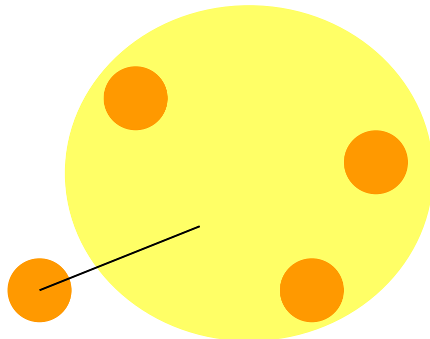
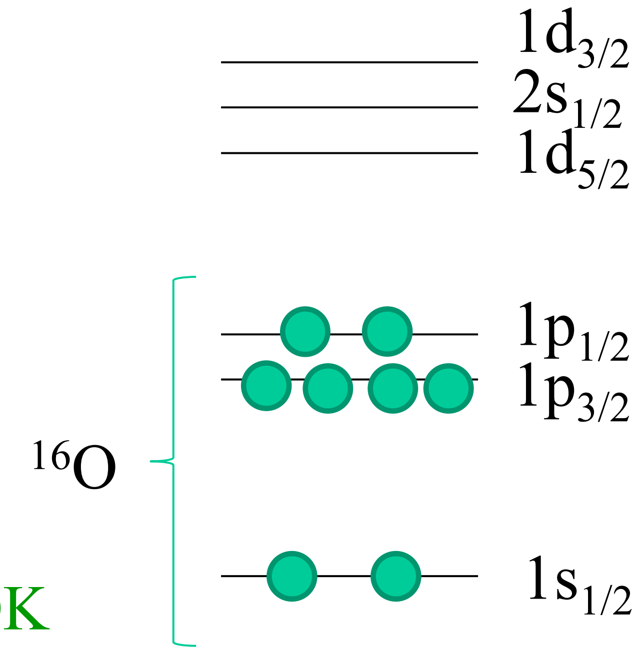
対相関

単純な平均場近似(独立粒子描像):





閉殻の場合はOK

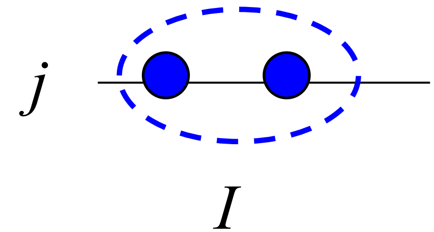
開殻原子核では「対相関」が重要

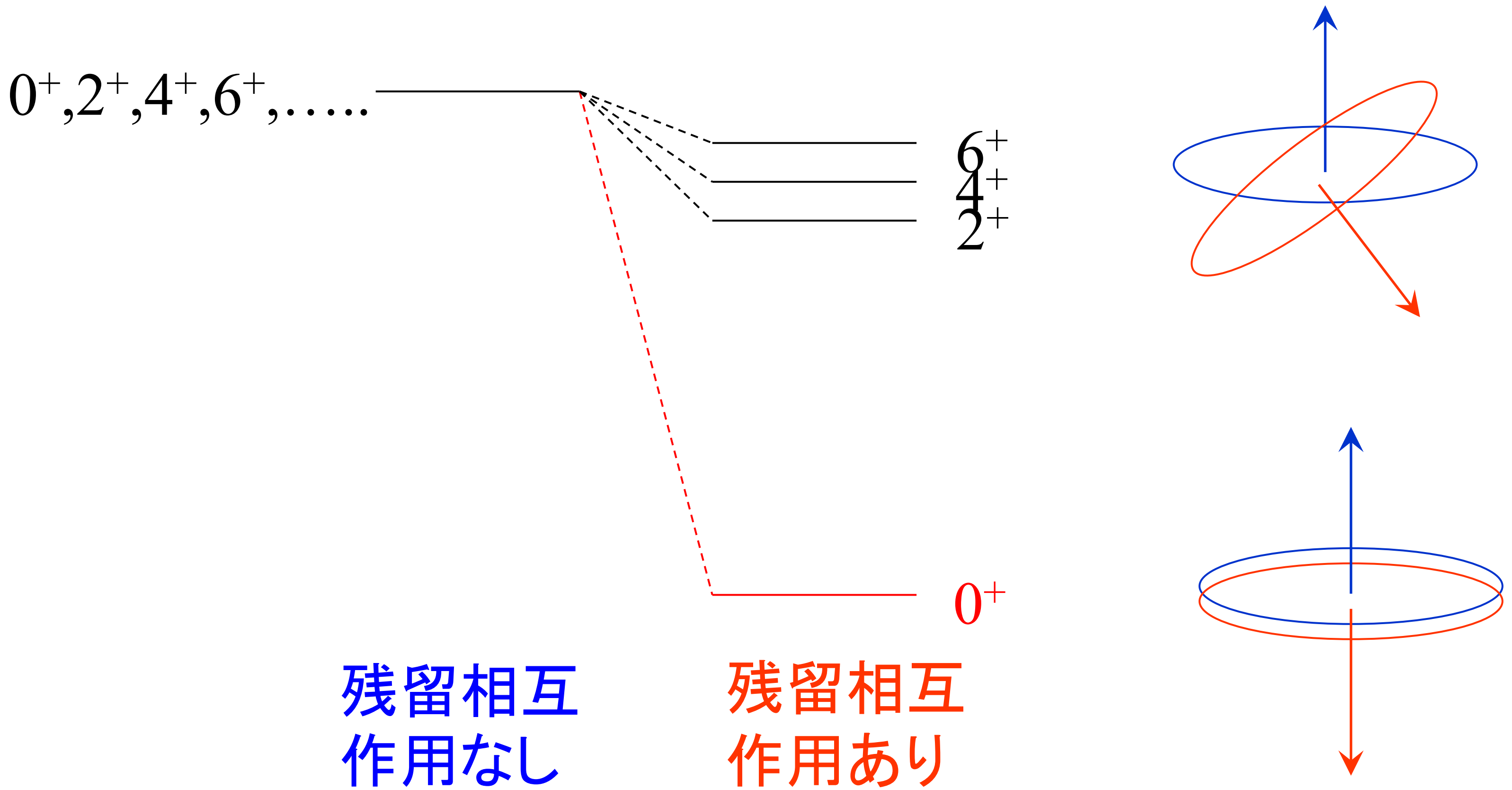
$$H = \sum_i T_i + \sum_{i<j} v_{ij} \to H = \sum_i (T_i + V_i) + \cancel{\underbrace{\sum_{i<j} v_{ij} - \sum_i V_i}}$$

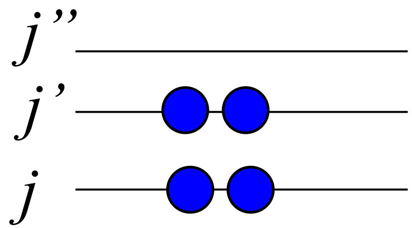
平均からのずれ

（残留相互作用）

$$v_{\mathrm{res}}(\boldsymbol{r},\boldsymbol{r}') \quad \sim -g\,\delta(\boldsymbol{r}-\boldsymbol{r}')$$

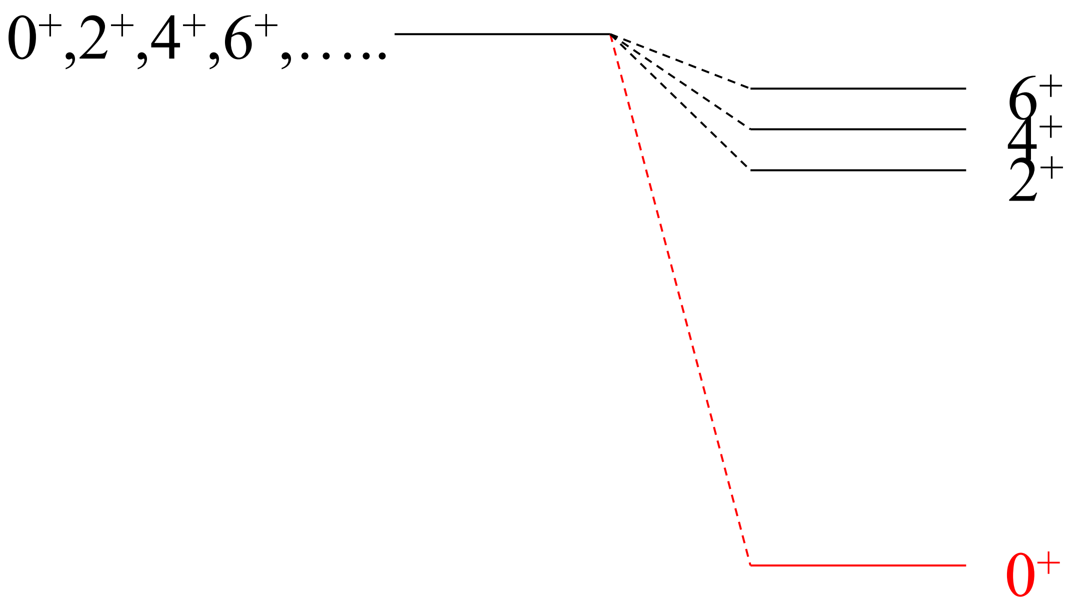






複数個のレベルに
複数個のペアがある問題

$$v_{\rm res}(\boldsymbol{r},\boldsymbol{r}') \quad \sim -g\,\delta(\boldsymbol{r}-\boldsymbol{r}')$$



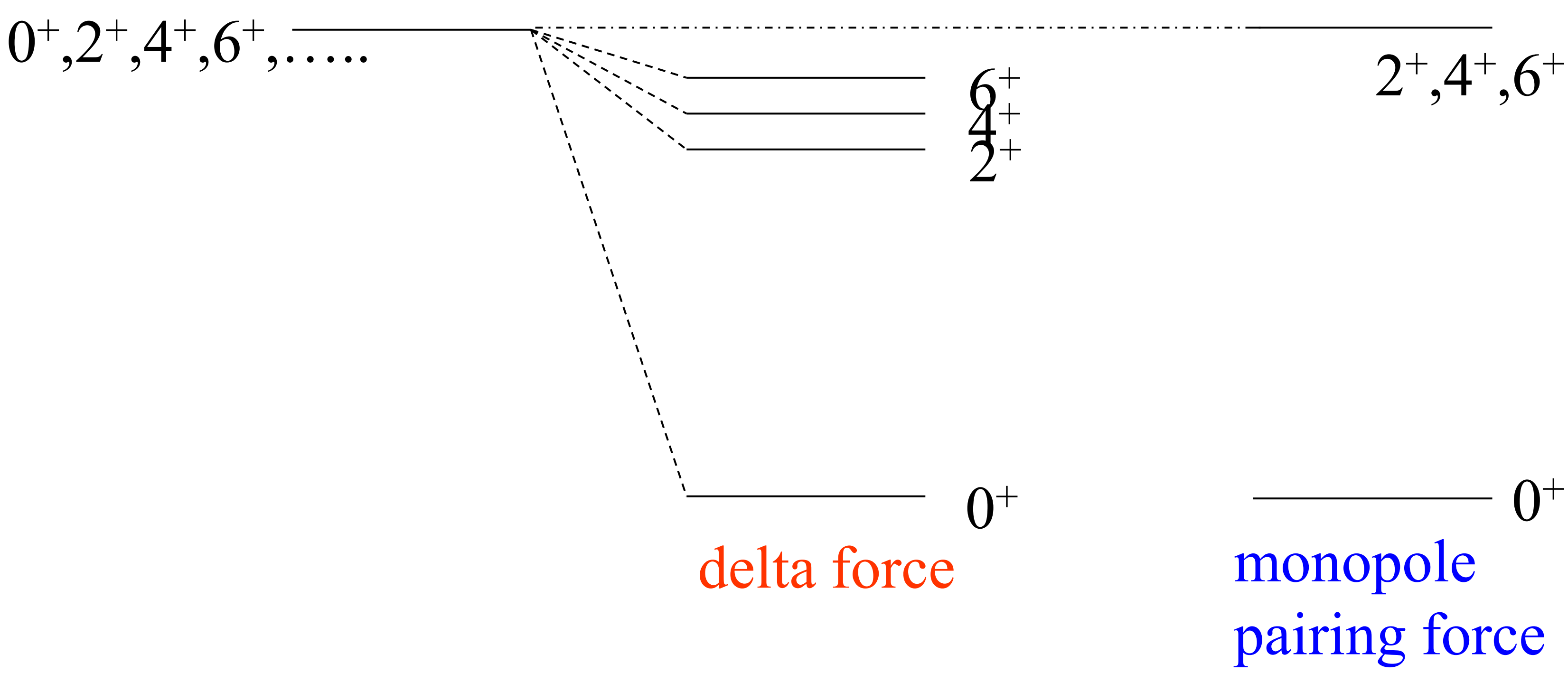
デルタ関数のままでもいいが、説明を簡単にするためにもう少し簡単にした相互作用を導入する。

Simplified pairing interaction

$$\boxed{V = -G\,P^\dagger P; \qquad P^\dagger = \sum_{\nu>0} a^\dagger_\nu a^\dagger_{\bar{\nu}}}$$

$\bar{\nu}$: the time reversed state of ν

e.g., $|\nu\rangle = |njlm\rangle, \quad |\bar{\nu}\rangle = |njl-m\rangle$

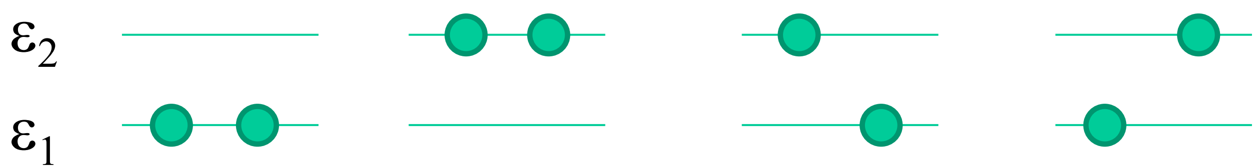


Simplified pairing interaction

$$\boxed{V = -G\,P^\dagger P; \qquad P^\dagger = \sum_{\nu>0} a^\dagger_\nu a^\dagger_{\bar{\nu}}}$$

$\bar{\nu}$: the time reversed state of ν

$$H = \sum_k \epsilon_k (a^\dagger_k a_k + a^\dagger_{\bar{k}} a_{\bar{k}}) - G \left(\sum_{k>0} a^\dagger_k a^\dagger_{\bar{k}} \right) \left(\sum_{k>0} a_{\bar{k}} a_k \right)$$

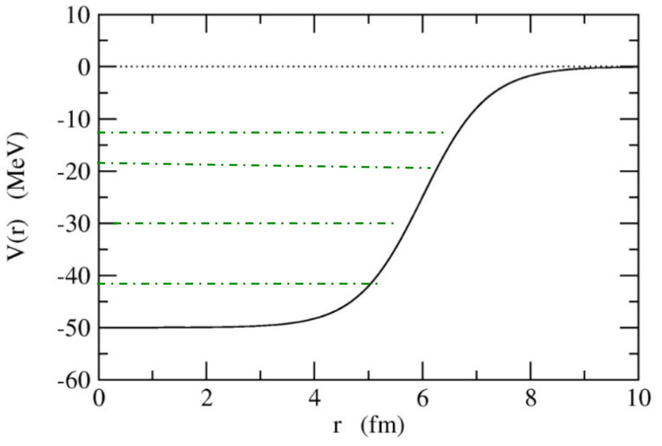


$$H = \begin{pmatrix} 2\epsilon_1 - G & -G & 0 & 0 \\ -G & 2\epsilon_2 - G & 0 & 0 \\ 0 & 0 & \epsilon_1 + \epsilon_2 & 0 \\ 0 & 0 & 0 & \epsilon_1 + \epsilon_2 \end{pmatrix}$$

$$\rightarrow \quad \Psi_{\text{g.s.}} = C_1 \Psi_1 + C_2 \Psi_2$$

HF+BCS theory

①平均場近似をして核子の感じるポテンシャルを求める
（平均的な振る舞いをまず決める）



$$H=\sum_k \epsilon_k(a_k^\dagger a_k + a_{\bar{k}}^\dagger a_{\bar{k}}) - G\left(\sum_{k>0} a_k^\dagger a_{\bar{k}}^\dagger\right)\left(\sum_{k>0} a_{\bar{k}} a_k\right)$$

②各準位の占有確率を決める。
決め方は、残留相互作用も含めてエネルギーが最小になるようにする。

$$H = \sum_{\nu} \epsilon_\nu (a^\dagger_\nu a_\nu + a^\dagger_{\bar{\nu}} a_{\bar{\nu}}) \underbrace{- G \left(\sum_{\nu>0} a^\dagger_\nu a^\dagger_{\bar{\nu}} \right) \left(\sum_{\nu>0} a_{\bar{\nu}} a_\nu \right)}_{\substack{\text{2体の相互作用} \\ \to \text{ １体近似をする}}}$$

cf. HF potential

$$V_H(\boldsymbol{r}) = \int v(\boldsymbol{r}, \boldsymbol{r}') \rho_{\mathsf{HF}}(\boldsymbol{r}') d\boldsymbol{r}$$

Solve the pairing Hamiltonian

$$\boxed{H = \sum_\nu \epsilon_\nu (a^\dagger_\nu a_\nu + a^\dagger_{\bar\nu} a_{\bar\nu}) - G \left(\sum_{\nu>0} a^\dagger_\nu a^\dagger_{\bar\nu} \right) \left(\sum_{\nu>0} a_{\bar\nu} a_\nu \right)}$$

in the mean-field approximation

- Mean-field approximation:

$$V = -G\, P^\dagger P \to -G \left(\langle P^\dagger \rangle P + P^\dagger \langle P \rangle \right) = -\Delta (P^\dagger + P)$$

$\Longleftrightarrow$ particle number violation

we consider $H' = H - \lambda\hat{N}$ instead of H :

$$
\begin{aligned}
H' &= \sum_{k>0}(\epsilon_k - \lambda)(a_k^\dagger a_k + a_{\bar{k}}^\dagger a_{\bar{k}}) - G\hat{P}^\dagger\hat{P} \\
&\to \sum_{k>0}(\epsilon_k - \lambda)(a_k^\dagger a_k + a_{\bar{k}}^\dagger a_{\bar{k}}) - \Delta(\hat{P}^\dagger + \hat{P}) \\
&= \sum_{k>0}(\epsilon_k - \lambda)(a_k^\dagger a_k + a_{\bar{k}}^\dagger a_{\bar{k}}) - \Delta\sum_{k>0}(a_k^\dagger a_{\bar{k}}^\dagger + a_{\bar{k}} a_k)
\end{aligned}
$$

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$$
\begin{aligned}
H' &= \sum_{k>0}(\epsilon_k - \lambda)(a^\dagger_k a_k + a^\dagger_{\bar{k}} a_{\bar{k}}) - G\hat{P}^\dagger \hat{P} \\
&\to \sum_{k>0}(\epsilon_k - \lambda)(a^\dagger_k a_k + a^\dagger_{\bar{k}} a_{\bar{k}}) - \Delta(\hat{P}^\dagger + \hat{P}) \\
&= \sum_{k>0}(\epsilon_k - \lambda)(a^\dagger_k a_k + a^\dagger_{\bar{k}} a_{\bar{k}}) - \Delta\sum_{k>0}(a^\dagger_k a^\dagger_{\bar{k}} + a_{\bar{k}} a_k)
\end{aligned}
$$

- Transform *H'* in a form of

$$
H' = \sum_{k>0} E_k(\alpha^\dagger_k \alpha_k + \alpha^\dagger_{\bar{k}} \alpha_{\bar{k}})
$$

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&\to \sum_{k>0}(\epsilon_k - \lambda)(a_k^\dagger a_k + a_{\bar{k}}^\dagger a_{\bar{k}}) - \Delta(\hat{P}^\dagger + \hat{P}) \\
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\end{aligned}
$$

- Transform *H'* in a form of

$$H' = \sum_{k>0} E_k(\alpha_k^\dagger \alpha_k + \alpha_{\bar{k}}^\dagger \alpha_{\bar{k}})$$

g.s.: $\alpha_k|BCS\rangle = 0$

1st excited state: $|1_k\rangle = \alpha_k^\dagger|BCS\rangle$ at E_k

…. and so on.

Bogoliubov transformation

$$\boxed{\alpha^\dagger_\nu = u_\nu a^\dagger_\nu - v_\nu a_{\bar{\nu}}, \qquad \alpha^\dagger_{\bar{\nu}} = u_\nu a^\dagger_{\bar{\nu}} + v_\nu a_\nu}$$

(Quasi-particle operator)

or $$a^\dagger_\nu = u_\nu \alpha^\dagger_\nu + v_\nu \alpha_{\bar{\nu}}, \qquad a^\dagger_{\bar{\nu}} = u_\nu \alpha^\dagger_{\bar{\nu}} + -v_\nu \alpha_\nu$$

(note)

$$\{\alpha_\nu, \alpha_{\nu'}\} = 0, \qquad \{\alpha_\nu, \alpha^\dagger_{\nu'}\} = \delta_{\nu,\nu'}$$

$$\longrightarrow \quad u_\nu^2 + v_\nu^2 = 1$$

Bogoliubov transformation

$$\boxed{\alpha^\dagger_\nu = u_\nu a^\dagger_\nu - v_\nu a_{\bar{\nu}}, \qquad \alpha^\dagger_{\bar{\nu}} = u_\nu a^\dagger_{\bar{\nu}} + v_\nu a_\nu}$$

(Quasi-particle operator)

or $$a^\dagger_\nu = u_\nu \alpha^\dagger_\nu + v_\nu \alpha_{\bar{\nu}}, \qquad a^\dagger_{\bar{\nu}} = u_\nu \alpha^\dagger_{\bar{\nu}} + -v_\nu \alpha_\nu$$

$$\begin{aligned} H' &= \sum_{k>0} (\epsilon_k - \lambda)(a^\dagger_k a_k + a^\dagger_{\bar{k}} a_{\bar{k}}) - \Delta \sum_{k>0} (a^\dagger_k a^\dagger_{\bar{k}} + a_{\bar{k}} a_k) \\ &\rightarrow \end{aligned}$$

using the quasi-particle operators:

$$
\begin{aligned}
H' &\sim \sum_{k>0} (\epsilon_k - \lambda)(a_k^\dagger a_k + a_{\bar{k}}^\dagger a_{\bar{k}}) - \Delta \sum_{k>0} (a_k^\dagger a_{\bar{k}}^\dagger + a_{\bar{k}} a_k) \\
&= \sum_{k>0} [(\epsilon_k - \lambda)(u_k^2 - v_k^2) + 2\Delta u_k v_k](\alpha_k^\dagger \alpha_k + \alpha_{\bar{k}}^\dagger \alpha_{\bar{k}}) \\
&\quad + \sum_{k>0} [2(\epsilon_k - \lambda) u_k v_k - \Delta(u_k^2 - v_k^2)](\alpha_k^\dagger \alpha_{\bar{k}}^\dagger + \alpha_{\bar{k}} \alpha_k)
\end{aligned}
$$

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$$\begin{aligned} H' &\sim \sum_{k>0} (\epsilon_k - \lambda)(a_k^\dagger a_k + a_{\bar{k}}^\dagger a_{\bar{k}}) - \Delta \sum_{k>0} (a_k^\dagger a_{\bar{k}}^\dagger + a_{\bar{k}} a_k) \\ &= \sum_{k>0} [(\epsilon_k - \lambda)(u_k^2 - v_k^2) + 2\Delta u_k v_k](\alpha_k^\dagger \alpha_k + \alpha_{\bar{k}}^\dagger \alpha_{\bar{k}}) \\ &\quad + \sum_{k>0} [2(\epsilon_k - \lambda) u_k v_k - \Delta(u_k^2 - v_k^2)](\alpha_k^\dagger \alpha_{\bar{k}}^\dagger + \alpha_{\bar{k}} \alpha_k) \end{aligned}$$

if $\quad 2(\epsilon_k - \lambda) u_k v_k - \Delta(u_k^2 - v_k^2) = 0$

then $\quad H' = \sum_{k>0} E_k(\alpha_k^\dagger \alpha_k + \alpha_{\bar{k}}^\dagger \alpha_{\bar{k}})$

with $\quad E_k = (\epsilon_k - \lambda)(u_k^2 - v_k^2) + 2\Delta u_k v_k$

$$
\begin{aligned}
H' &= \sum_{k>0} [(\epsilon_k - \lambda)(u_k^2 - v_k^2) + 2\Delta u_k v_k](\alpha_k^\dagger \alpha_k + \alpha_{\bar{k}}^\dagger \alpha_{\bar{k}}) \\
&\quad + \sum_{k>0} [2(\epsilon_k - \lambda)u_k v_k - \Delta(u_k^2 - v_k^2)](\alpha_k^\dagger \alpha_{\bar{k}}^\dagger + \alpha_{\bar{k}} \alpha_k)
\end{aligned}
$$

$$
\begin{cases}
0 = 2(\epsilon_k - \lambda)u_k v_k - \Delta(u_k^2 - v_k^2) \\
1 = u_k^2 + v_k^2
\end{cases}
$$

$$
\boxed{
\begin{aligned}
u_\nu^2 &= \frac{1}{2}\left(1 + \frac{\epsilon_\nu - \lambda}{\sqrt{(\epsilon_\nu - \lambda)^2 + \Delta^2}}\right) \\
v_\nu^2 &= \frac{1}{2}\left(1 - \frac{\epsilon_\nu - \lambda}{\sqrt{(\epsilon_\nu - \lambda)^2 + \Delta^2}}\right)
\end{aligned}
}
$$

$$
\begin{aligned}
H' &= \sum_{k>0} [(\epsilon_k - \lambda)(u_k^2 - v_k^2) + 2\Delta u_k v_k](\alpha_k^\dagger \alpha_k + \alpha_{\bar{k}}^\dagger \alpha_{\bar{k}}) \\
&\quad + \sum_{k>0} [2(\epsilon_k - \lambda) u_k v_k - \Delta(u_k^2 - v_k^2)](\alpha_k^\dagger \alpha_{\bar{k}}^\dagger + \alpha_{\bar{k}} \alpha_k)
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$$

$$
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\end{aligned}
}
$$

$$
\begin{aligned}
E_k &= (\epsilon_k - \lambda)(u_k^2 - v_k^2) + 2\Delta u_k v_k \\
&= \sqrt{(\epsilon_k - \lambda)^2 + \Delta^2}
\end{aligned}
$$

$$\boxed{\begin{aligned} H' &= \sum_{k>0} E_k(\alpha_k^\dagger \alpha_k + \alpha_{\bar{k}}^\dagger \alpha_{\bar{k}}) \\ E_k &= \sqrt{(\epsilon_k - \lambda)^2 + \Delta^2} \end{aligned}}$$

Ground state wave function:

$$\alpha_k|BCS\rangle = 0$$

$$\begin{aligned} |BCS\rangle &\propto \prod_{\nu>0} \alpha_\nu \alpha_{\bar{\nu}} |0\rangle \\ &= \prod_{\nu>0} v_\nu \left(u_\nu + v_\nu\, a_\nu^\dagger a_{\bar{\nu}}^\dagger\right) \Big| 0 \Big\rangle \end{aligned}$$

$$\boxed{|BCS\rangle = \prod_{\nu>0} \left(u_\nu + v_\nu\, a_\nu^\dagger a_{\bar{\nu}}^\dagger\right) \Big| 0 \Big\rangle}$$

Ground state wave function:

$$\alpha_k|BCS\rangle = 0$$

$$\boxed{|BCS\rangle = \prod_{\nu>0}\left(u_\nu + v_\nu\, a^\dagger_\nu a^\dagger_{\bar\nu}\right)\big|0\big\rangle}$$

(note) $\left\langle BCS|a^\dagger_\nu a_\nu|BCS\right\rangle = |v_\nu|^2$: occupation probability

(note)

$$E'_{\mathrm{BCS}} = \langle BCS|H'|BCS\rangle \sim 2\sum_{\nu>0}(\epsilon_\nu - \lambda)v_\nu^2 - \frac{\Delta^2}{G}$$

Gap equation

$$\begin{cases} u_\nu^2 = \dfrac{1}{2}\left(1 + \dfrac{\epsilon_\nu - \lambda}{E_\nu}\right) \\ v_\nu^2 = \dfrac{1}{2}\left(1 - \dfrac{\epsilon_\nu - \lambda}{E_\nu}\right) \end{cases}$$

$$E_\nu = \sqrt{(\epsilon_\nu - \lambda)^2 + \Delta^2}$$

$$\boxed{\begin{aligned} \Delta &= G\langle BCS|\hat{P}|BCS\rangle = G\sum_{\nu>0} u_\nu v_\nu \\ &= \frac{G}{2}\sum_{\nu>0}\frac{\Delta}{E_\nu} \end{aligned}}$$

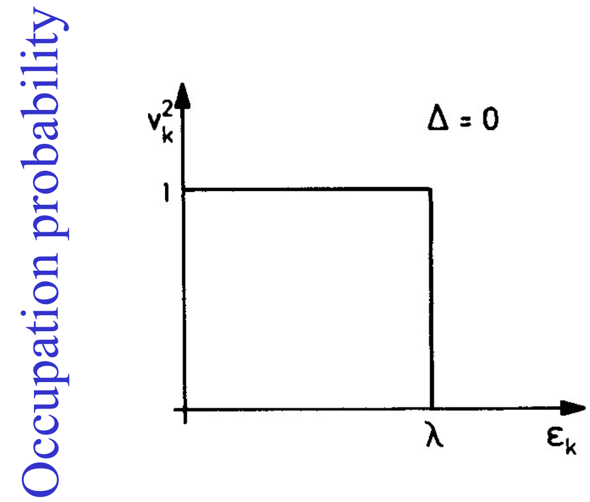
(Gap equation)

$$N = 2\sum_{\nu>0} v_\nu^2 \qquad \leftarrow \lambda$$

$$\Delta = \frac{G}{2} \sum_{\nu > 0} \frac{\Delta}{E_\nu}$$

i) Trivial solution: always exists

$$\Delta = 0$$



$$\Delta = G \sum_{\nu > 0} u_\nu v_\nu$$

$$\begin{aligned} v_\nu^2 &= 1 \quad (\epsilon_\nu \le \lambda) \\ &= 0 \quad (\epsilon_\nu > \lambda) \end{aligned}$$

$$|\Psi\rangle = \prod_{\nu > 0} a_\nu^\dagger a_{\bar{\nu}}^\dagger |0\rangle$$

$$\Delta = \frac{G}{2} \sum_{\nu > 0} \frac{\Delta}{E_\nu}$$

i) Trivial solution: always exists

$$\Delta = 0$$

G a/o $N \longrightarrow$ large

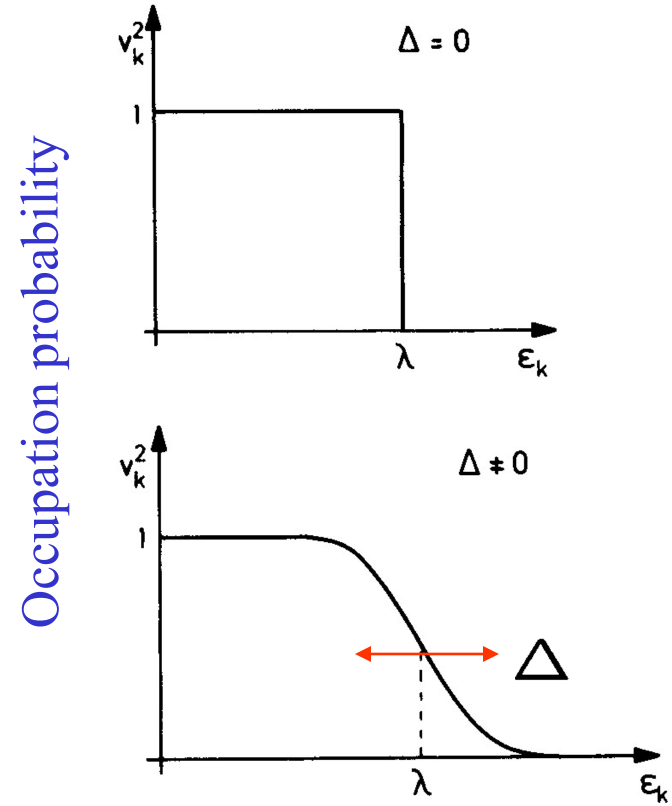
ii) Superfluid solution

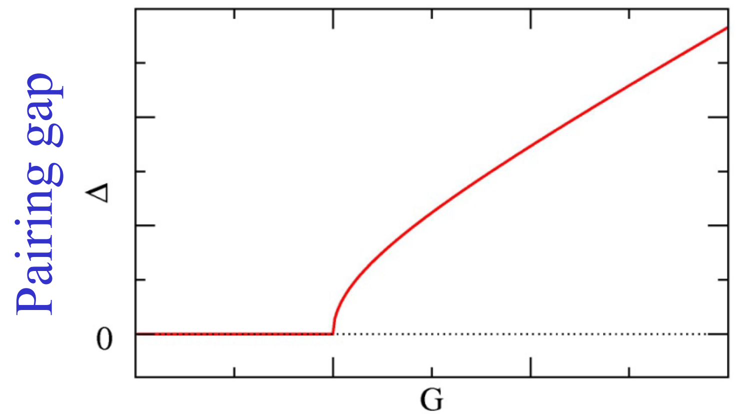
$$\Delta \neq 0$$

$$v_\nu^2 < 1$$

$$|BCS\rangle = \prod_{\nu>0} \left(u_\nu + v_\nu \, a_\nu^\dagger a_{\bar{\nu}}^\dagger \right) \Big| 0 \Big\rangle$$

Number fluctuation





Normal-Superfulid phase transition

Quasi-particle excitations

$$H = \sum_{\nu} \epsilon_k (a_k^\dagger a_k + a_{\bar{k}}^\dagger a_{\bar{k}}) - G \left(\sum_{k>0} a_k^\dagger a_{\bar{k}}^\dagger \right) \left(\sum_{k>0} a_{\bar{k}} a_k \right)$$

ハミルトニアンを書き直すと:

$$H \sim E_{BCS} + \sum_k E_k \, \alpha_k^\dagger \alpha_k$$

$$E_k = \sqrt{(\epsilon_k - \lambda)^2 + \Delta^2}$$

$$\alpha_\nu^\dagger = u_\nu a_\nu^\dagger - v_\nu a_{\bar{\nu}}, \qquad \alpha_{\bar{\nu}}^\dagger = u_\nu a_{\bar{\nu}}^\dagger + v_\nu a_\nu$$

（ボゴリューボフ変換）

Quasi-particle excitations

$$H \sim E_{BCS} + \sum_k E_k \,\alpha_k^\dagger \alpha_k$$

$$\alpha_\nu^\dagger = u_\nu a_\nu^\dagger - v_\nu a_{\bar{\nu}}, \qquad \alpha_{\bar{\nu}}^\dagger = u_\nu a_{\bar{\nu}}^\dagger + v_\nu a_\nu$$

（ボゴリューボフ変換）

基底状態: $|\mathrm{BCS}\rangle$

1準粒子状態: $\alpha_k^\dagger|\mathrm{BCS}\rangle$ 　　奇核に対応

2準粒子状態: $\alpha_k^\dagger \alpha_{k'}^\dagger|\mathrm{BCS}\rangle$

- N +/- 2 の原子核
- 同じ原子核の励起状態に対応

Quasi-particle excitations

$$H \sim E_{BCS} + \sum_k E_k \, \alpha_k^\dagger \alpha_k$$

$$\alpha_\nu^\dagger = u_\nu a_\nu^\dagger - v_\nu a_{\bar{\nu}}, \qquad \alpha_{\bar{\nu}}^\dagger = u_\nu a_{\bar{\nu}}^\dagger + v_\nu a_\nu$$

（ボゴリューボフ変換）

基底状態: $|\mathrm{BCS}\rangle$

1準粒子状態: $\alpha_k^\dagger|\mathrm{BCS}\rangle$ 　　奇核に対応

2準粒子状態: $\alpha_k^\dagger \alpha_{k'}^\dagger|\mathrm{BCS}\rangle$

- N +/- 2 の原子核
- 同じ原子核の励起状態に対応

$$E_k = \sqrt{(\epsilon_k - \lambda)^2 + \Delta^2} \geq \Delta$$

（エネルギー・ギャップ）

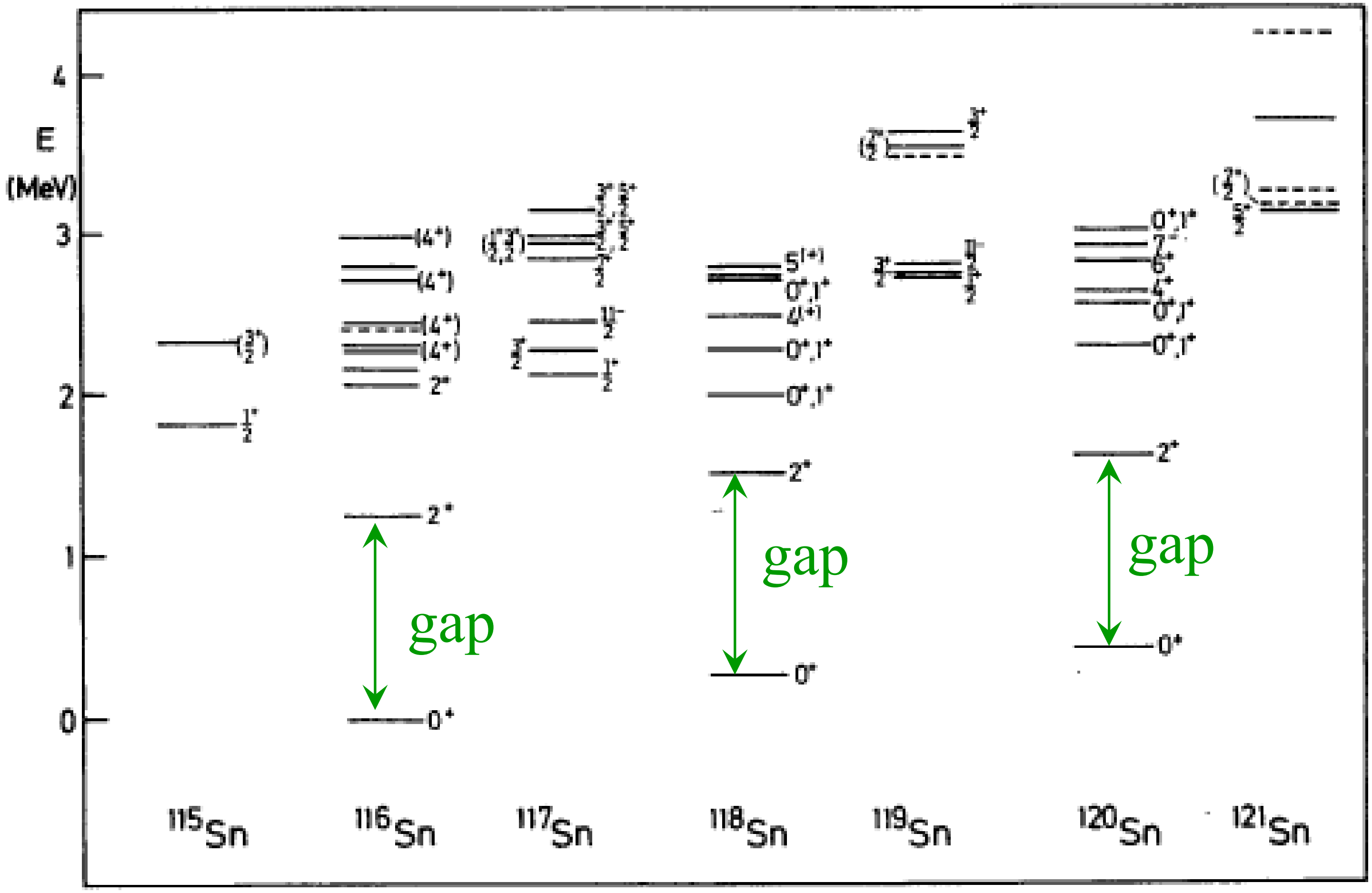


Figure 6.1. Excitation spectra of the $_{50}$Sn isotopes.

Ring-Schuck

Effects of pairing on moment of inertia

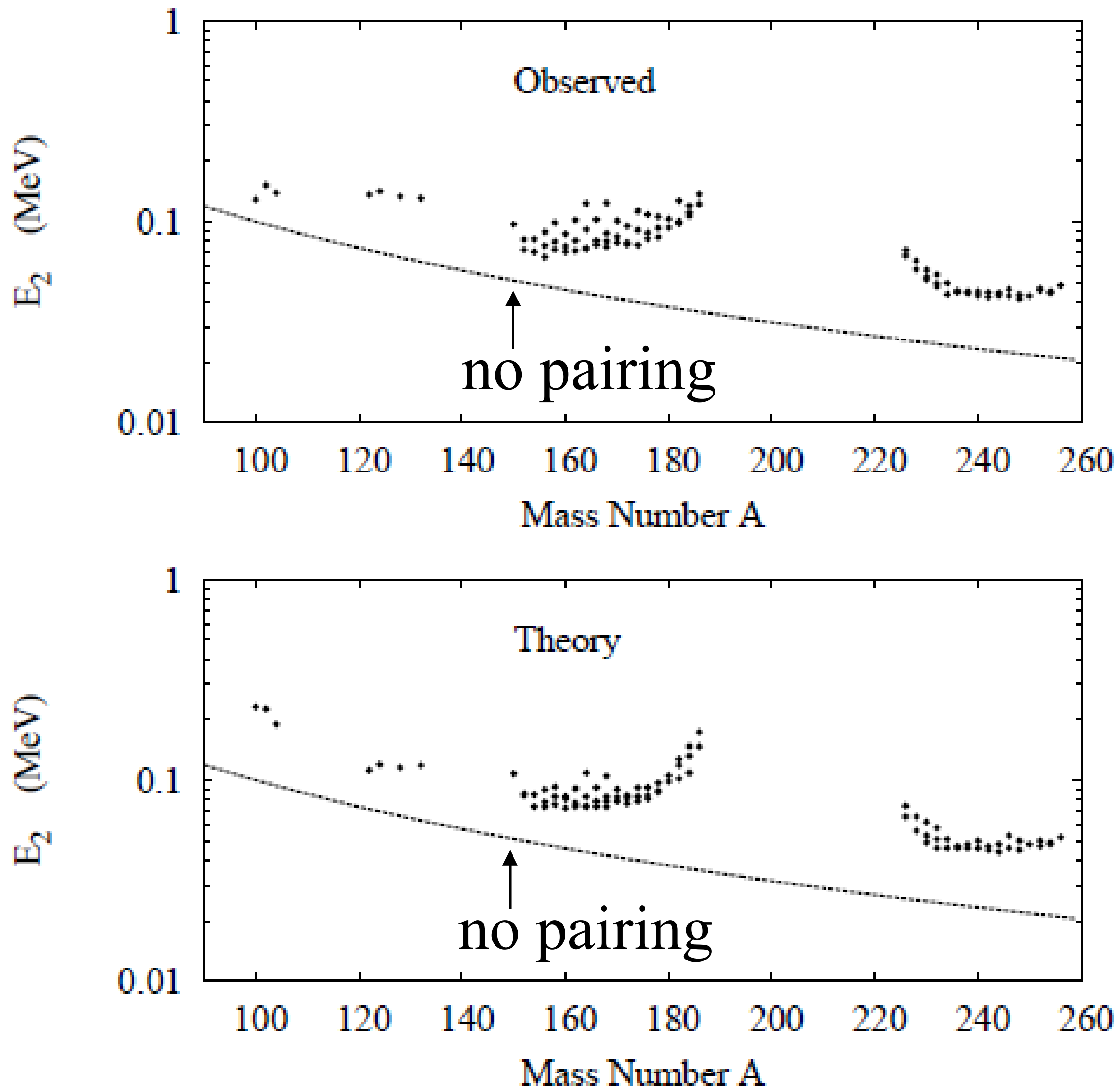


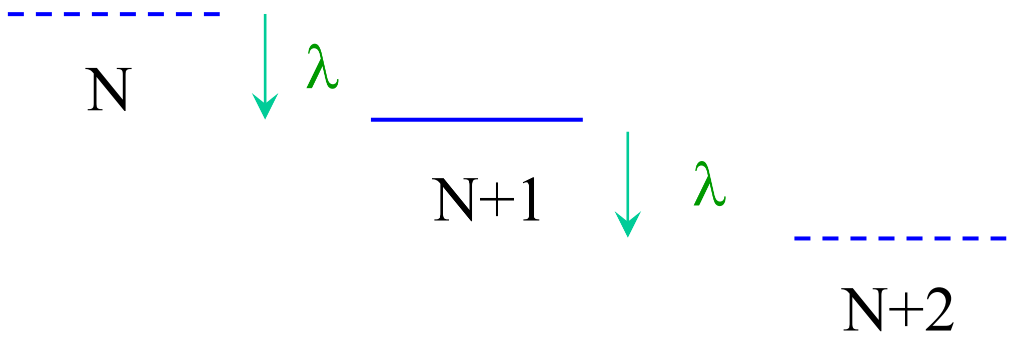
Fig. 9. Excitation energy of the first 2^+ state in deformed nuclei. The line shows the prediction assuming a rigid rotor.

$$E_I = \frac{I(I+1)\hbar^2}{2\mathcal{J}}$$

G.F. Bertsch, in "Fifty years of nuclear BCS"

Even-odd mass difference and pairing gap

$$\begin{aligned} E(N+2,Z) &= E(N,Z)+2\lambda \\ E(N+1,Z) &= E(N,Z)+\lambda+\Delta \end{aligned}$$

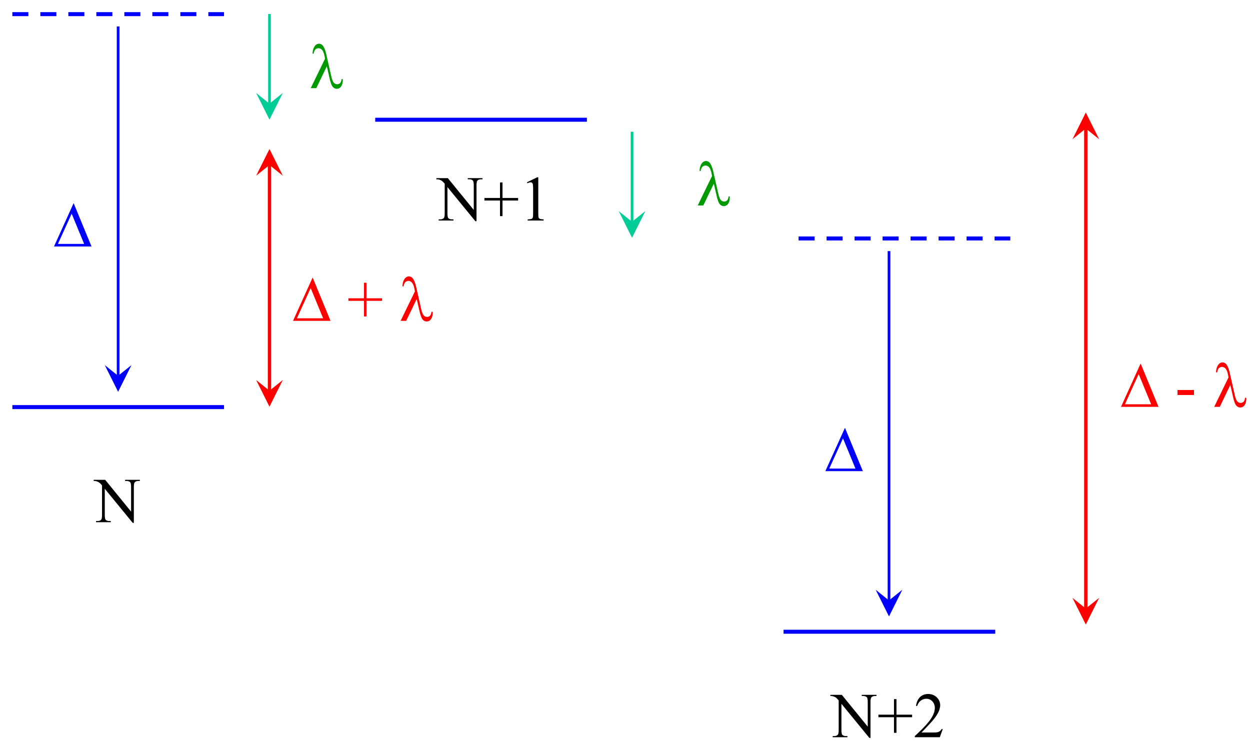


(note) $\lambda < 0$

$$\boxed{-\Delta_n \sim [E(N+2,Z)-2E(N+1,Z)+E(N,Z)]/2}$$

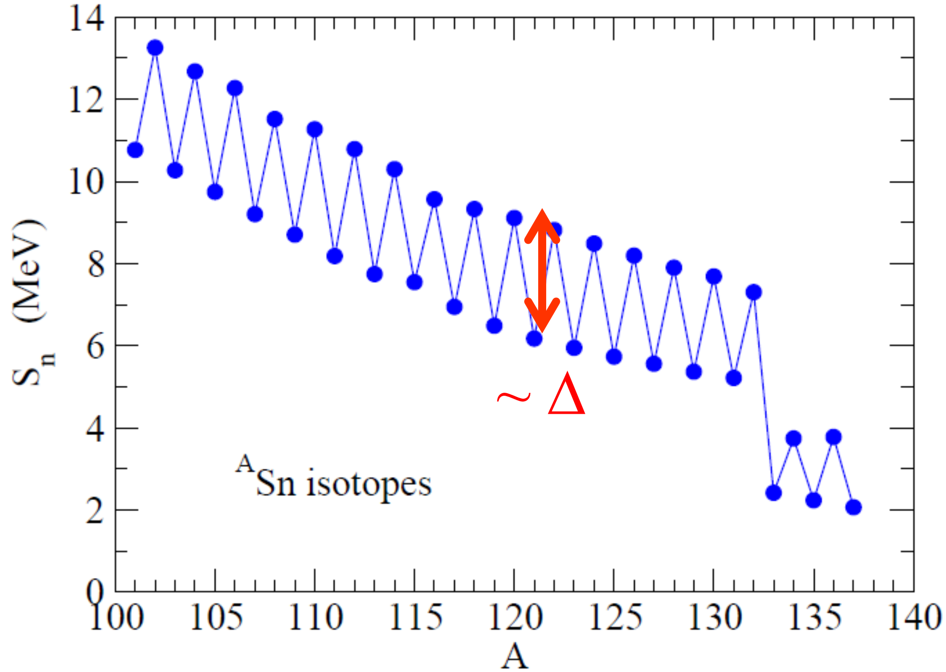
Even-odd mass difference and pairing gap

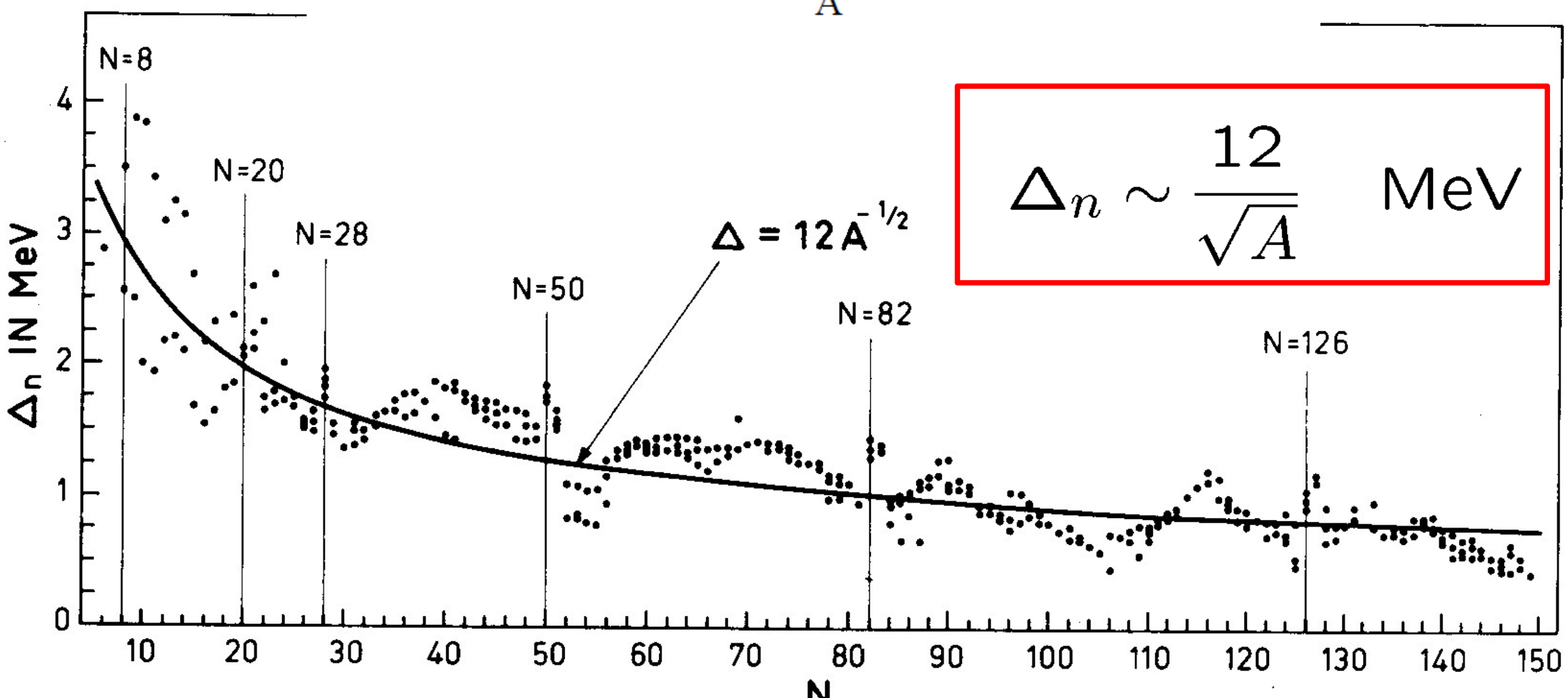
$$E(N+2,Z) = E(N,Z) + 2\lambda$$
$$E(N+1,Z) = E(N,Z) + \lambda + \Delta$$



(note) $\lambda < 0$

$$\boxed{-\Delta_n \sim [E(N+2,Z) - 2E(N+1,Z) + E(N,Z)]/2}$$





$$\Delta_n \sim \frac{12}{\sqrt{A}} \quad \text{MeV}$$

粒子数射影法

$$|BCS\rangle = \prod_{\nu>0}\left(u_\nu + v_\nu\, a_\nu^\dagger a_{\bar{\nu}}^\dagger\right)\Big|\,0\Big\rangle$$

様々な粒子数の状態が混ざっている $\quad |BCS\rangle = \sum_{N_k} C_{N_k}|N_k\rangle$

ただし、平均値だけは正しく設定されている:

$$\langle BCS|\hat{N}|BCS\rangle = N$$

粒子数射影: $\quad \hat{P}_N|BCS\rangle = C_N|N\rangle$

$$\hat{P}_N = \frac{1}{2\pi}\int_0^{2\pi} d\phi\, e^{i(\hat{N}-N)\phi}$$