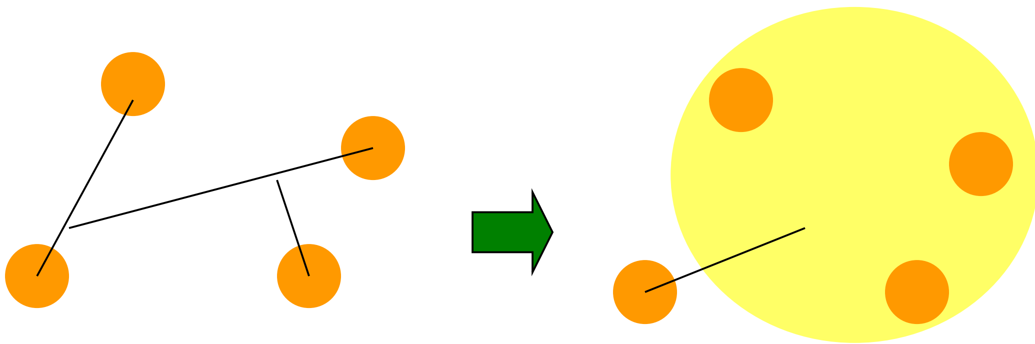
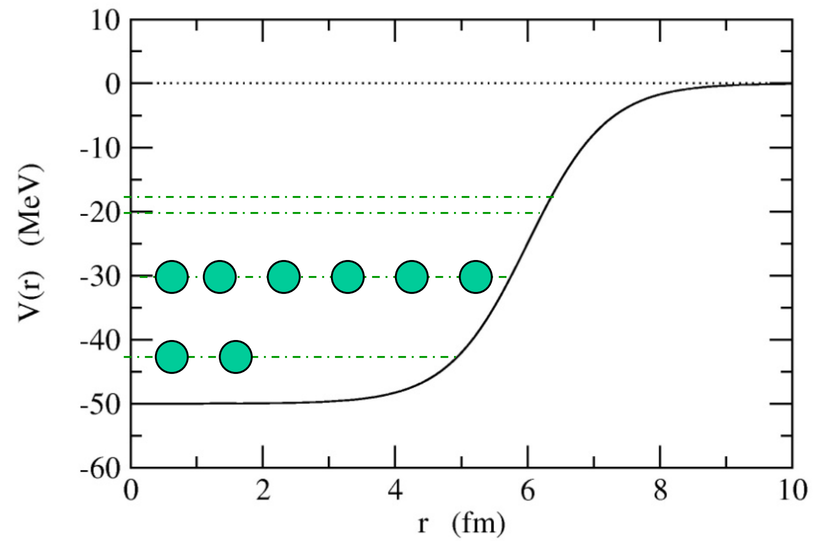
Pairing Correlations

Mean-field approximation





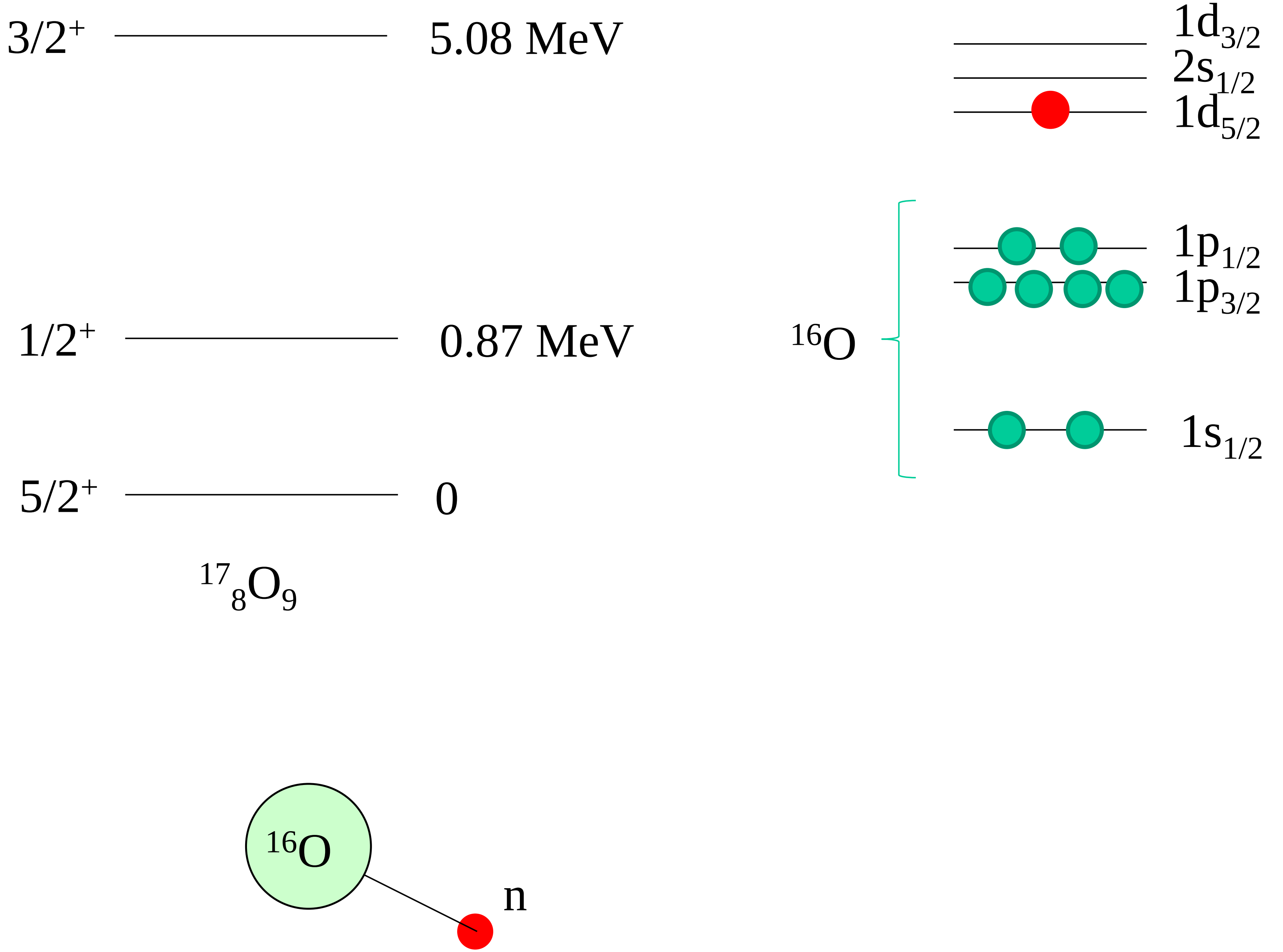
independent particle motion
in a potential well

$\longrightarrow$

nn interaction:
only through a mean-field
potential

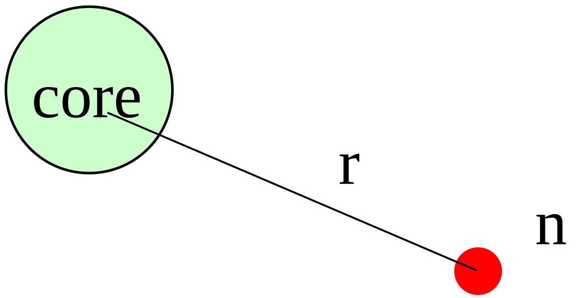
$$
\begin{aligned}
\Psi(1,2,\cdots,A) &= \mathcal{A}[\psi_1(1)\psi_2(2)\cdots\psi_A(A)] \\
&= \frac{1}{\sqrt{A!}}\begin{vmatrix} \psi_1(1) & \psi_2(1) & \cdots & \psi_A(1) \\ \psi_1(2) & \psi_2(2) & \cdots & \psi_A(2) \\ \vdots & & & \\ \psi_1(A) & \psi_2(A) & \cdots & \psi_A(A) \end{vmatrix}
\end{aligned}
$$

Slater determinant: antisymmetrization due to the Pauli principle

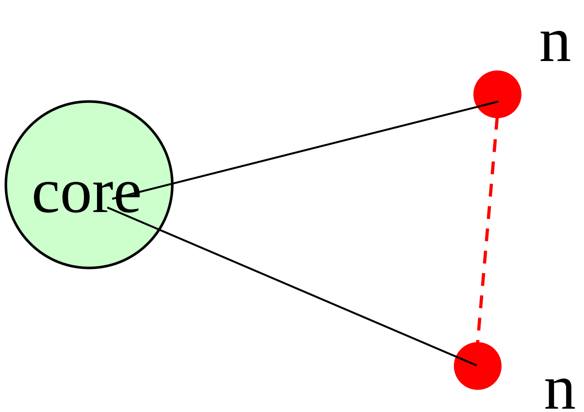


3/2+
5.08 MeV
1/2+
0.87 MeV
5/2+
0
$^{17}_{8}O_9$
$1d_{3/2}$
$2s_{1/2}$
$1d_{5/2}$
$1p_{1/2}$
$1p_{3/2}$
^{16}O
$1s_{1/2}$
^{16}O
n

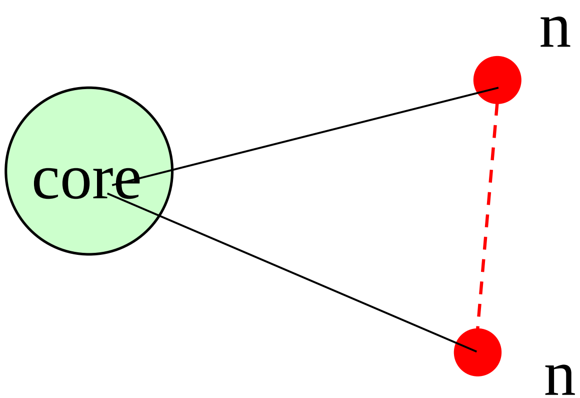
Pairing correlation



What if two neutrons are put outside the core nucleus?

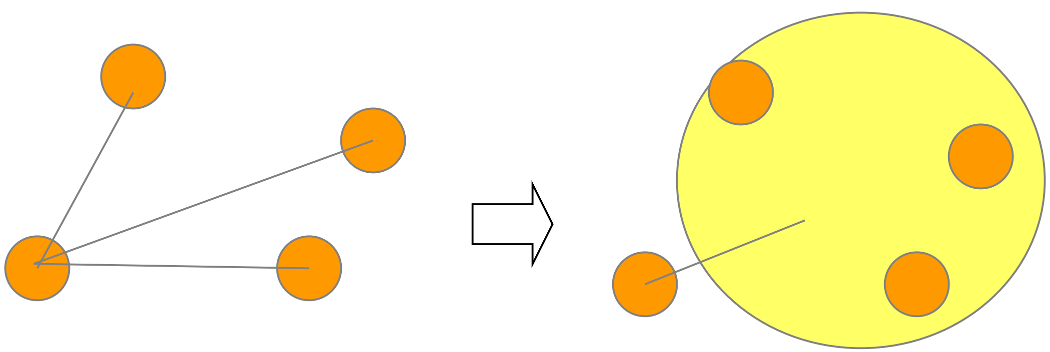


What is the influence of the interaction between the two neutrons?



What is the influence of the interaction between the two neutrons?

Mean-field theory



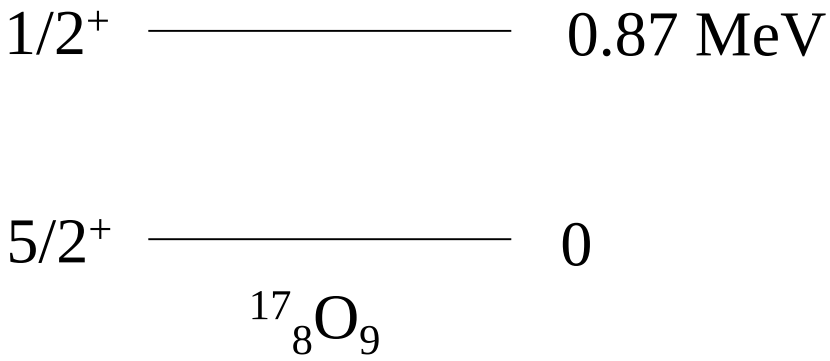
treat the interaction among particles only on average

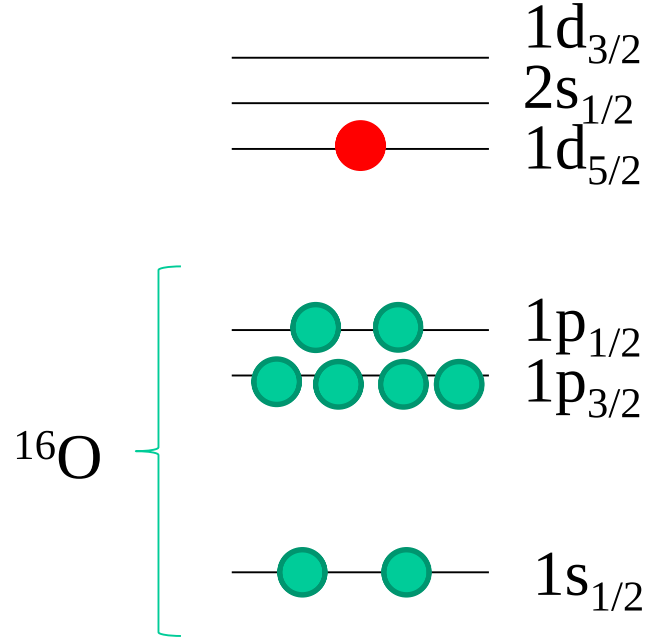
the pure mean-field picture

→ the interaction between the two neutrons :

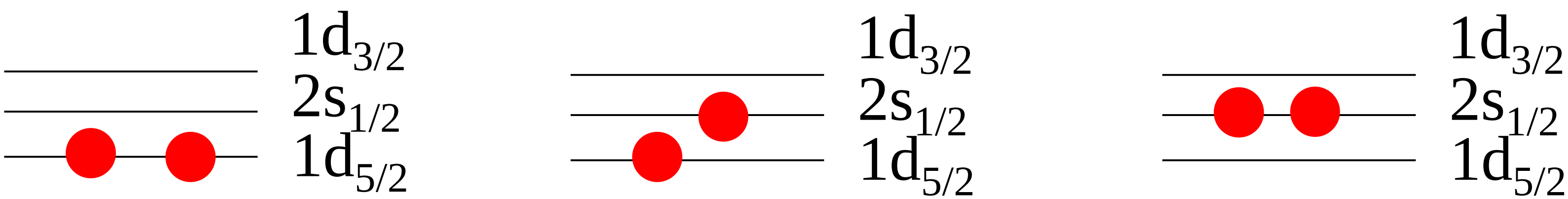
only through the mean-field potential,

(the two neutrons: uncorrelated).





^{18}O states in the pure MF picture



$E = 0$

$I^\pi = d_{5/2} \text{ x } d_{5/2} = 0^+, 2^+, 4^+$

$E = 0.87$ MeV

$I^\pi = d_{5/2} \text{ x } s_{1/2} = 2^+, 3^+$

$E = 0.87 \text{ x } 2 = 1.74$ MeV

$I^\pi = s_{1/2} \text{ x } s_{1/2} = 0^+$

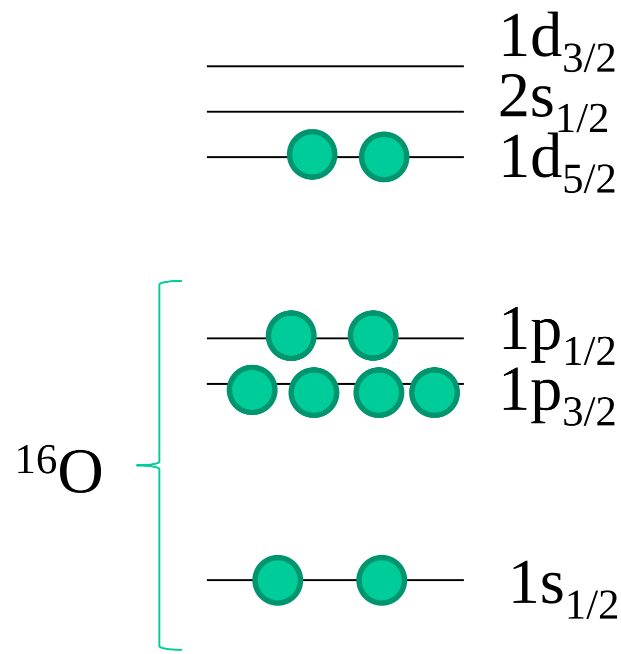
at least 6 levels below 2 MeV (?)

pure mean-field approximation:

0^+		1.74 MeV
$2^+,3^+$		0.78 MeV
$0^+,2^+,4^+$		0

${}^{18}_{8}O_{10}$

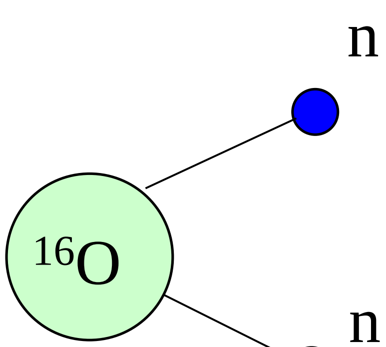
6 levels



in reality:

2^+		1.98 MeV
0^+		0

${}^{18}_{8}O_{10}$



only two levels!

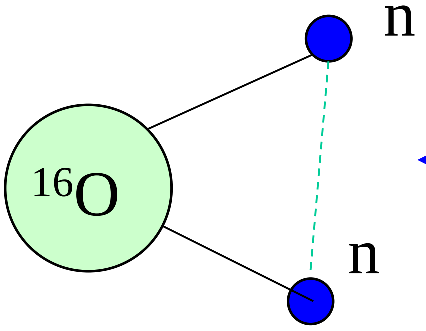
what is going on?

$$H=\sum_i T_i+\sum_{i<j}v_{ij}\to H=\sum_i(T_i+V_i)+\underbrace{\cancel{\sum_{i<j}v_{ij}-\sum_i V_i}}$$

deviation from the average
(residual interaction)

Can the residual interaction be neglected completely?

→ "no" for open-shell nuclei (pairing correlation)



Pairing correlation

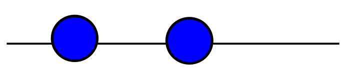
$$H = \sum_{i=1}^{A}\left(-\frac{\hbar^2}{2m}\boldsymbol{\nabla}_i^2 + V_{\rm HF}(i)\right) + \underline{\underline{\frac{1}{2}\sum_{i,j}^{A} v(\boldsymbol{r}_i,\boldsymbol{r}_j) - \sum_i V_{\rm HF}(i)}}$$

$$v_{\rm res}(\boldsymbol{r},\boldsymbol{r}')$$

A delta function interaction for a residual interaction:
(an extremely short range interaction)

$$\begin{aligned} v_{\rm res}(\boldsymbol{r},\boldsymbol{r}') &\sim -g\,\delta(\boldsymbol{r}-\boldsymbol{r}') \\ &= -g\frac{\delta(r-r')}{rr'}\sum_{\lambda\mu} Y^*_{\lambda\mu}(\hat{\boldsymbol{r}})Y_{\lambda\mu}(\hat{\boldsymbol{r}}') \end{aligned}$$

Estimate the effect of $v_{\rm res}$ using the perturbation theory:

l 

unperturbative wave function:
two neutrons in a angular momentum l state
with the total angular momentum L

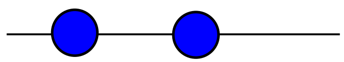
$$\boxed{|(ll)LM\rangle = \sum_{m,m'}\langle lml m'|LM\rangle\psi_{lm}(\boldsymbol{r})\psi_{lm'}(\boldsymbol{r}')}$$

Pairing correlations

$$v_{\text{res}}(\boldsymbol{r},\boldsymbol{r}') \sim -g\,\delta(\boldsymbol{r}-\boldsymbol{r}')$$

$$= -g\frac{\delta(r-r')}{rr'}\sum_{\lambda\mu} Y^*_{\lambda\mu}(\hat{\boldsymbol{r}})Y_{\lambda\mu}(\hat{\boldsymbol{r}}')$$

l



$$\boxed{|(ll)LM\rangle = \sum_{m,m'}\langle lmlm'|LM\rangle\psi_{lm}(\boldsymbol{r})\psi_{lm'}(\boldsymbol{r}')}$$

The energy change due to the residual interaction:

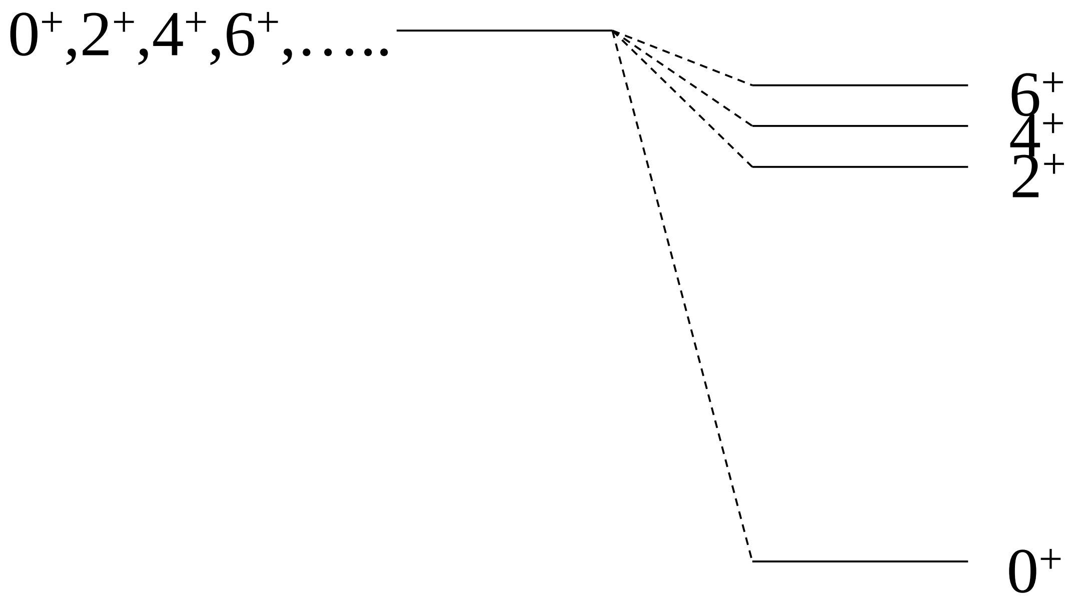
$$\Delta E_L = \langle (ll)LM|v_{\text{res}}|(ll)LM\rangle$$

$$= -g\,I_r^{(l)}\,\frac{(2l+1)^2}{4\pi}\begin{pmatrix} l & l & L \\ 0 & 0 & 0 \end{pmatrix}^2$$

$$\psi_{lm}(\boldsymbol{r}) = R_l(r)Y_{lm}(\hat{\boldsymbol{r}}) \qquad I_r^{(l)} = \int_0^\infty r^2dr\,(R_l(r))^4$$

$$\Delta E_L = -g\, I_r^{(l)} \frac{(2l+1)^2}{4\pi} \begin{pmatrix} l & l & L \\ 0 & 0 & 0 \end{pmatrix}^2 \equiv -g\, I_r^{(l)} \frac{A(ll;L)}{4\pi}$$

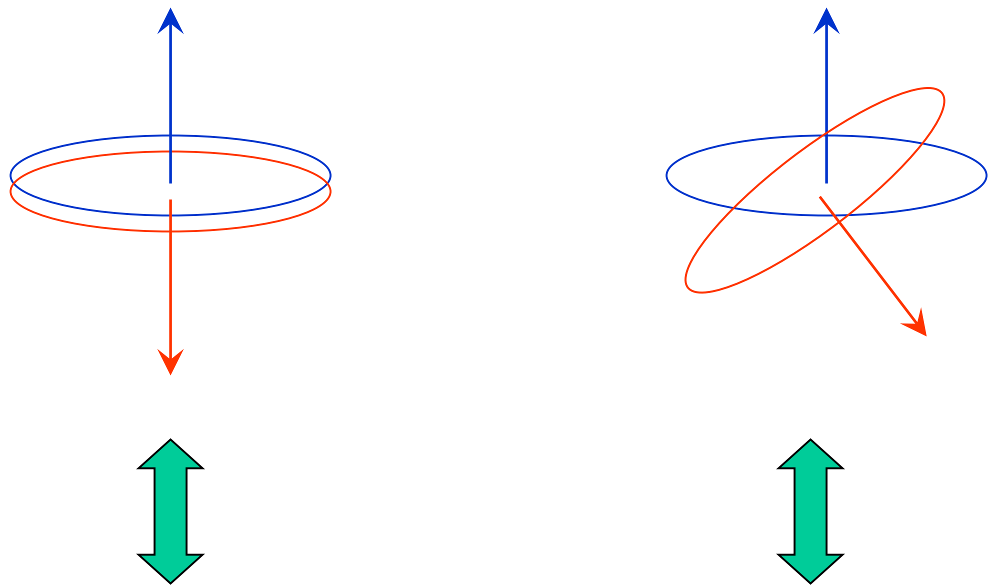
$A(ll;L)$	L=0	L=2	L=4	L=6	L=8
l = 2	5.00	1.43	1.43	---	---
l = 3	7.00	1.87	1.27	1.63	---
l = 4	9.00	2.34	1.46	1.26	1.81



without residual interaction

with residual interaction

Simple interpretation:



$L = 0$ pair $\qquad\qquad$ $L \neq 0$ pair

$\boxed{\text{The spatial overlap is the largest for an } L = 0 \text{ pair.}}$

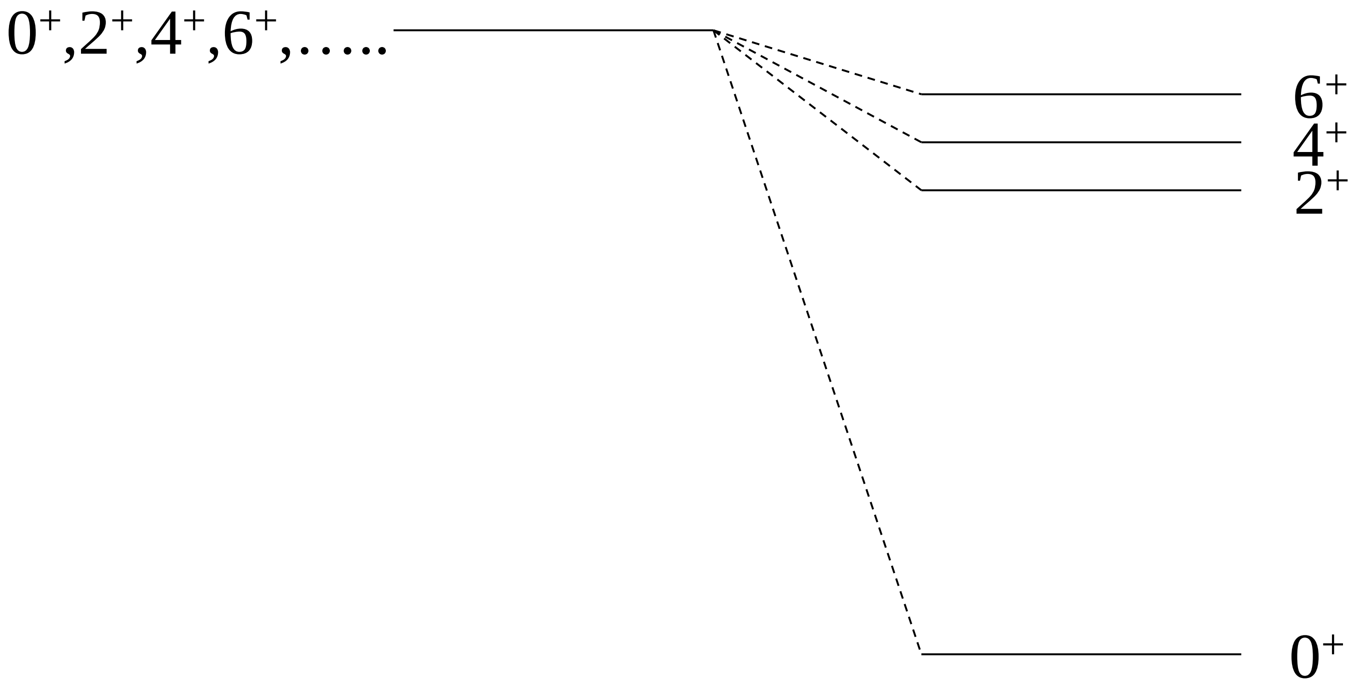
“Pairing Correlation”

(note) The $L=2l$ pair is unfavoured due to the Pauli principle.

(note)

$$\psi(l^2; L=0) = \sum_\mu \langle l\mu l-\mu | L=0,0\rangle \, Y_{l\mu}(\hat{\boldsymbol{r}}_1) Y_{l-\mu}(\hat{\boldsymbol{r}}_2) = Y_{l0}(\theta_{12})/\sqrt{4\pi}$$

“Pairing Correlation”



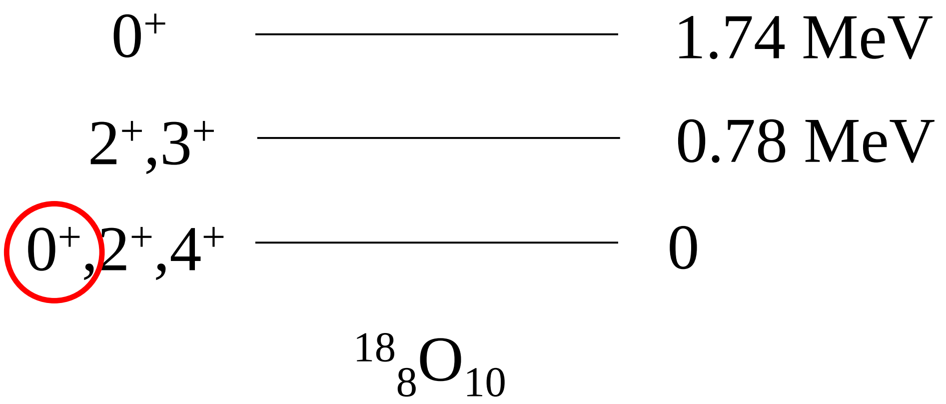
without residual interaction

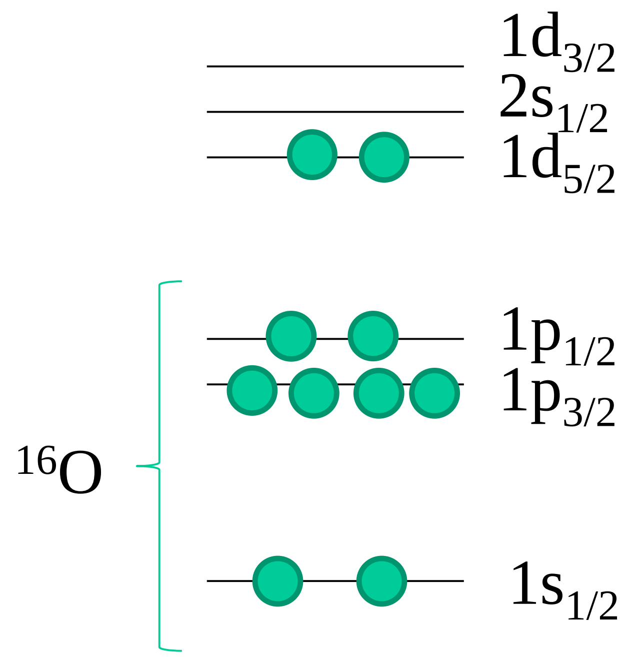
with residual interaction

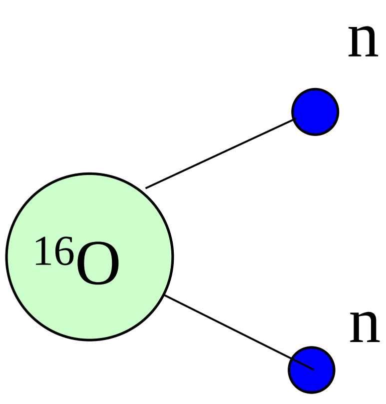
The ground state spin of nuclei

- Even-even nuclei: 0^+
- Even-odd nuclei: the spin of the valence particle

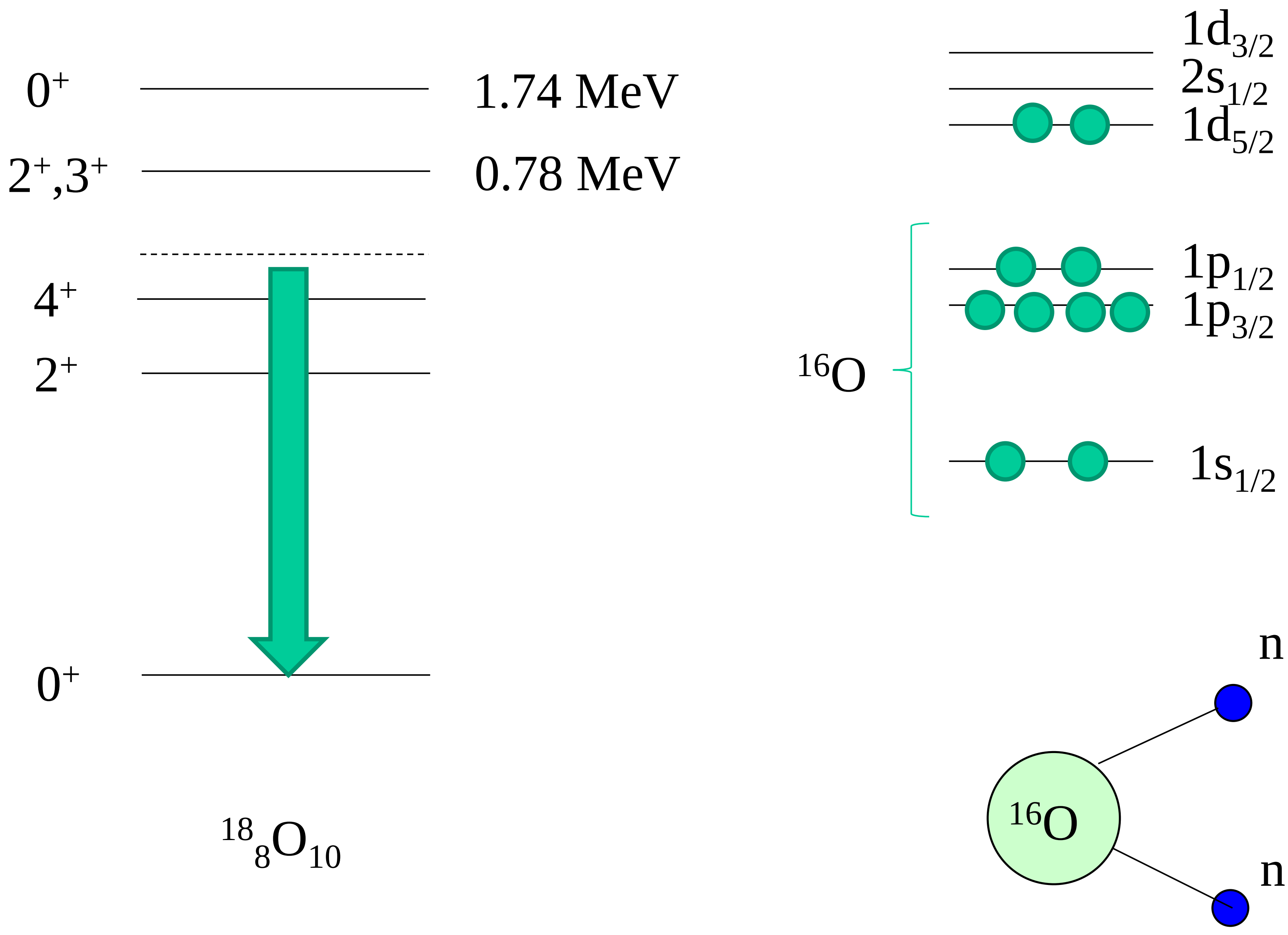
pure mean-field approximation:







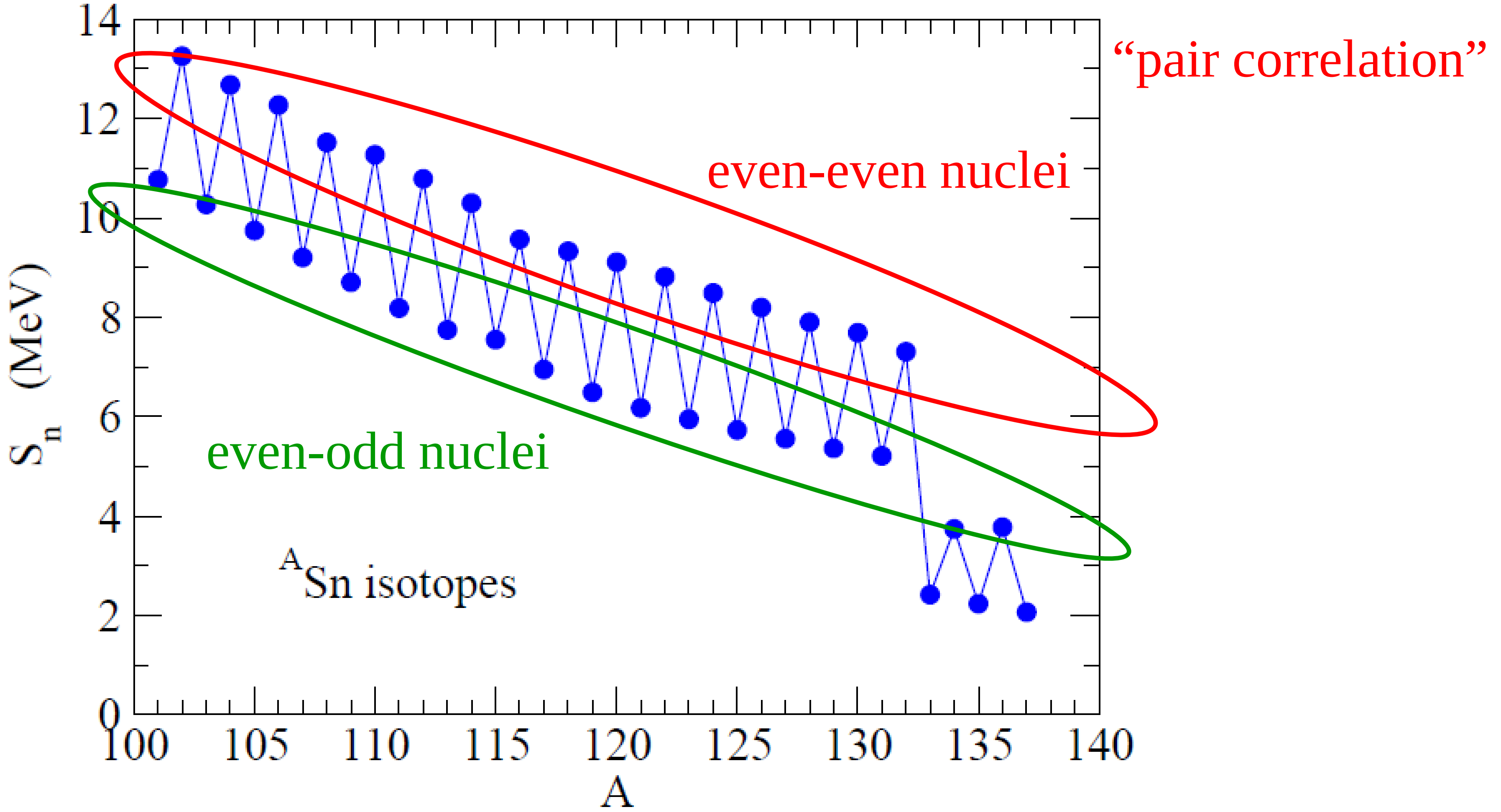
pure mean-field approximation:



Pairing energy

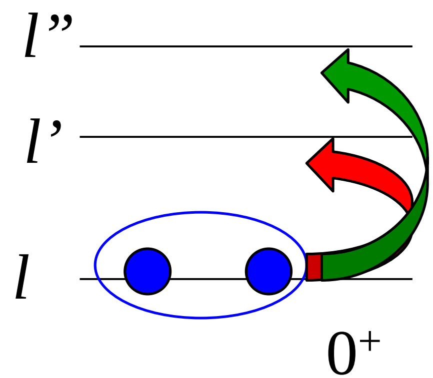
A larger energy required to remove one neutron from even number than from odd number

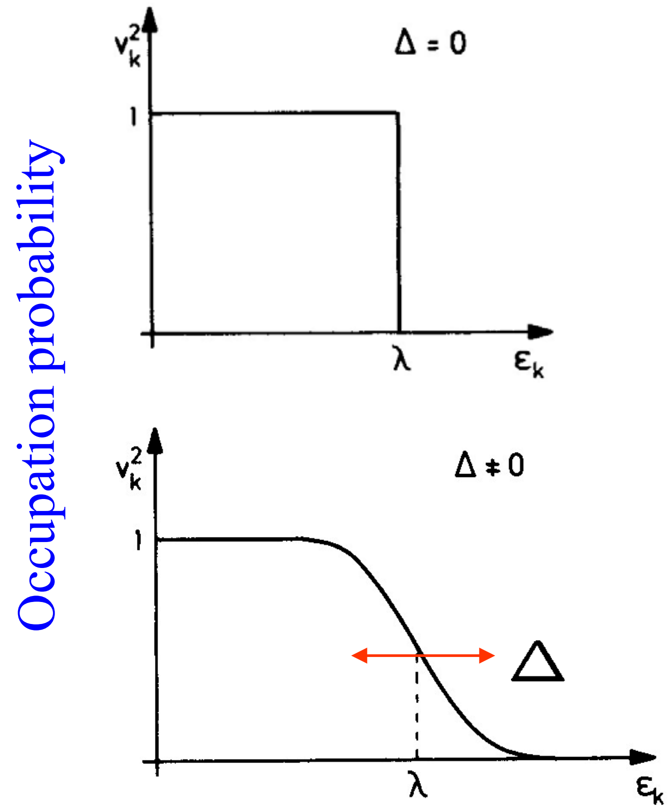
even-odd staggering



1n separation energy: $S_n\,(A,Z) = B(A,Z) - B(A-1,Z)$

Wave functions:





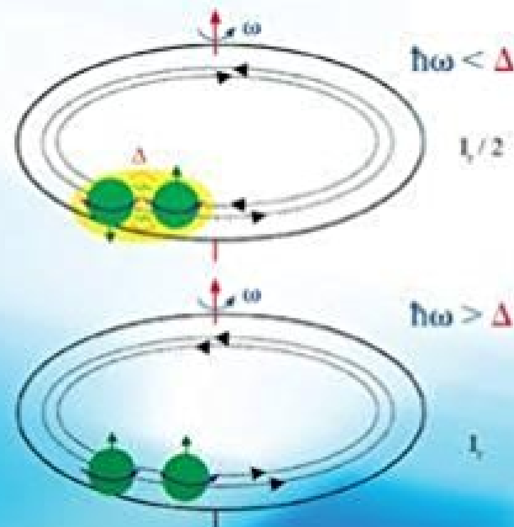
$$
\begin{aligned}
|\Psi_{0+}\rangle &= |(ll)L=0\rangle \\
&+\sum_{l'} \frac{\langle (l'l')L=0|v_{\rm res}|(ll)L=0\rangle}{2\epsilon_l - 2\epsilon_{l'}} |(l'l')L=0\rangle + \cdots
\end{aligned}
$$

Each orbit is occupied only partially.

cf. BCS theory (super fluidity/super conductivity)

Fifty Years of Nuclear BCS

Pairing in Finite Systems

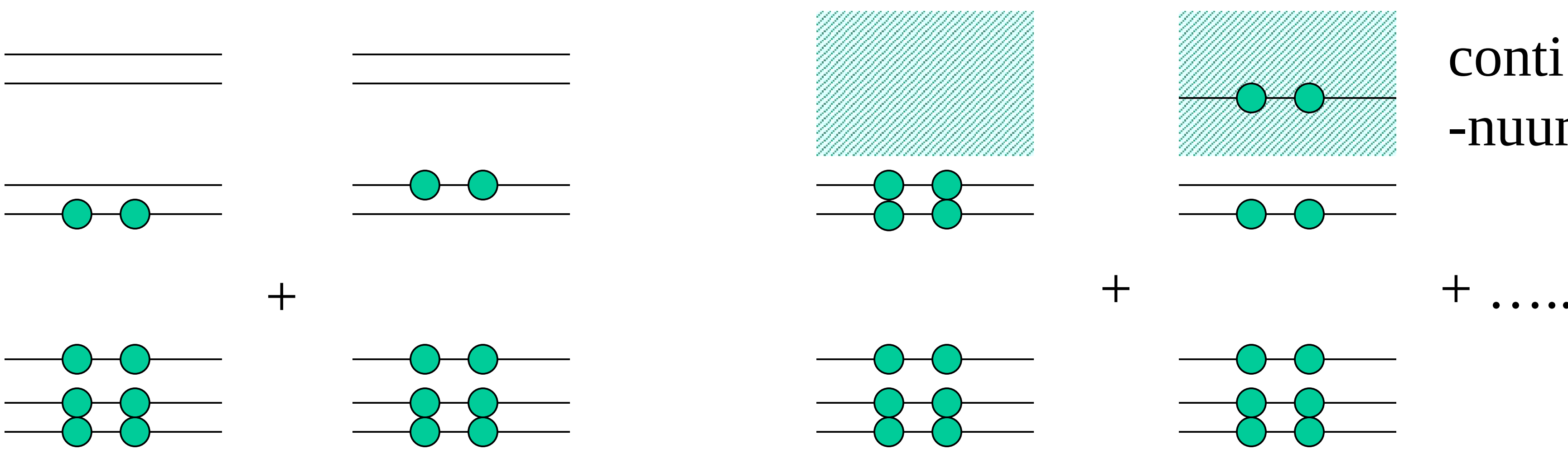


Ricardo A Broglia
Vladimir Zelevinsky
editors

World Scientific

Role of redidual interaction

$$H=\sum_i T_i+\sum_{i<j} v_{ij} \rightarrow H=\sum_i (T_i+V_i)+\underbrace{\sum_{i<j} v_{ij}-\sum_i V_i}_{\text{residual interaction (pairing)}}$$



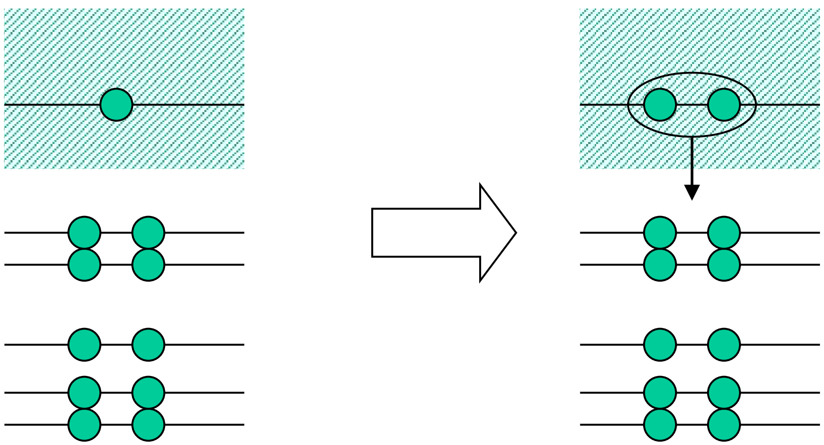
open shell nuclei

$\longrightarrow$ superfluidity

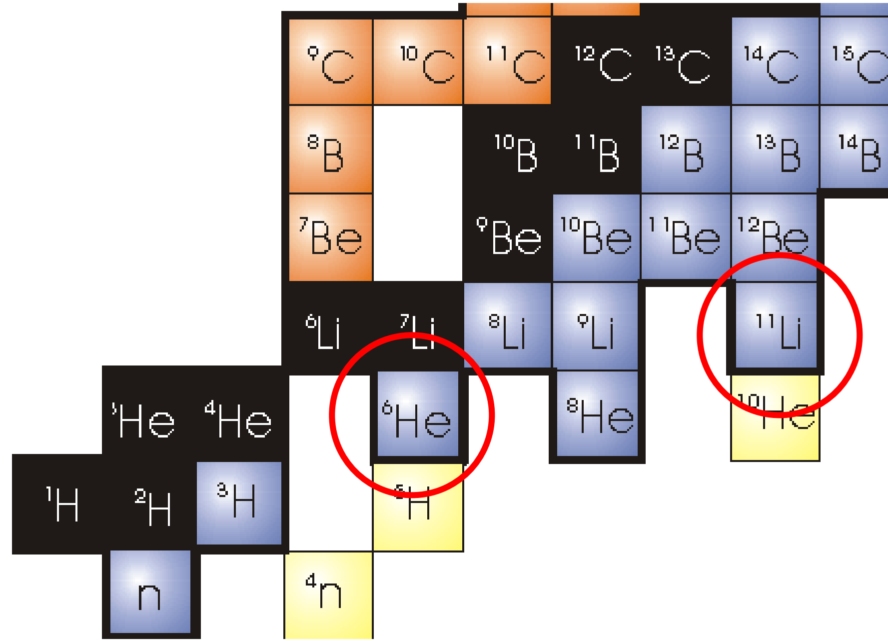
weakly bound nuclei

Borromean nucleus

residual interaction → attractive



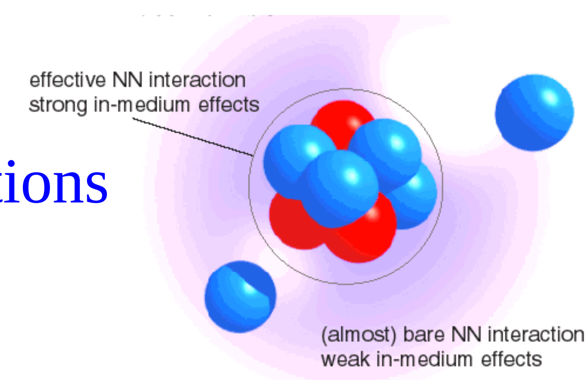
particle unstable particle stable



"Borromean nuclei"

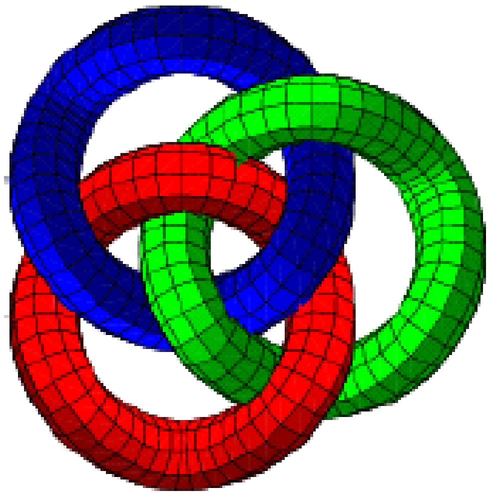
Structure of Borromean nuclei

✓non-trivial due to many-body correlations
✓has attracted lots of attention



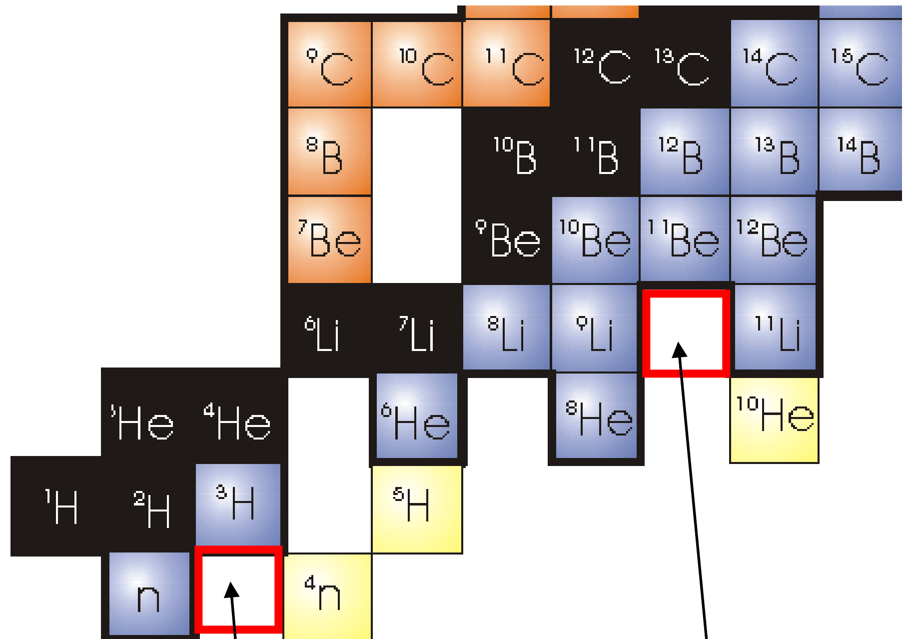
Borromean nuclei

n



n

^{9}Li



^{10}Li (^{9}Li+n)
does not exist

Borromean nuclei

^{2}n (n+n) does not exist

Another typical example: ^{6}He

What is “Borromean” ?

Even though three rings are tied together,
two rings can be separated once any of three is removed.

“Borromean rings”

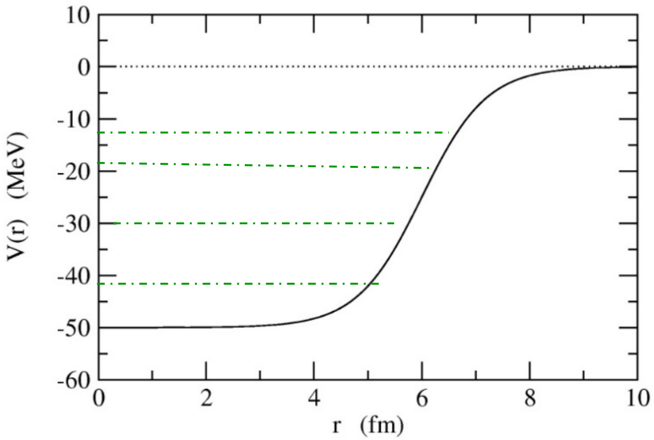
What is "Borromean" ?

Borromean islands
(northen Italy,
in Lake Maggiore)
near Milano

Crest of Borromeo Family
(13th century)

HF+BCS theory

① Mean-field approximation for a mean-field potential
(first, an average behavior)



② Next, an occupation probability for each level
based on the variational principle including the residual interaction

$$\boxed{|BCS\rangle = \prod_{k>0}\left(u_k + v_k\, a_k^\dagger a_{\bar{k}}^\dagger\right)\left|0\right\rangle}$$

$$a_k^\dagger = a_{jlm}^\dagger, \;\; a_{\bar{k}}^\dagger = (-)^{l+j-m} a_{jl-m}^\dagger$$

$$\left\langle BCS|a_k^\dagger a_k|BCS\right\rangle = |v_k|^2$$

occupation probability

For the Hamiltonian:

$$\boxed{H = \sum_{\nu} \epsilon_k (a_k^\dagger a_k + a_{\bar{k}}^\dagger a_{\bar{k}}) - G \left(\sum_{k>0} a_k^\dagger a_{\bar{k}}^\dagger \right) \left(\sum_{k>0} a_{\bar{k}} a_k \right)}$$

$$\boxed{\begin{aligned} u_\nu^2 &= \frac{1}{2}\left(1 + \frac{\epsilon_\nu - \lambda}{E_k}\right) \\ v_\nu^2 &= \frac{1}{2}\left(1 - \frac{\epsilon_\nu - \lambda}{E_k}\right) \\ E_k &= \sqrt{(\epsilon_k - \lambda)^2 + \Delta^2} \end{aligned}}$$

λ: chemical potential (Fermi energy)

$$\begin{aligned} \Delta &= G\langle BCS| \sum_{k>0} a_k^\dagger a_{\bar{k}}^\dagger |BCS\rangle = G \sum_{\nu>0} u_\nu v_\nu \\ &= \frac{G}{2} \sum_{\nu>0} \frac{\Delta}{E_\nu} \end{aligned}$$

pairing gap

i) Trivial solution: always exists

$$\Delta = 0$$

$$\begin{aligned} v_\nu^2 &= 1 \quad (\epsilon_\nu \le \lambda) \\ &= 0 \quad (\epsilon_\nu > \lambda) \end{aligned}$$

$$|\Psi\rangle = \prod_{\nu>0} a_\nu^\dagger a_{\bar\nu}^\dagger |0\rangle$$

G a/o $N \longrightarrow$ large

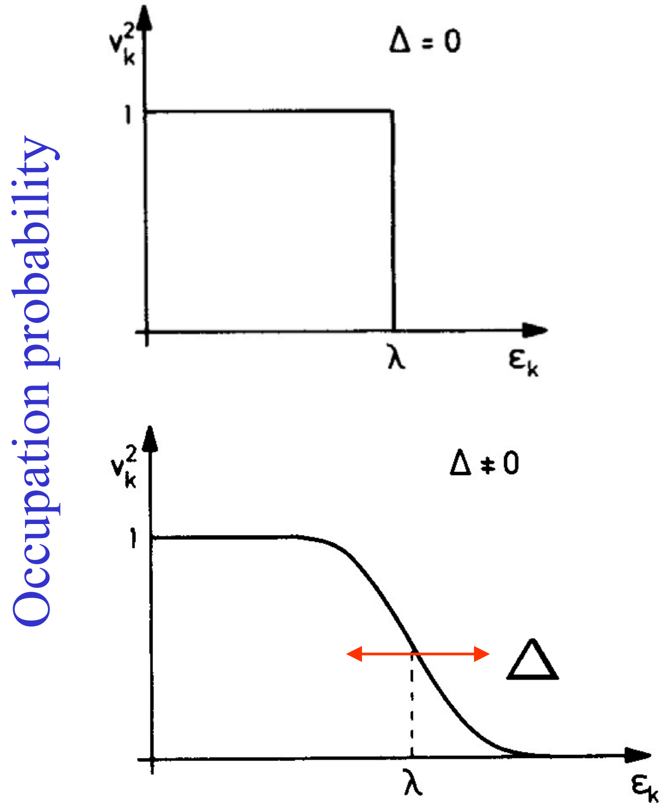
ii) Superfluid solution

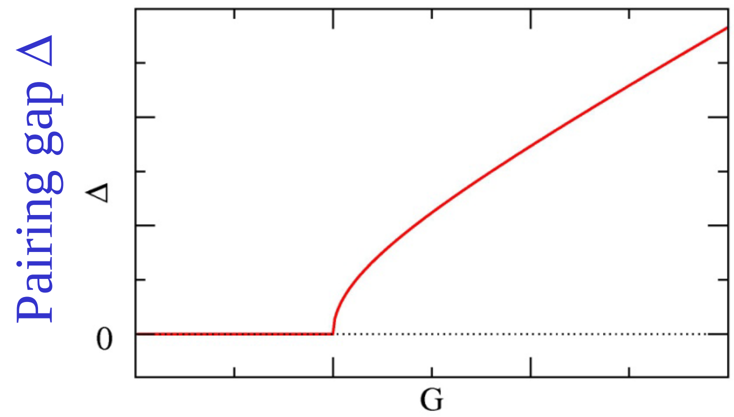
$$\Delta \neq 0$$

$$v_\nu^2 < 1$$

$$|BCS\rangle = \prod_{\nu>0} \left(u_\nu + v_\nu\, a_\nu^\dagger a_{\bar\nu}^\dagger\right) \Big| 0 \Big\rangle$$

the number fluctuation





Normal-Superfulid phase transition

Quasi-particle excitations

$$H \sim E_{BCS} + \sum_k E_k \alpha_k^\dagger \alpha_k \qquad \alpha_k|BCS\rangle = 0$$

$$E_k = \sqrt{(\epsilon_k - \lambda)^2 + \Delta^2}$$

$$\alpha_\nu^\dagger = u_\nu a_\nu^\dagger - v_\nu a_{\bar{\nu}}, \qquad \alpha_{\bar{\nu}}^\dagger = u_\nu a_{\bar{\nu}}^\dagger + v_\nu a_\nu$$

(Bogoliubov transformation)

(note) $E_k = \sqrt{(\epsilon_k - \lambda)^2 + \Delta^2} \geq \Delta$ (energy gap)

- ✓ a nucleus with N+1 nucleons: $\alpha_\nu^\dagger|BCS\rangle$
- ✓ excited states of the same nucleus: $\alpha_\nu^\dagger\alpha_{\nu'}^\dagger|BCS\rangle$

(note) $\alpha^\dagger\alpha^\dagger \sim a^\dagger a^\dagger + a^\dagger a + a a^\dagger + aa$

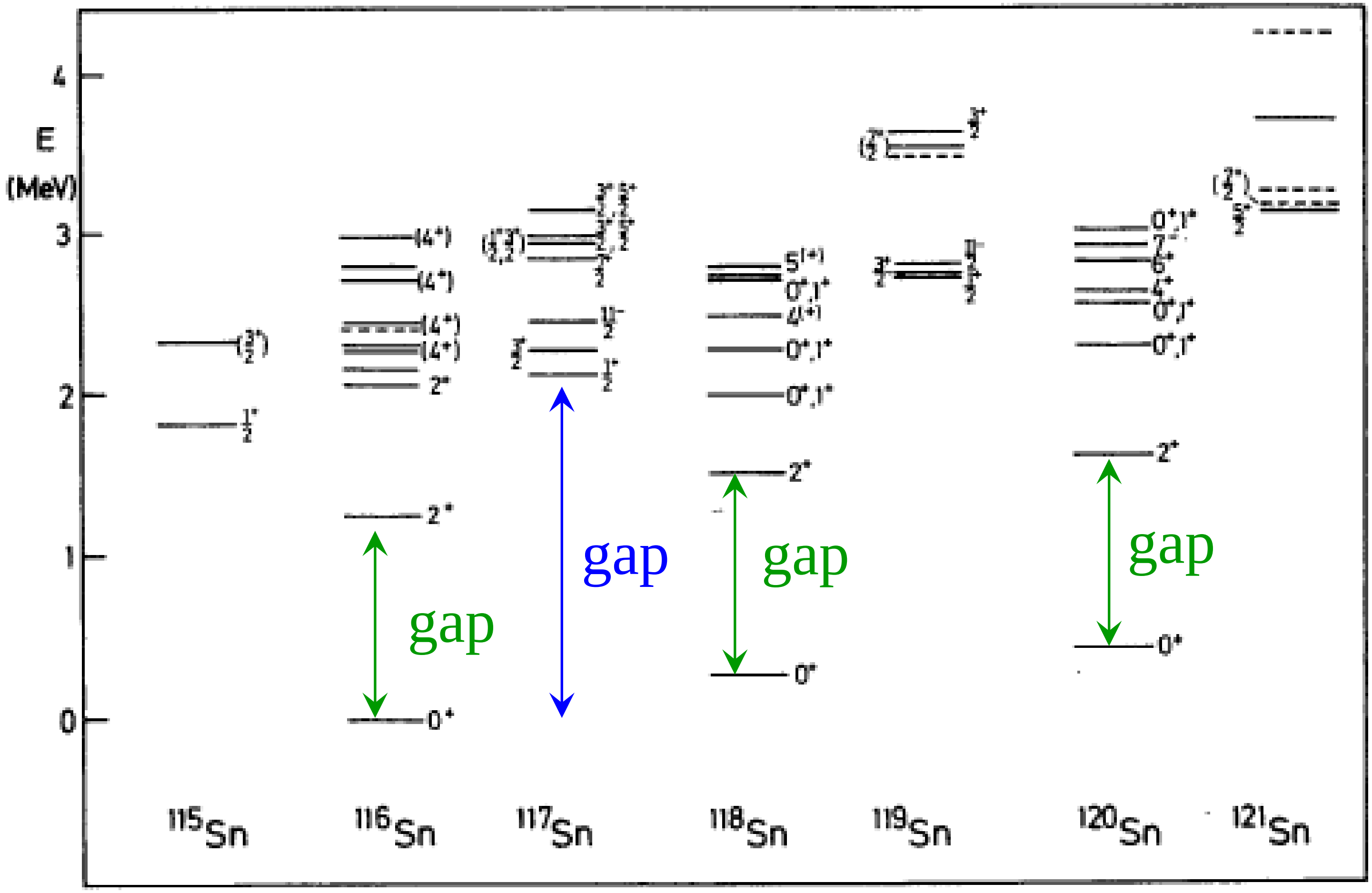
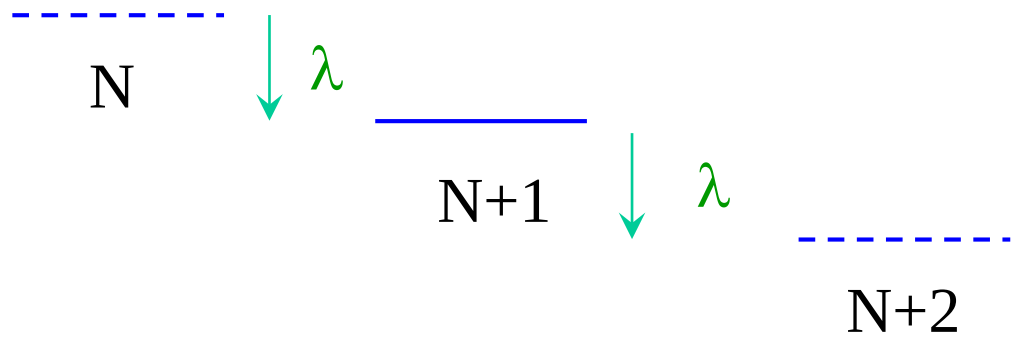


Figure 6.1. Excitation spectra of the $_{50}$Sn isotopes.

Ring-Schuck

Even-odd mass difference and pairing gap

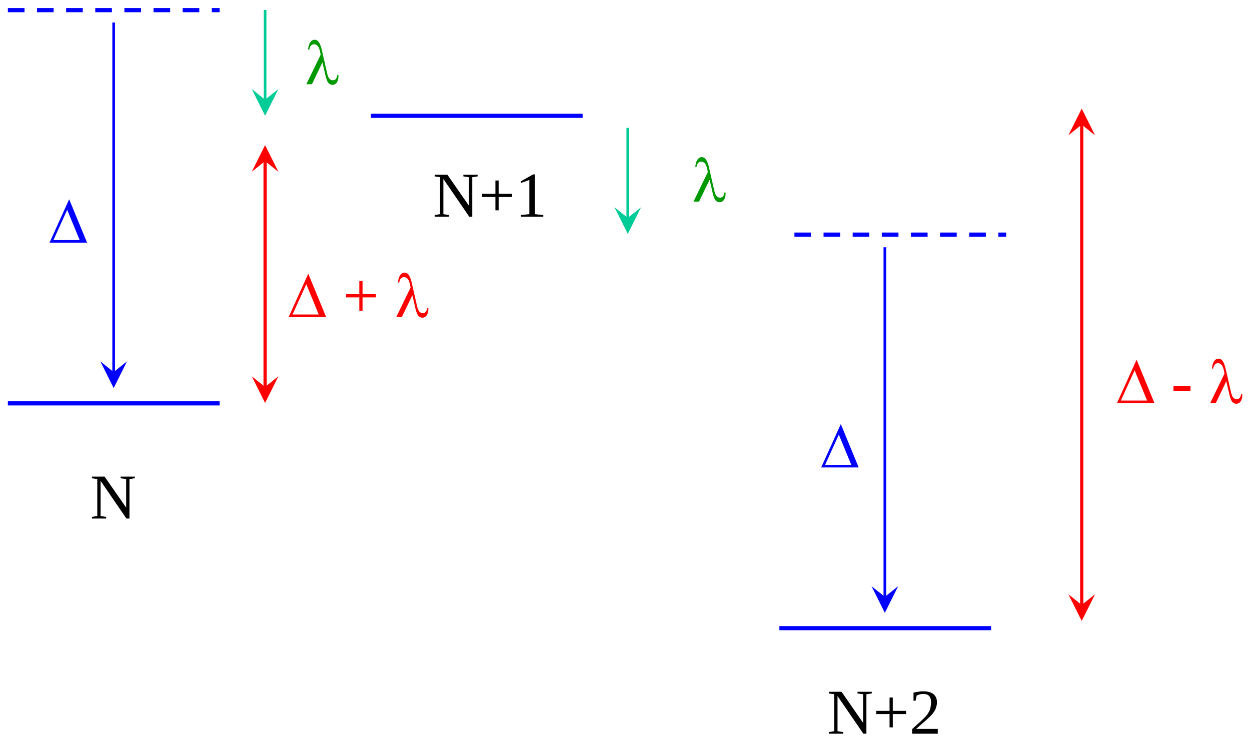
$$
\begin{aligned}
E(N+2,Z) &= E(N,Z)+2\lambda \\
E(N+1,Z) &= E(N,Z)+\lambda+\Delta
\end{aligned}
$$



$$\boxed{-\Delta_n \sim [E(N+2,Z)-2E(N+1,Z)+E(N,Z)]/2}$$

Even-odd mass difference and pairing gap

$$E(N+2,Z) = E(N,Z) + 2\lambda$$
$$E(N+1,Z) = E(N,Z) + \lambda + \Delta$$



(note) $\lambda < 0$

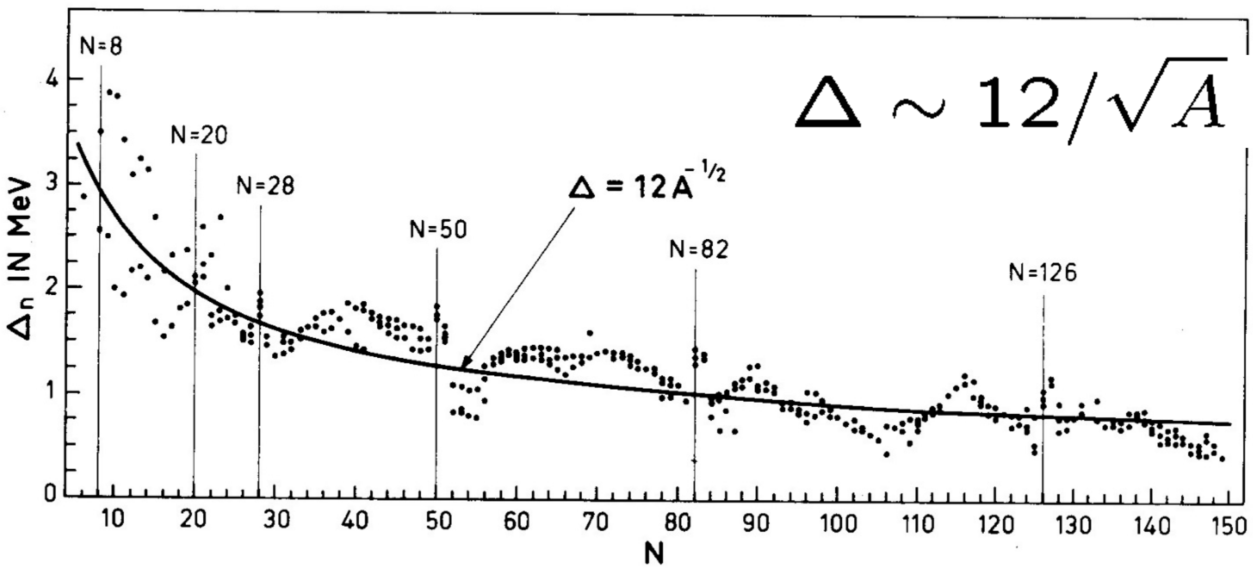
$$\boxed{-\Delta_n \sim [E(N+2,Z) - 2E(N+1,Z) + E(N,Z)]/2}$$

Even-odd mass difference and pairing gap

$$
\begin{aligned}
B_{\text{pair}} &= \Delta \quad &\text{(for even} - \text{even)}\\
&= 0 \quad &\text{(for even} - \text{odd)}\\
&= -\Delta \quad &\text{(for odd} - \text{odd)}
\end{aligned}
$$

$$
\begin{aligned}
E(N+2,Z) &= E(N,Z) + 2\lambda\\
E(N+1,Z) &= E(N,Z) + \lambda + \Delta
\end{aligned}
$$

$$\boxed{-\Delta_n \sim [E(N+2,Z) - 2E(N+1,Z) + E(N,Z)]/2}$$



$$\Delta \sim 12/\sqrt{A} \quad \text{(MeV)}$$

Bohr-Mottelson ('69)

Hartree-Fock-Bogoliubov (HFB) Theory

HF+BCS method: two-step procedure

(first MF potential, then occupation probabilities)

$$\psi_k(\boldsymbol{r}), u_k, v_k$$

improvement: MF and occ. prob. at the same time

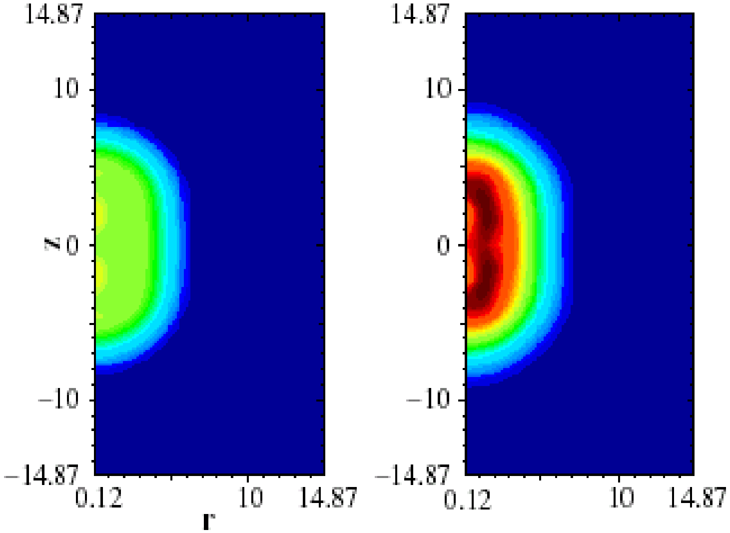
Hartree-Fock-Bogoliubov (HFB) theory:

wave function+occupation probabilities

$$U_k(\boldsymbol{r}), V_k(\boldsymbol{r})$$

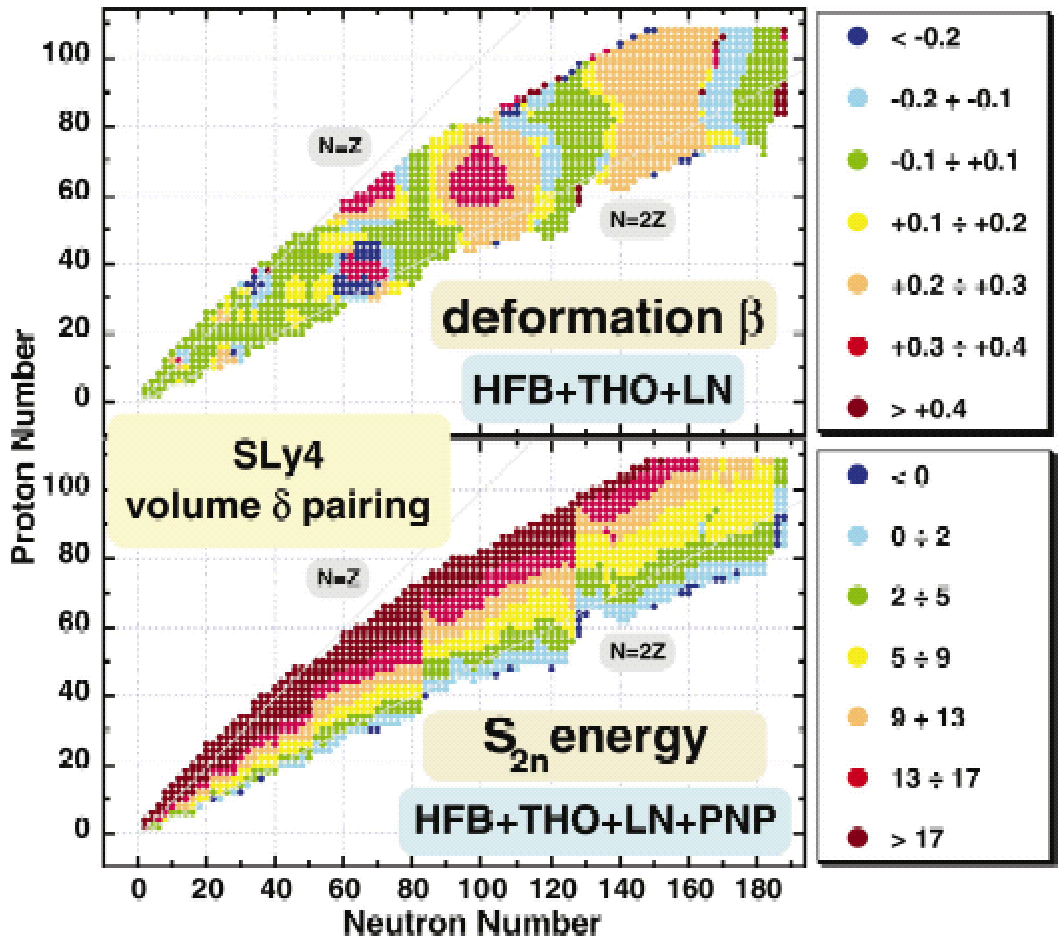
Application of the HFB method

Density of ^{110}Zr (SHFB-SLy4)



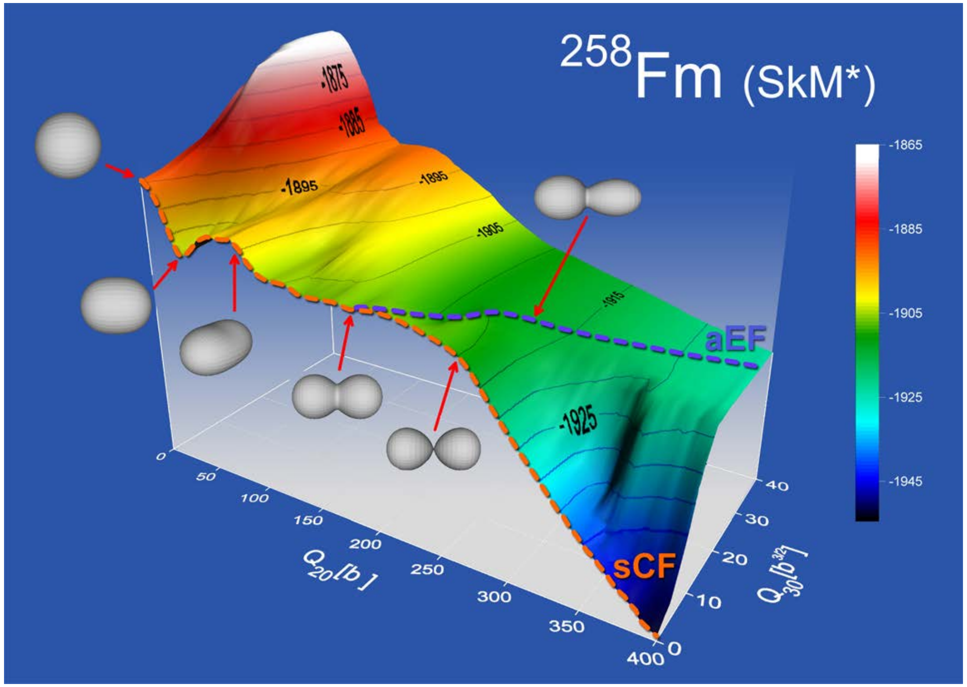
A. Blazkiewicz et al., PRC71('05)054231

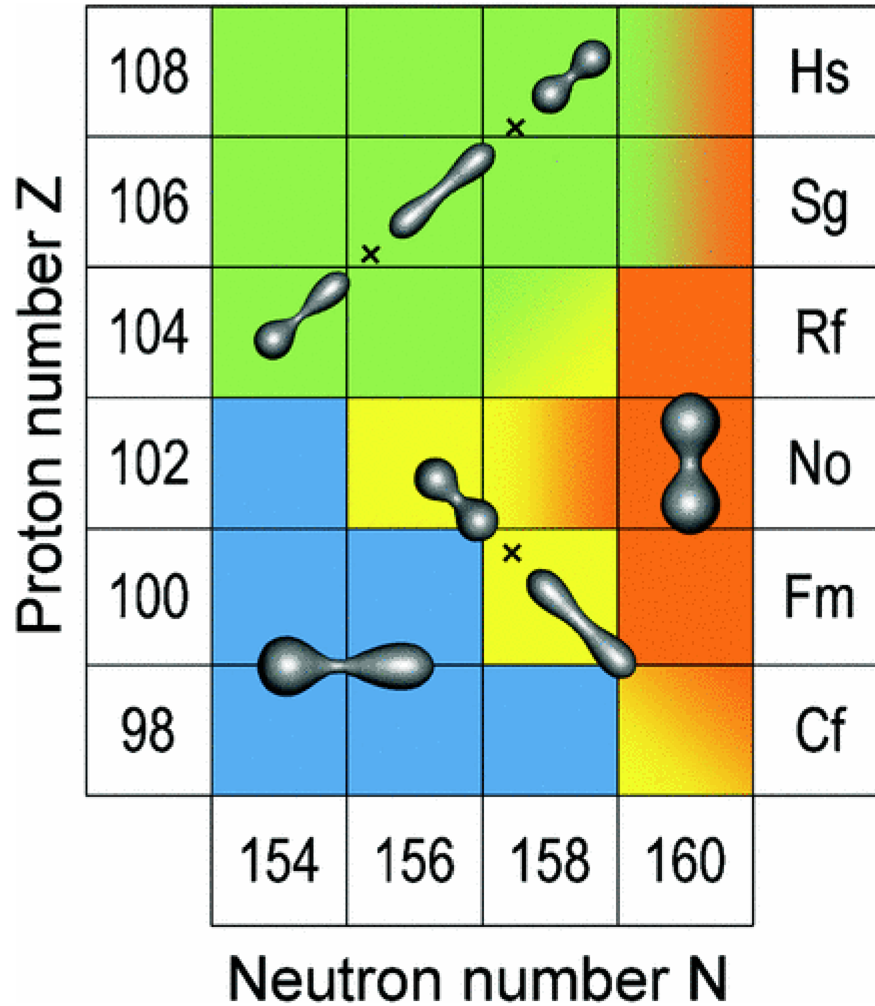
Systematics of β_2 and S_{2n}



M.V. Stoitsov et al., PRC68('03)054312

potential energy surface for fission process





A. Staszczak, A. Baran, J. Dobaczewski,
and W. Nazarewicz, PRC80 (‘09) 014309