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$$\Delta E_L = \langle (\ell\ell) LM | -g\,\delta(\mathbf{r}-\mathbf{r}') | (\ell\ell) LM \rangle$$

$$= -g\, I_r^{(\ell)} \frac{(2\ell+1)^2}{4\pi} \begin{pmatrix} \ell & \ell & L \\ 0 & 0 & 0 \end{pmatrix}^2$$ の証明

$$\Psi_{LM}(\mathbf{r},\mathbf{r}') = \sum_{m,m'} \langle \ell m \ell m' | LM \rangle\, \varphi_{\ell m}(\mathbf{r})\, \varphi_{\ell m'}(\mathbf{r}')$$

$$\Delta E_L = -g \int d\mathbf{r}\; \Psi^*_{LM}(\mathbf{r},\mathbf{r})\, \Psi_{LM}(\mathbf{r},\mathbf{r})$$

$$= -g \sum_{m_1,m_1'} \sum_{m_2,m_2'} \langle \ell m_1 \ell m_1' | LM \rangle \langle \ell m_2 \ell m_2' | LM \rangle \cdot I_r^{(\ell)}$$

$$\times \int d\hat{\mathbf{r}}\; Y^*_{\ell m_1}(\hat{\mathbf{r}})\, Y^*_{\ell m_1'}(\hat{\mathbf{r}})\, Y_{\ell m_2}(\hat{\mathbf{r}})\, Y_{\ell m_2'}(\hat{\mathbf{r}})$$

(note) $$Y_{\ell m_2}(\hat{\mathbf{r}})\, Y_{\ell m_2'}(\hat{\mathbf{r}}) = \sum_{L_2 M_2} \frac{\hat{\ell}^2}{\sqrt{4\pi}\,\hat{L}_2} \langle \ell 0 \ell 0 | L_2 0 \rangle \times \langle \ell m_2 \ell m_2' | L_2 M_2 \rangle\, Y_{L_2 M_2}(\hat{\mathbf{r}})$$

$$Y^*_{\ell m_1}(\hat{\mathbf{r}})\, Y^*_{\ell m_1'}(\hat{\mathbf{r}}) = \sum_{L_1 M_1} \frac{\hat{\ell}^2}{\sqrt{4\pi}\,\hat{L}_1} \langle \ell 0 \ell 0 | L_1 0 \rangle \times \langle \ell m_1 \ell m_1' | L_1 M_1 \rangle\, Y^*_{L_1 M_1}(\hat{\mathbf{r}})$$

$$= -g\, I_r \sum_{m_1,m_1'} \sum_{m_2,m_2'} \langle \ell m_1 \ell m_1' | LM \rangle \langle \ell m_2 \ell m_2' | LM \rangle$$

$$\times \sum_{L_1 M_1} \frac{\hat{\ell}^2}{\sqrt{4\pi}\,\hat{L}_1} \langle \ell 0 \ell 0 | L_1 0 \rangle \langle \ell m_1 \ell m_1' | L_1 M_1 \rangle$$

$$\times \frac{\hat{\ell}^2}{\sqrt{4\pi}\,\hat{L}_1} \langle \ell 0 \ell 0 | L_1 0 \rangle \langle \ell m_2 \ell m_2' | L_1 M_1 \rangle$$

$$= -g\, I_r \cdot \frac{\hat{\ell}^4}{4\pi \cancel{\hat{L}^2}} \underbrace{\langle \ell 0 \ell 0 | L 0 \rangle^2}_{= \cancel{\hat{L}^2} \begin{pmatrix} \ell & \ell & L \\ 0 & 0 & 0 \end{pmatrix}^2} = -g\, I_r \frac{(2\ell+1)^2}{4\pi} \begin{pmatrix} \ell & \ell & L \\ 0 & 0 & 0 \end{pmatrix}^2 .$$