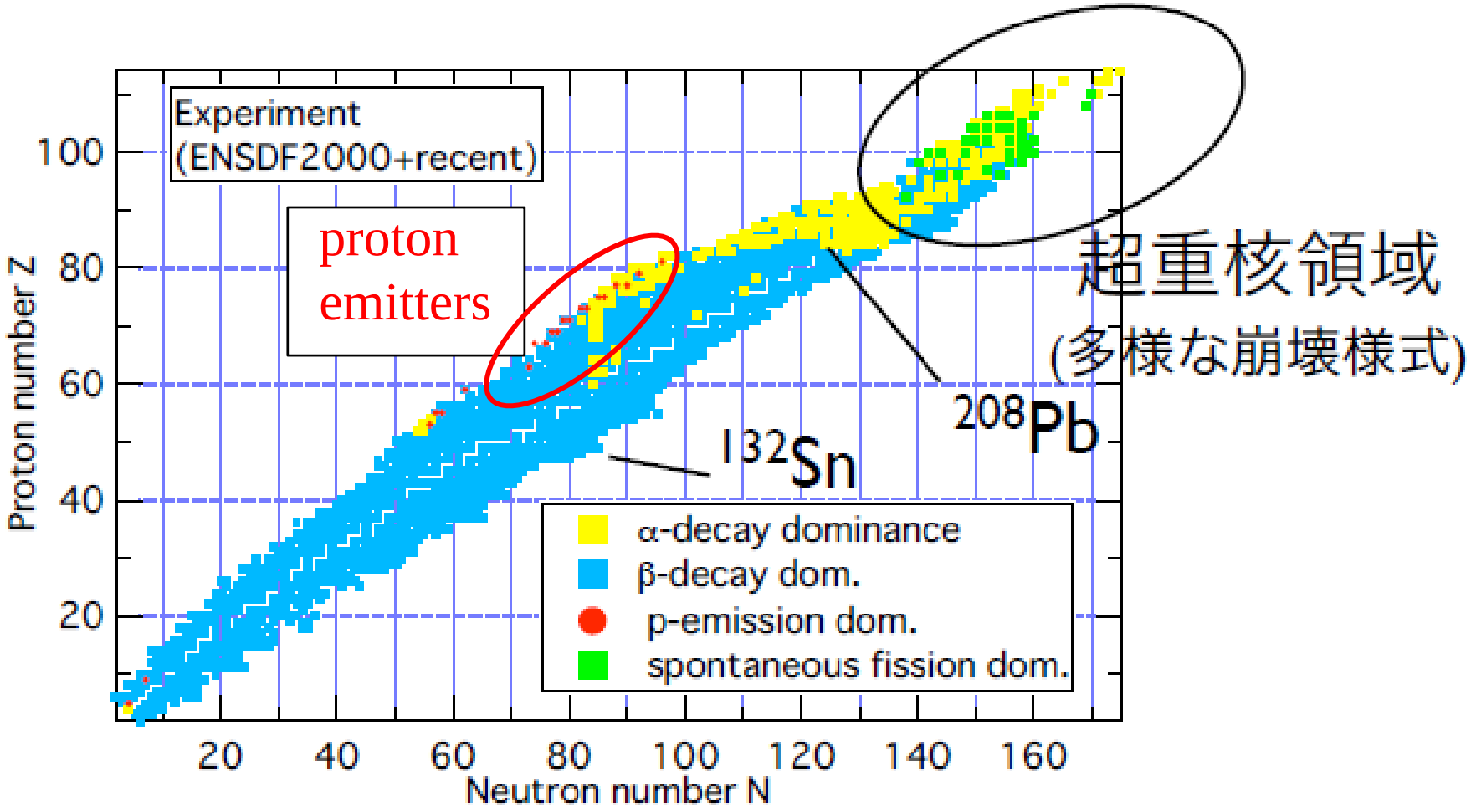
proton: trapped by the Coul. barrier
(the barrier has to be penetrated)

a very narrow (long-lived) resonance state



many (g.s.) proton emitters have been discovered

exp. observables: proton energy E_p and decay half-life $T_{1/2}$



slide: H. Koura (JAEA)

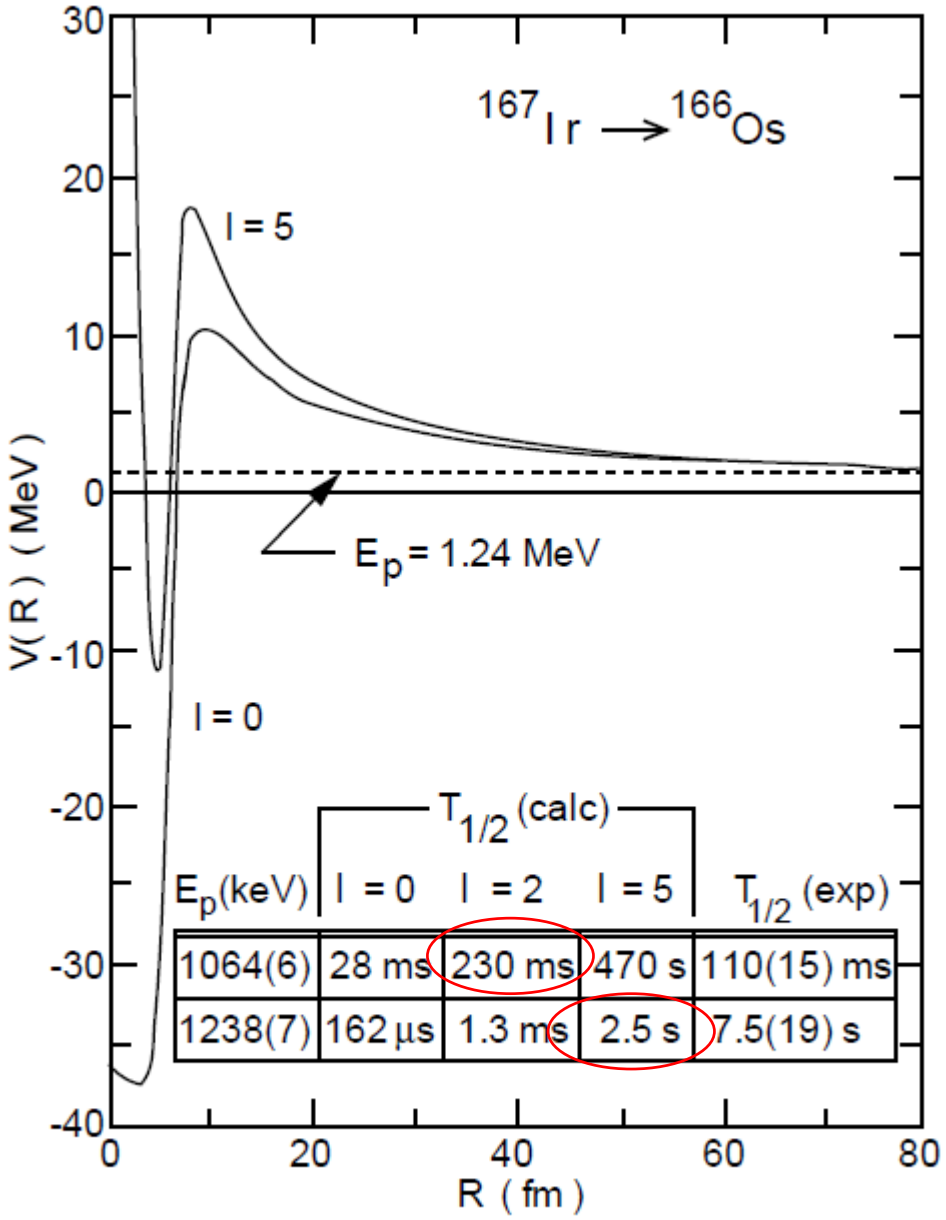


Figure 3 Proton-nucleus potential calculated for the proton emitter ^{167}Ir. The *inset* shows the proton-decay half-lives calculated using the WKB approximation for three values of the angular momentum ℓ, compared to the experimental values for the ground and isomeric transitions.

Typical values in the A ~ 150-160 region

V_b ~ 10 MeV (l=0)

E_p ~ 1 MeV

R_{turn}: 80~100 fm

Γ : 10^{-18}~10^{-22} MeV

$T_{1/2}$: 100 µs~1 sec

a characteristic feature:
half-life: sensitive to l

↓

a single-particle l in proton-rich nuclei can be determined by a measurement of p-emission

P.J. Woods and C.N. Davids,
Annu. Rev. Nucl. Part. Sci. 47 (‘97)541

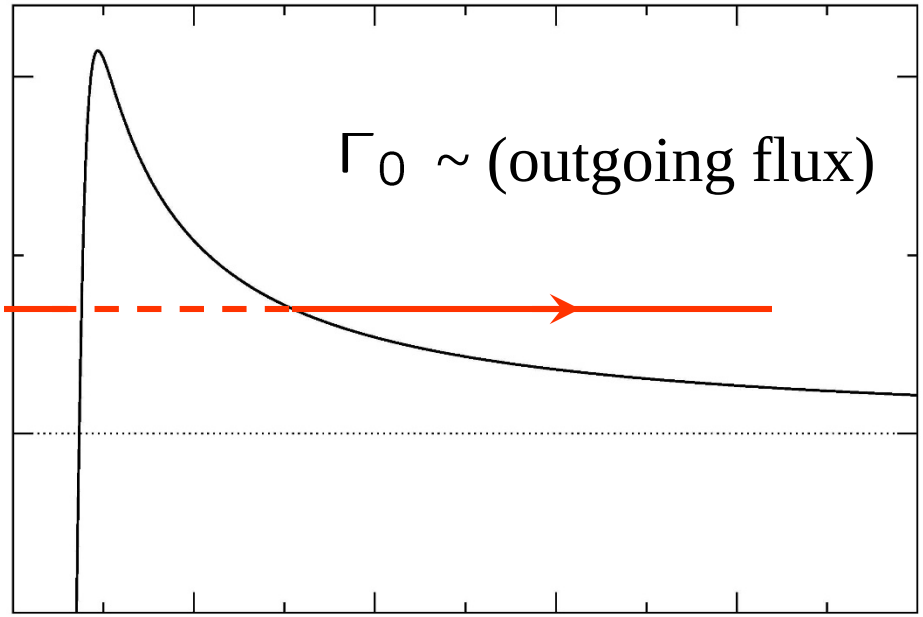
Differences from α decays

1. Smaller reduced mass μ

- Much stronger l dependence (centrifugal potential)
- Spectroscopic information for proton s.p. states can be extracted

2. Much simpler spectroscopic factor

$$^{A+1}_{Z+1}X \rightarrow ^{A}_{Z}X + p$$



$$\Gamma = \Gamma_0 \cdot S$$
$$S = |\langle (A+1)|(A)+1\rangle|^2$$

Simple estimation:

$$|(A+1)\rangle = \alpha^\dagger |(A)\rangle$$
$$\alpha^\dagger = ua^\dagger - va$$
$$|(A)+1\rangle = a^\dagger |(A)\rangle$$

$$\rightarrow \quad S = u^2$$

Spectroscopic factor

Decay half-life:

$$T_{1/2} = \frac{\hbar}{\Gamma} \log 2 = \frac{\hbar}{S\Gamma_0} \log 2$$

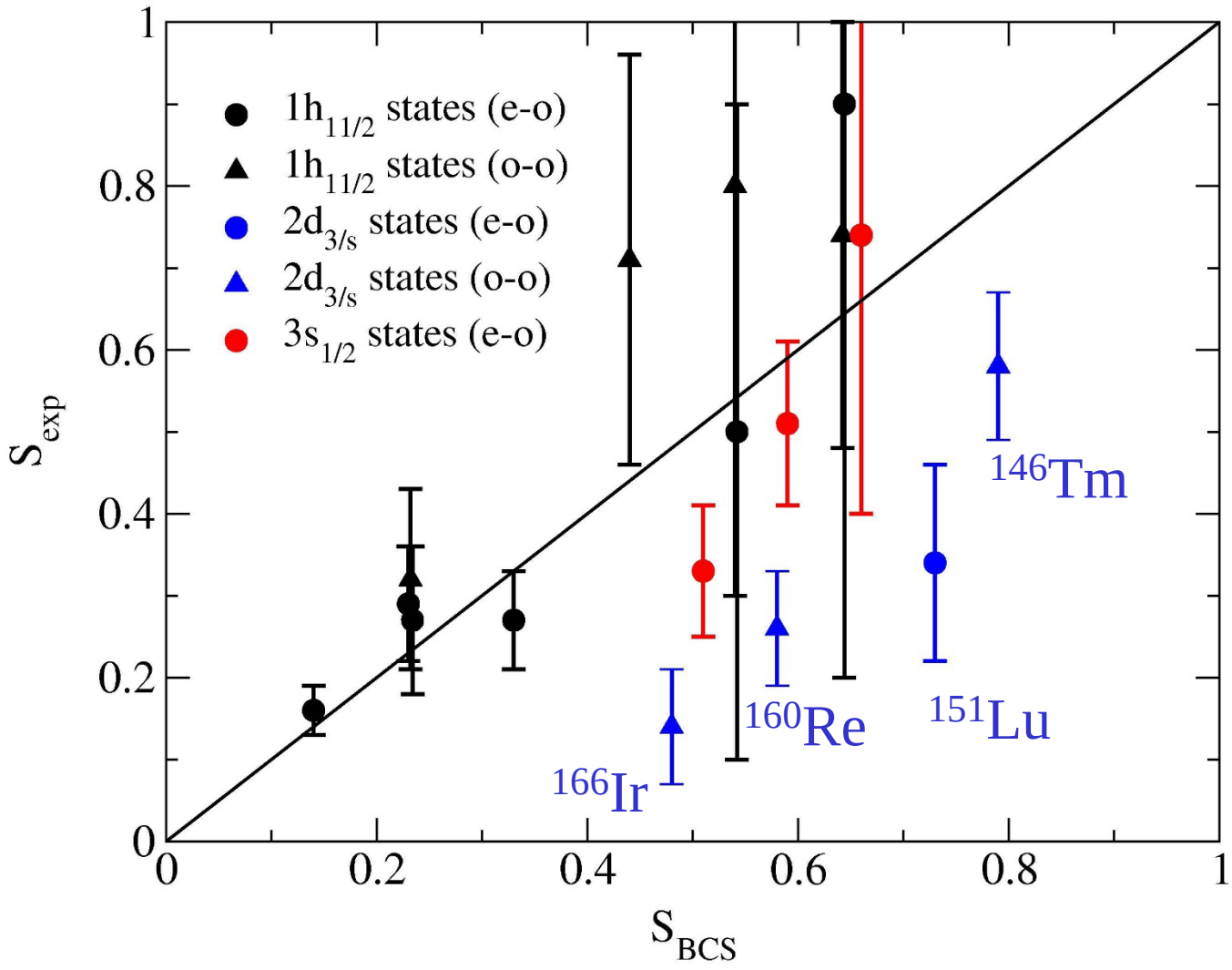
Experimental spectroscopic factor:

$$S_{\mathrm{exp}} \equiv \frac{T^{\mathrm{th}}_{1/2}}{T^{\mathrm{exp}}_{1/2}} = \left(\frac{\hbar}{\Gamma_0} \log 2 \right) \Big/ T^{\mathrm{exp}}_{1/2}$$

$\longleftrightarrow$ Theoretical spectroscopic factor

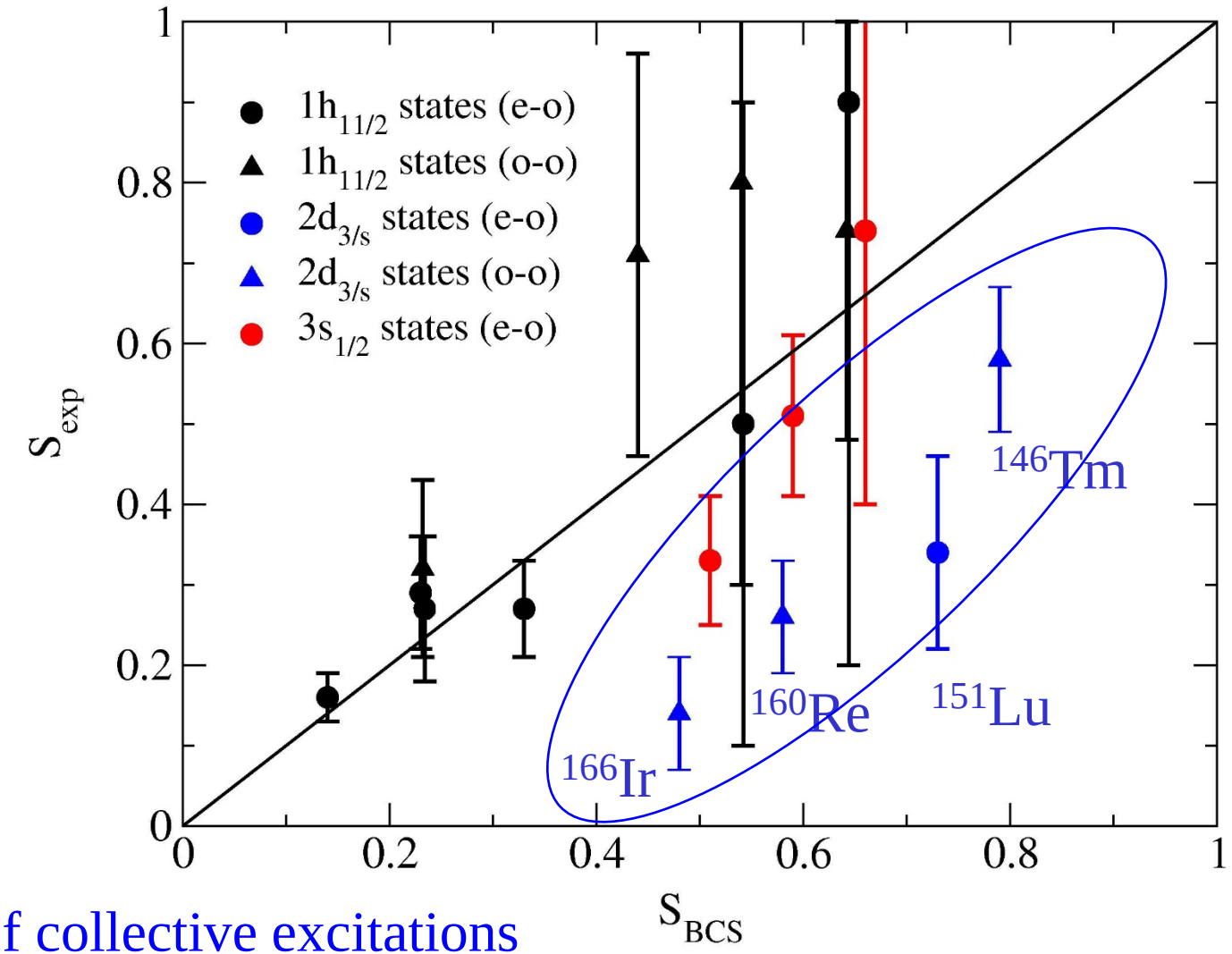
$$S_{\mathrm{th}} = u^2 \quad \text{or} \quad S_{\text{shell model}}$$

Prediction of potential model



Calculations: S. Aberg et al., PRC56(’97)1762
(spherical optical pot. description)

Prediction of potential model

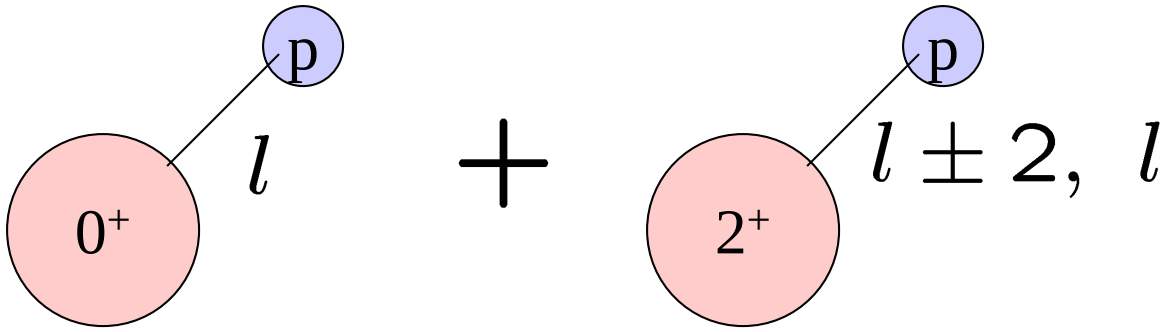


role of collective excitations

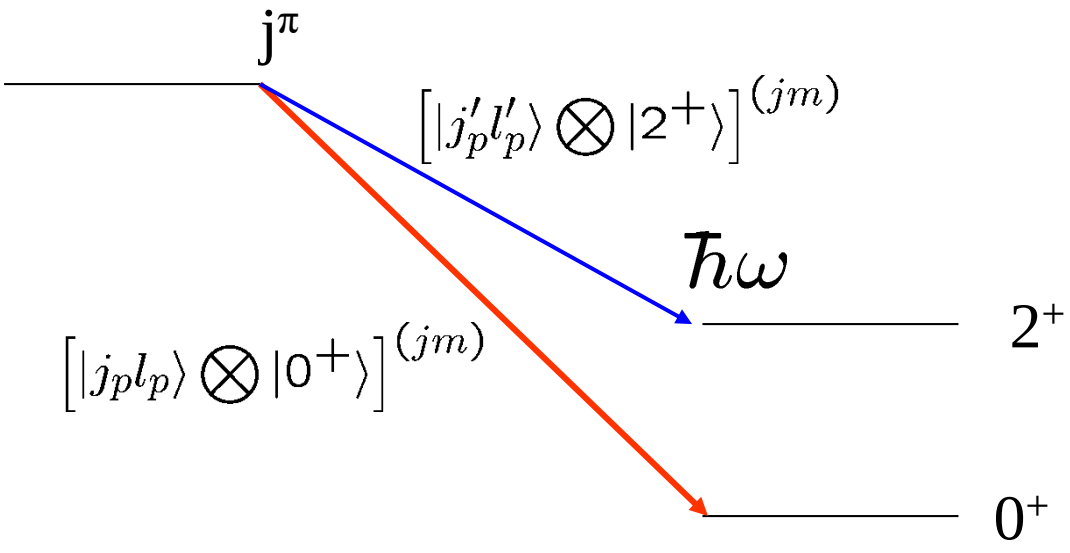
K.H., PRC64('01)041304(R)

C.N. Davids and H. Esbensen, PRC64('01)034317

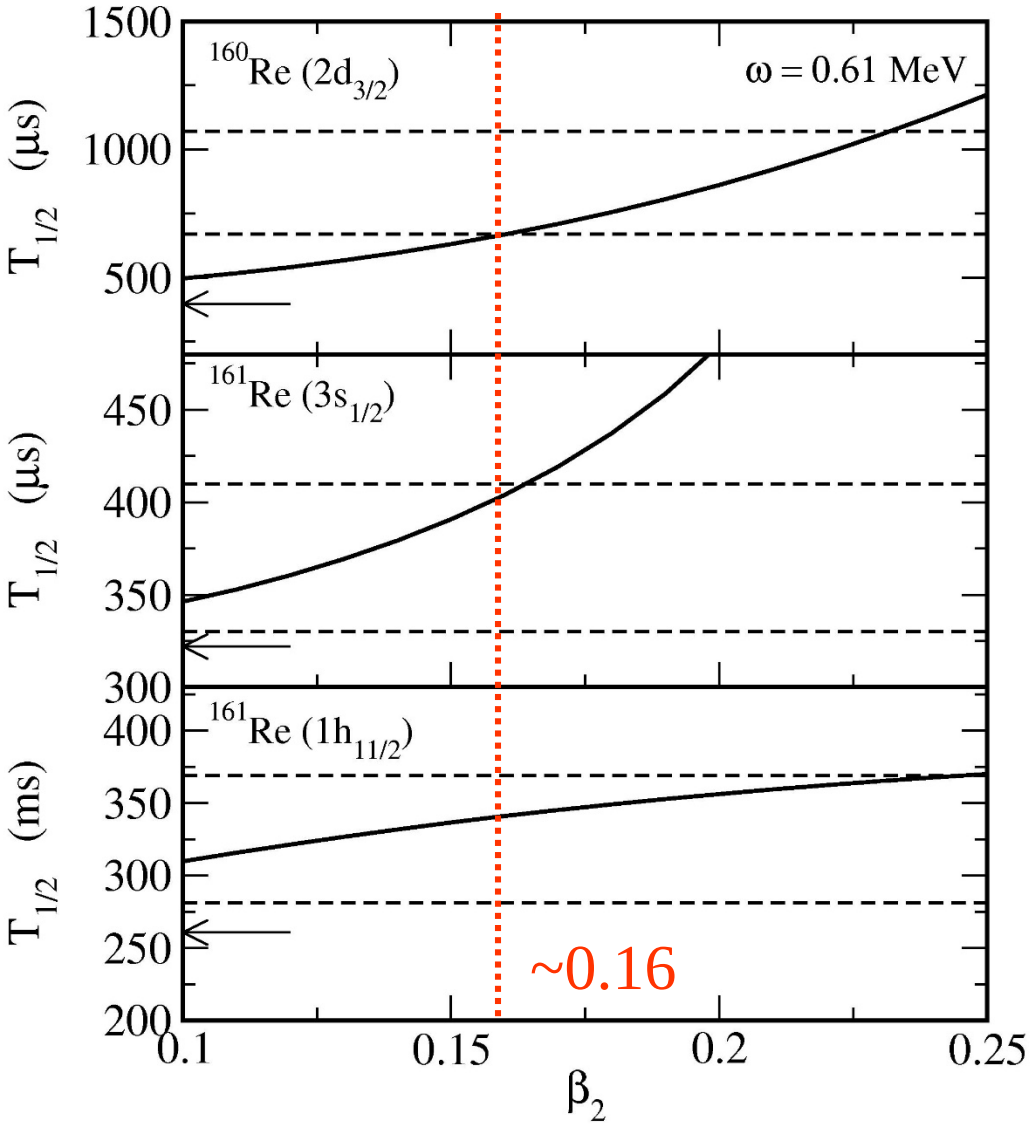
Role of particle-vibration coupling



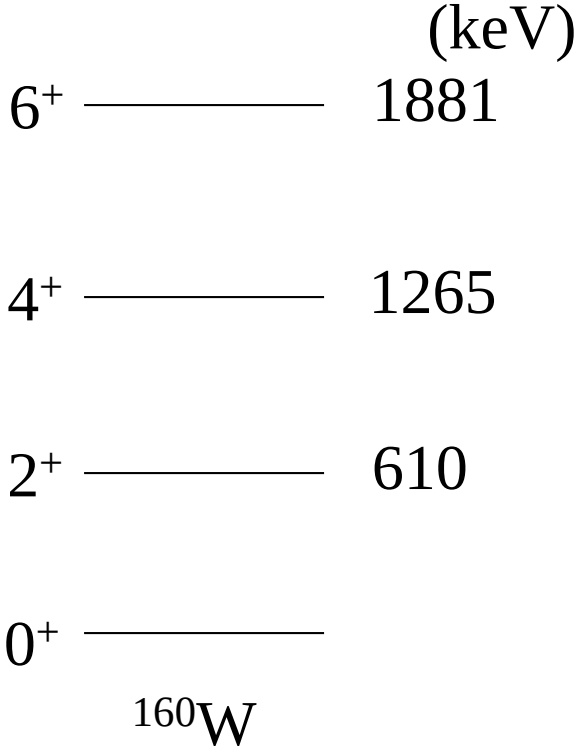
$$|jm\rangle = \sum_{l_p, j_p} \sum_{n, I} \left[|j_p l_p\rangle \bigotimes |nI\rangle \right]^{(jm)}$$



$^{160,161}_{75}\mathrm{Re} \rightarrow p + {}^{160,159}_{74}\mathrm{W}$



K.H., PRC64('01)041304



A. Keenen et al.,
PRC63('01)064309