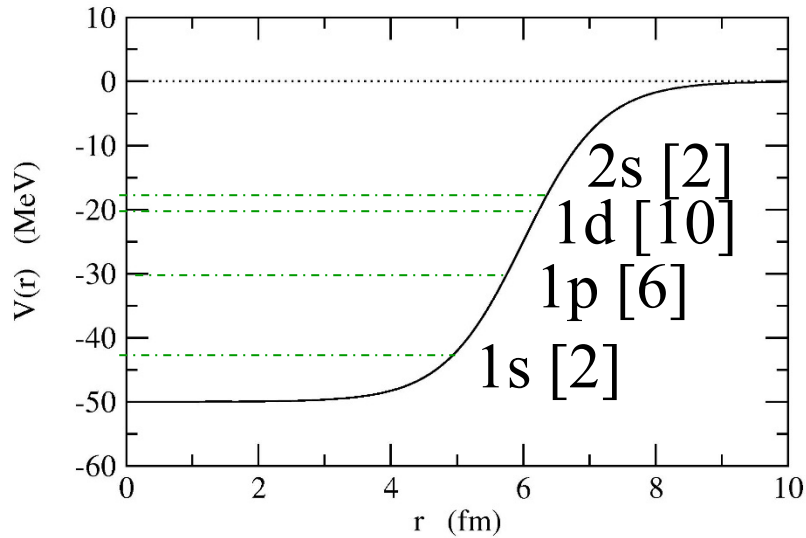


Extra binding for N or $Z = 2, 8, 20, 28, 50, 82, 126$ (magic numbers)

An interpretation: independent particle motion in a potential well

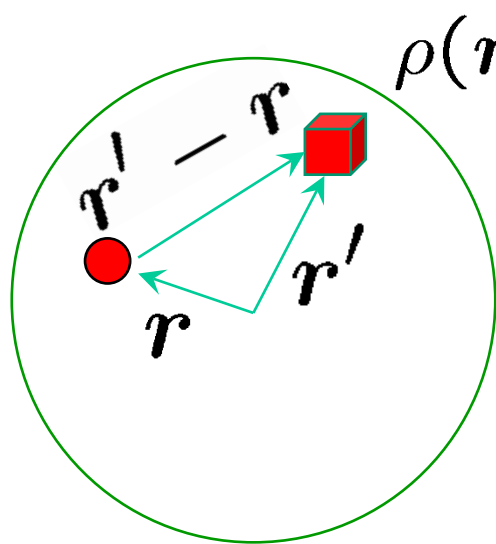


+ spin-orbit interaction

how to construct the potential well?

Mean-field (Hartree-Fock) Theory

interaction for a nucleon inside a nucleus:



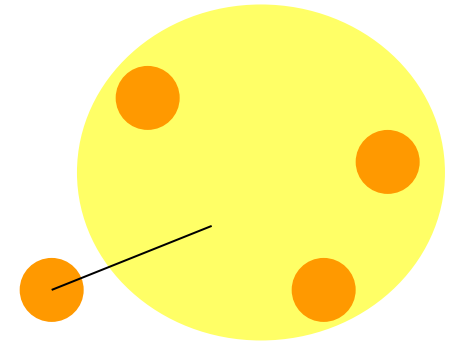
$$v(\mathbf{r}' - \mathbf{r}) \cdot \rho(\mathbf{r}')d\mathbf{r}'$$

the number of nucleon
at \mathbf{r}'

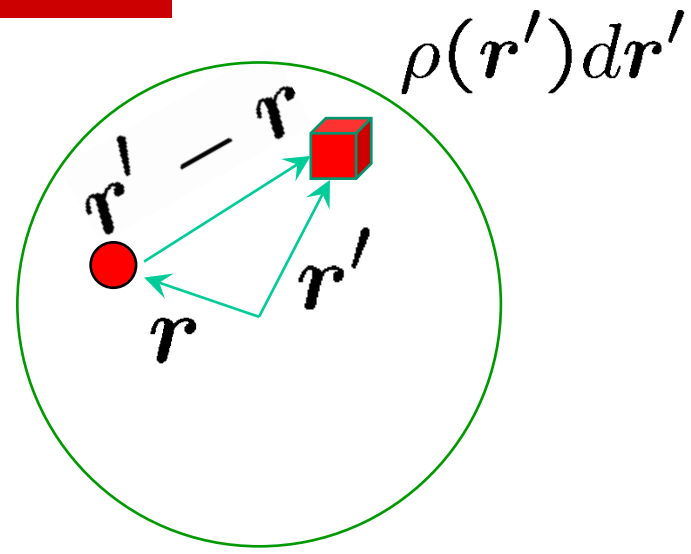
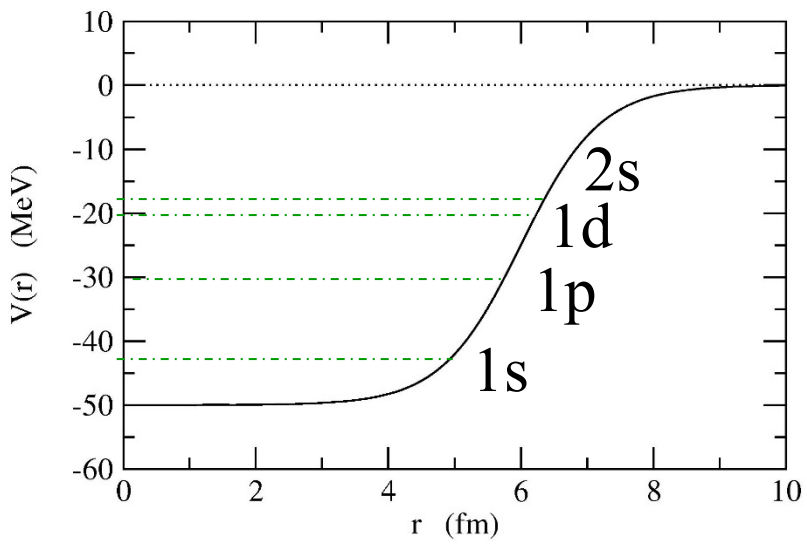
naively speaking,

$$V(\mathbf{r}) \sim \int v(\mathbf{r} - \mathbf{r}')\rho(\mathbf{r}')d\mathbf{r}'$$

平均場



Mean-field (Hartree-Fock) Theory

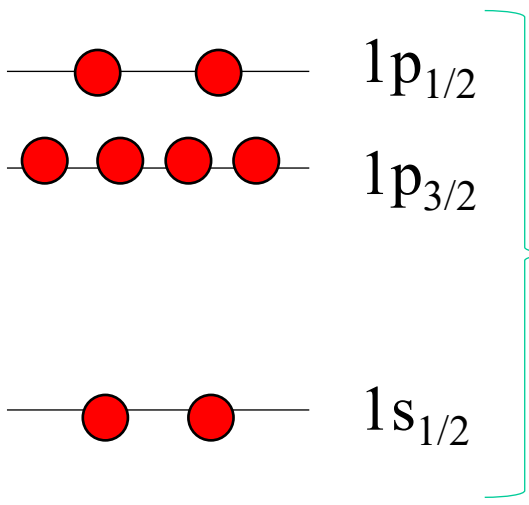


naively speaking,

$$V(\mathbf{r}) \sim \int v(\mathbf{r} - \mathbf{r}')\rho(\mathbf{r}')d\mathbf{r}'$$

independent motion

$$\rho(\mathbf{r}) = \sum_i |\psi_i(\mathbf{r})|^2$$



shell model

Mean-field (Hartree-Fock) Theory

$$0 = \left[-\frac{\hbar^2}{2m} \nabla^2 + \int v(\mathbf{r} - \mathbf{r}') \left(\sum_j |\psi_j(\mathbf{r}')|^2 \right) d\mathbf{r}' - \epsilon_i \right] \psi_i(\mathbf{r})$$

the potential depends on the solutions

→ **self-consistent solutions**

Iteration: $\{\psi_i\} \rightarrow \rho \rightarrow V \rightarrow \{\psi_i\} \rightarrow \dots$

Mean-field (Hartree-Fock) Theory

$$\begin{aligned} 0 &= \left[-\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{r}) - \epsilon_i \right] \psi_i(\mathbf{r}) \\ &= \left[-\frac{\hbar^2}{2m} \nabla^2 + \int v(\mathbf{r} - \mathbf{r}') \left(\sum_j |\psi_j(\mathbf{r}')|^2 \right) d\mathbf{r}' - \epsilon_i \right] \psi_i(\mathbf{r}) \end{aligned}$$

the potential depends on the solutions

→ **self-consistent solutions**

$$\text{Iteration: } \{\psi_i\} \rightarrow \rho \rightarrow V \rightarrow \{\psi_i\} \rightarrow \dots$$

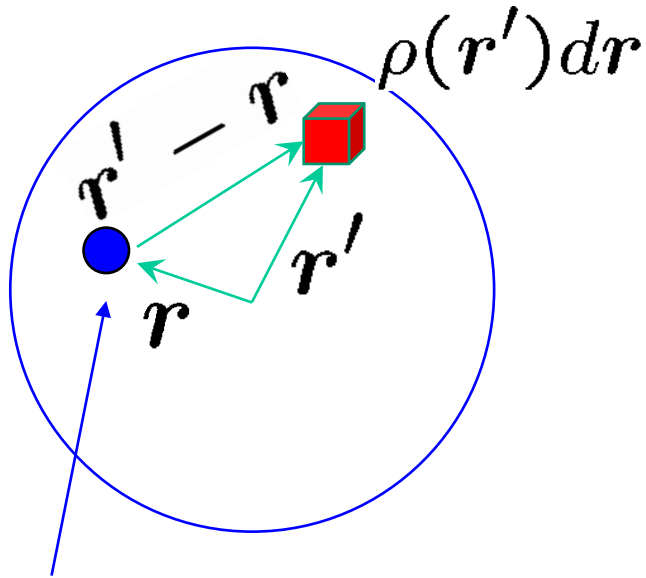
$$\rho(\mathbf{r}) = \sum_i |\psi_i(\mathbf{r})|^2, \quad V(\mathbf{r}) \sim \int v(\mathbf{r} - \mathbf{r}') \rho(\mathbf{r}') d\mathbf{r}'$$

repeat until the first and the last wave functions are the same.

“self-consistent solutions”

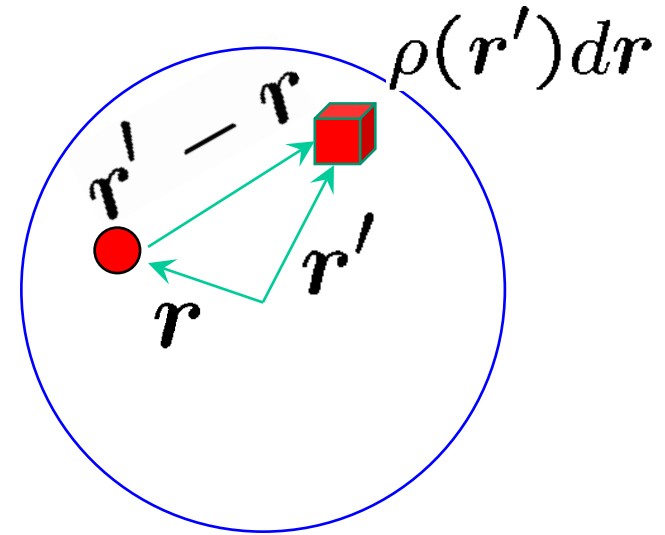
Mean-field (Hartree-Fock) Theory

electro-static potential



test charge

nucleus

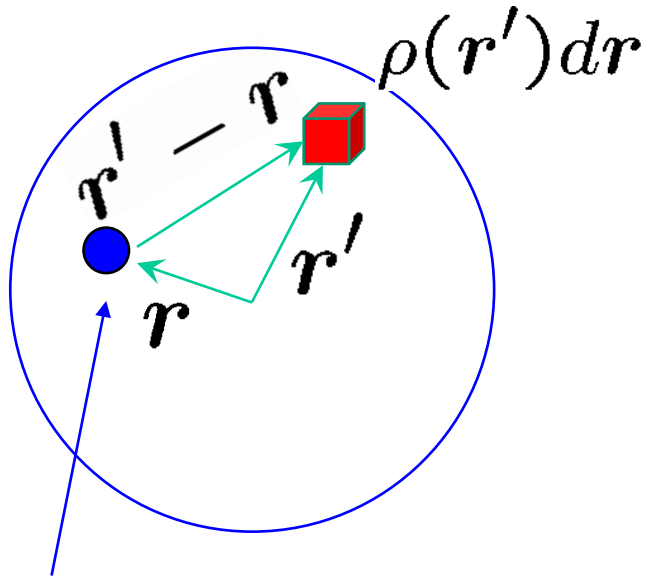


interaction between identical particles

$$V(r) \sim \int v(r - r') \rho(r') dr'$$

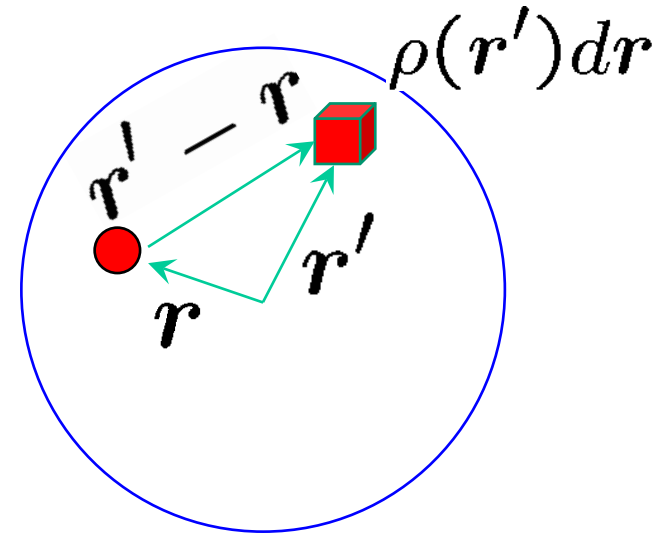
Mean-field (Hartree-Fock) Theory

electro-static potential



test charge

nucleus




interaction between identical particles
→ needs anti-symmetrization

$$V(\mathbf{r}) \sim \int v(\mathbf{r} - \mathbf{r}')\rho(\mathbf{r}')d\mathbf{r}'$$


anti-symmetrization

nucleon: fermion


$$\Psi(x_1, x_2, x_3 \dots) = -\Psi(x_2, x_1, x_3 \dots)$$

$$\psi_1(x_1)\psi_2(x_2) \rightarrow \frac{1}{\sqrt{2}}[\psi_1(x_1)\psi_2(x_2) - \psi_2(x_1)\psi_1(x_2)]$$


Slater determinat


$$0 = \left[-\frac{\hbar^2}{2m} \nabla^2 + \int v(\mathbf{r} - \mathbf{r}') \left(\sum_j |\psi_j(\mathbf{r}')|^2 \right) d\mathbf{r}' - \epsilon_i \right] \psi_i(\mathbf{r})$$

$$\psi_j^*(\mathbf{r}')\psi_j(\mathbf{r}')\psi_i(\mathbf{r}) \rightarrow \psi_j^*(\mathbf{r}')\psi_i(\mathbf{r}')\psi_j(\mathbf{r})$$

anti-symmetrization

nucleon: fermion


$$\Psi(x_1, x_2, x_3 \dots) = -\Psi(x_2, x_1, x_3 \dots)$$

$$\psi_1(x_1)\psi_2(x_2) \rightarrow \frac{1}{\sqrt{2}}[\psi_1(x_1)\psi_2(x_2) - \psi_2(x_1)\psi_1(x_2)]$$

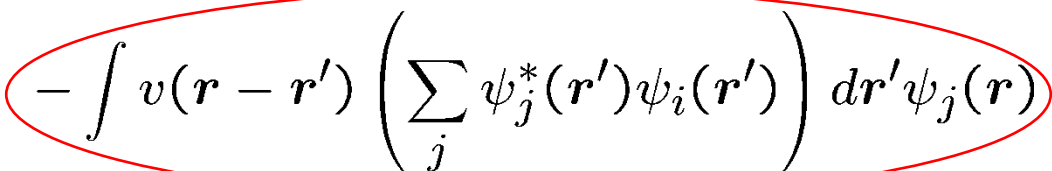


Slater determinat

$$0 = \left[-\frac{\hbar^2}{2m} \nabla^2 + \int v(\mathbf{r} - \mathbf{r}') \left(\sum_j |\psi_j(\mathbf{r}')|^2 \right) d\mathbf{r}' - \epsilon_i \right] \psi_i(\mathbf{r})$$

$$\psi_j^*(\mathbf{r}')\psi_j(\mathbf{r}')\psi_i(\mathbf{r}) \rightarrow \psi_j^*(\mathbf{r}')\psi_i(\mathbf{r}')\psi_j(\mathbf{r})$$

$$\rightarrow \left[-\frac{\hbar^2}{2m} \nabla^2 + \int v(\mathbf{r} - \mathbf{r}') \left(\sum_j |\psi_j(\mathbf{r}')|^2 \right) d\mathbf{r}' - \epsilon_i \right] \psi_i(\mathbf{r})$$


$$- \int v(\mathbf{r} - \mathbf{r}') \left(\sum_j \psi_j^*(\mathbf{r}')\psi_i(\mathbf{r}') \right) d\mathbf{r}' \psi_j(\mathbf{r})$$

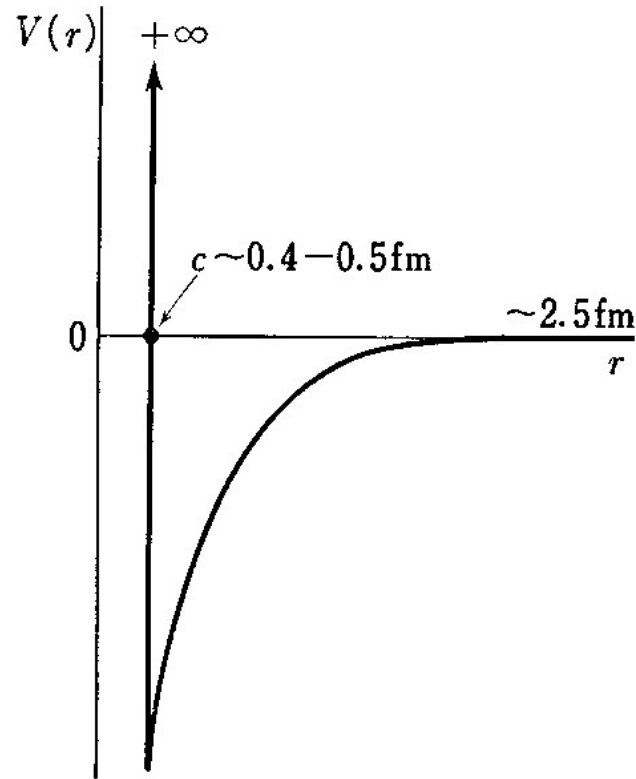
exchange term

anti-symmetrization

$$\begin{aligned} 0 &= \left[-\frac{\hbar^2}{2m} \nabla^2 + \int v(\mathbf{r} - \mathbf{r}') \left(\sum_j |\psi_j(\mathbf{r}')|^2 \right) d\mathbf{r}' - \epsilon_i \right] \psi_i(\mathbf{r}) \\ &\rightarrow \left[-\frac{\hbar^2}{2m} \nabla^2 + \int v(\mathbf{r} - \mathbf{r}') \left(\sum_j |\psi_j(\mathbf{r}')|^2 \right) d\mathbf{r}' - \epsilon_i \right] \psi_i(\mathbf{r}) \\ &\quad - \int v(\mathbf{r} - \mathbf{r}') \left(\sum_j \psi_j^*(\mathbf{r}') \psi_i(\mathbf{r}') \right) d\mathbf{r}' \psi_j(\mathbf{r}) \\ &= \left[-\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{r}) - \epsilon_i \right] \psi_i(\mathbf{r}) + \int d\mathbf{r}' V_{\text{NL}}(\mathbf{r}, \mathbf{r}') \psi_i(\mathbf{r}') \end{aligned}$$

non-local potential

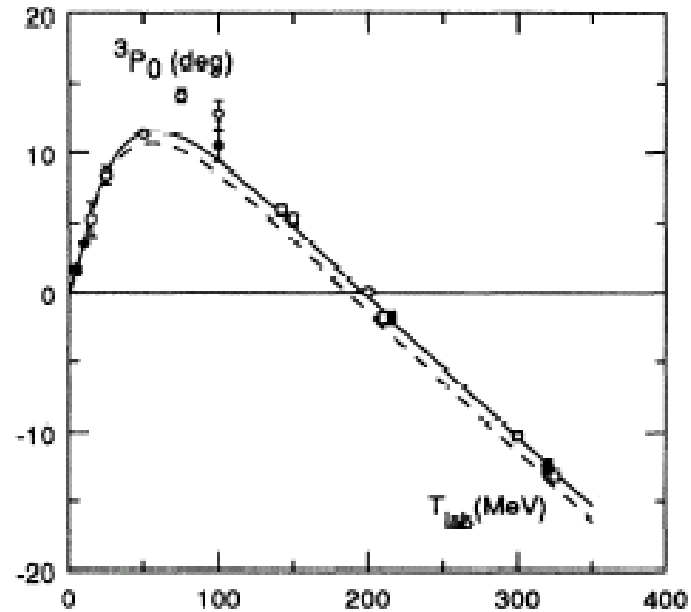
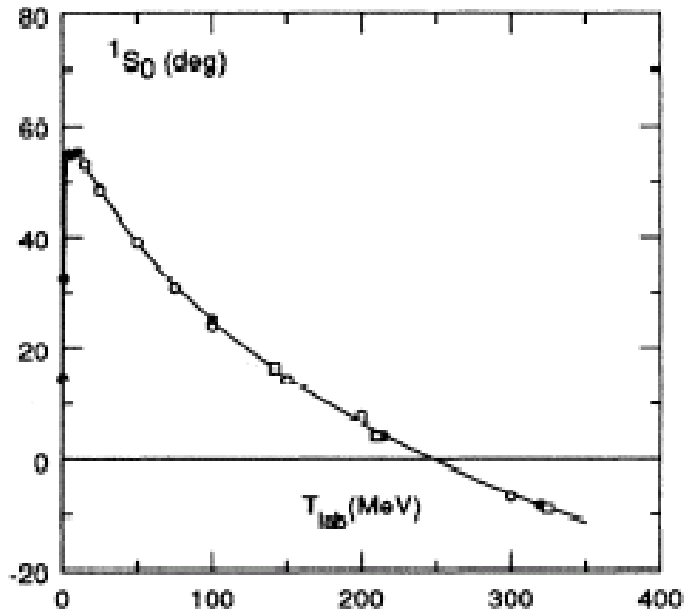
Bare nucleon-nucleon interaction



Existence of short range
repulsive core

Bare nucleon-nucleon interaction

Phase shift for p-p scattering



(V.G.J. Stoks et al., PRC48('93)792)

Phase shift: +ve \rightarrow -ve
at high energies

Phase shift:

Radial wave function

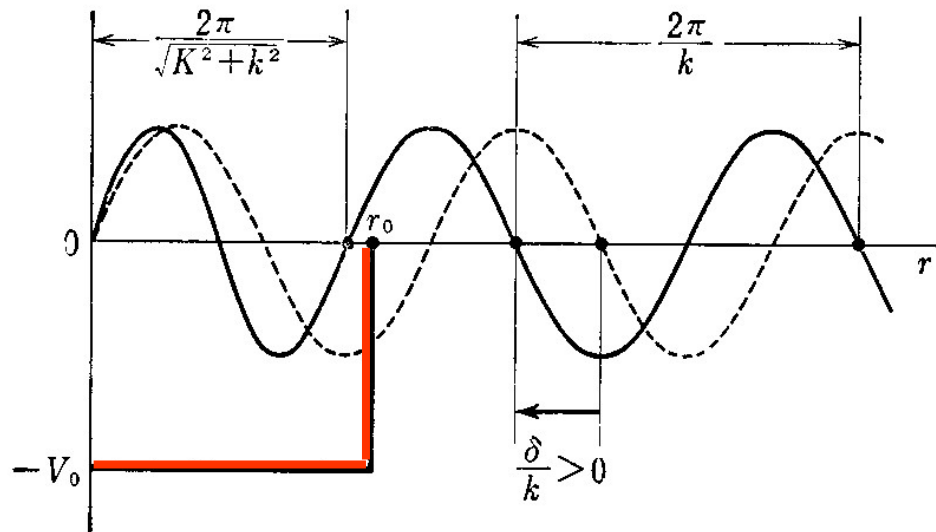
$$\Psi_l(r) = \frac{u_l(r)}{r} Y_{lm}(\hat{r})$$



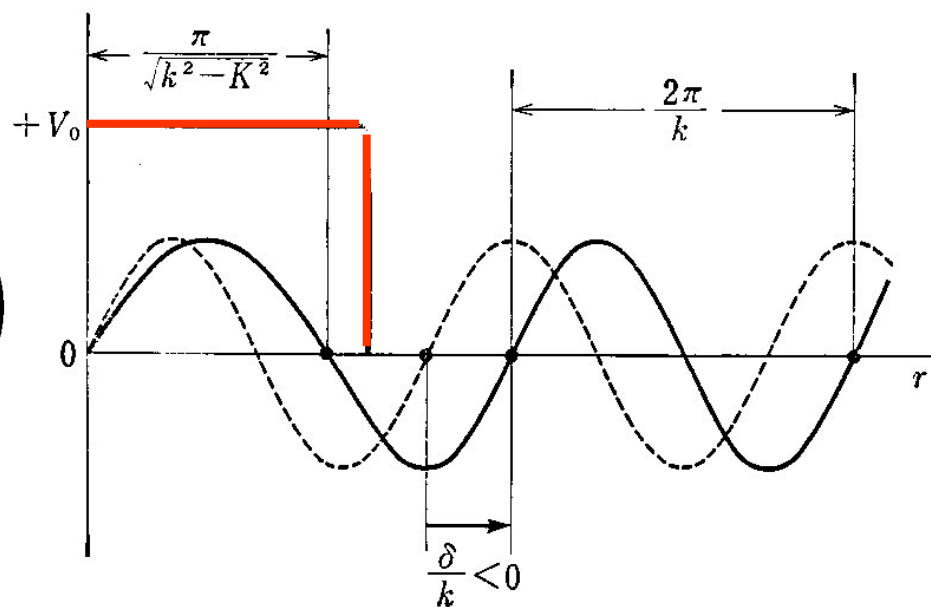
$$\left[-\frac{\hbar^2}{2m} \frac{d^2}{dr^2} + V(r) + \frac{l(l+1)\hbar^2}{2mr^2} - E \right] u_l(r) = 0$$

Asymptotic form:

$$u_l(r) \rightarrow \sin \left(kr - \frac{l\pi}{2} + \delta_l \right) \quad (r \rightarrow \infty)$$

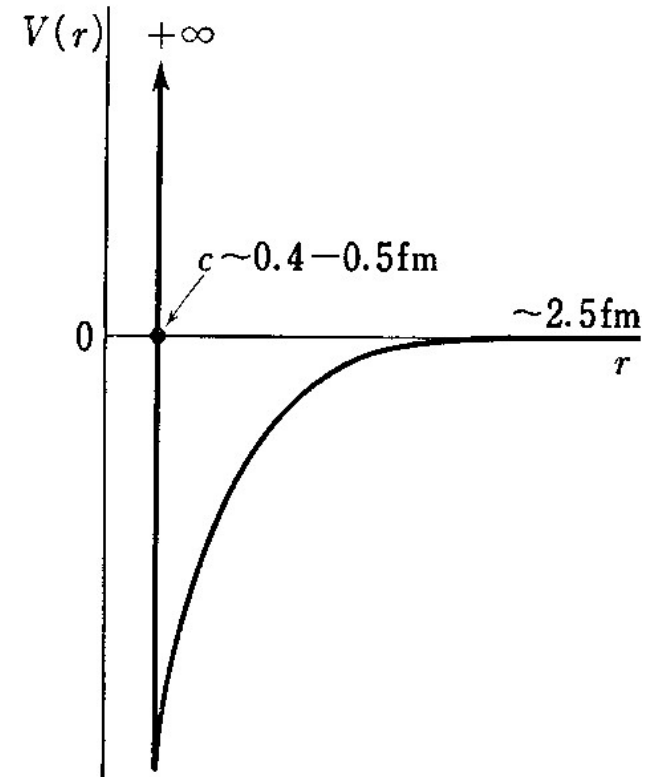
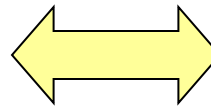
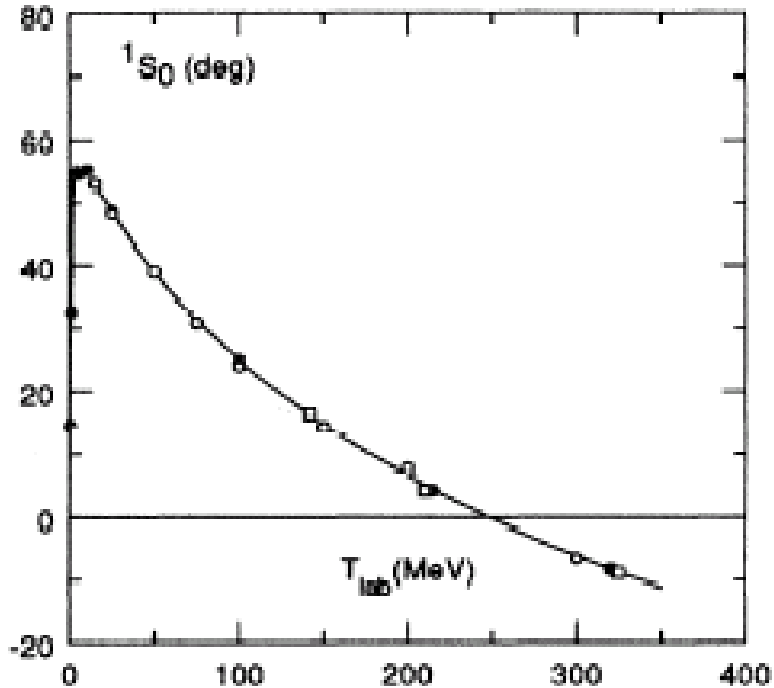


(a) 引力 attraction



(b) 斥力 repulsion

$\delta > 0 \rightarrow$ attraction
 $\delta < 0 \rightarrow$ repulsion



Phase shift: +ve \rightarrow -ve
at high energies

Existence of short range
repulsive core

Bruckner's G-matrix Nucleon-nucleon interaction *in medium*

Nucleon-nucleon interaction with a hard core

→ HF method: does not work

← Matrix elements: diverge

.....but the HF picture seems to work in nuclear systems

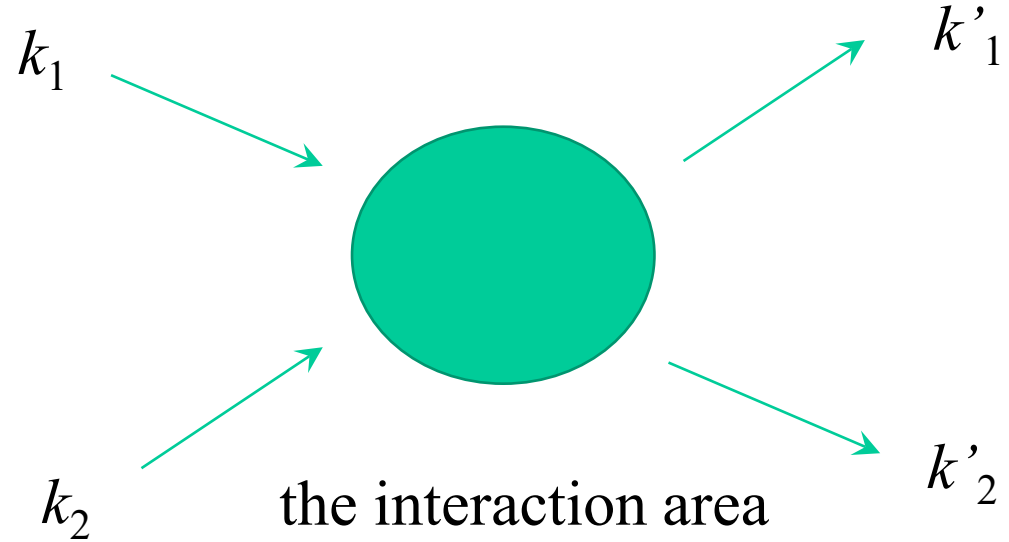
cf. magic numbers

Solution: a nucleon-nucleon interaction *in medium* (effective interaction) rather than a bare interaction



Bruckner's G-matrix

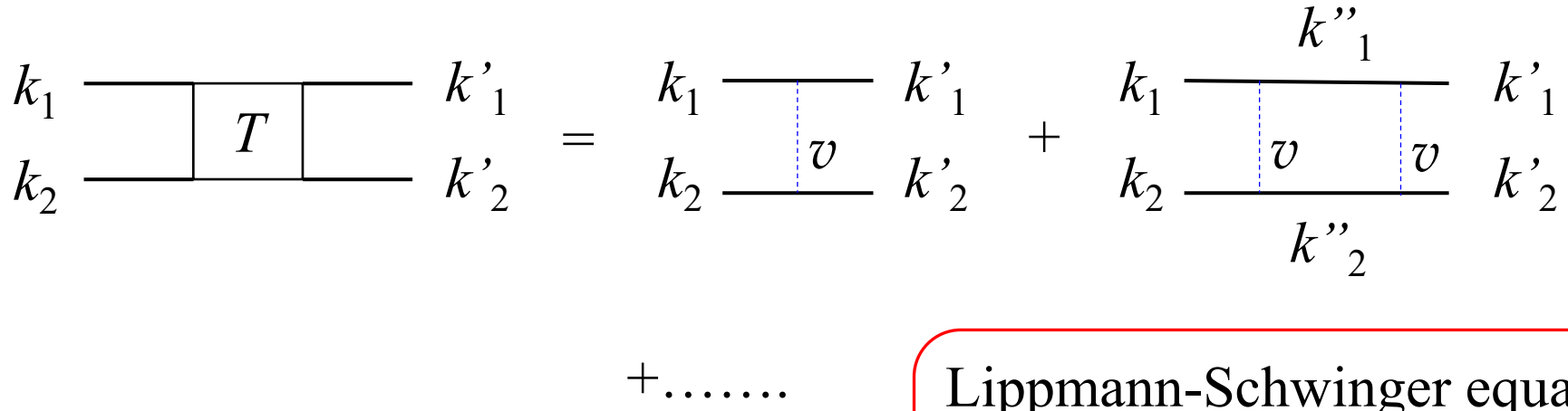
➤ two-body (multiple) scattering *in vacuum*



$$\begin{array}{c}
 k_1 \text{ --- } | \text{ } T \text{ --- } k'_1 \\
 k_2 \text{ --- } | \text{ } \text{ --- } k'_2
 \end{array}
 =
 \begin{array}{c}
 k_1 \text{ --- } | \text{ } \text{ --- } k'_1 \\
 k_2 \text{ --- } | \text{ } \text{ --- } k'_2 \\
 \text{v}
 \end{array}
 +
 \begin{array}{c}
 k_1 \text{ --- } | \text{ } k''_1 \text{ --- } | \text{ } k'_1 \\
 k_2 \text{ --- } | \text{ } \text{ --- } | \text{ } k'_2 \\
 \text{v} \quad \text{v} \\
 k''_2
 \end{array}
 + \dots$$

the 1st order
the 2nd order
higher orders

➤ two-body (multiple) scattering *in vacuum*



Lippmann-Schwinger equation

$$T = v + v \frac{1}{E - H_0} T$$

$$\left(-\frac{\hbar^2}{2m} \nabla^2 + V - E \right) \psi = 0$$

⇒
$$\left(-\frac{\hbar^2}{2m} \nabla^2 - E \right) \psi = -V \psi$$

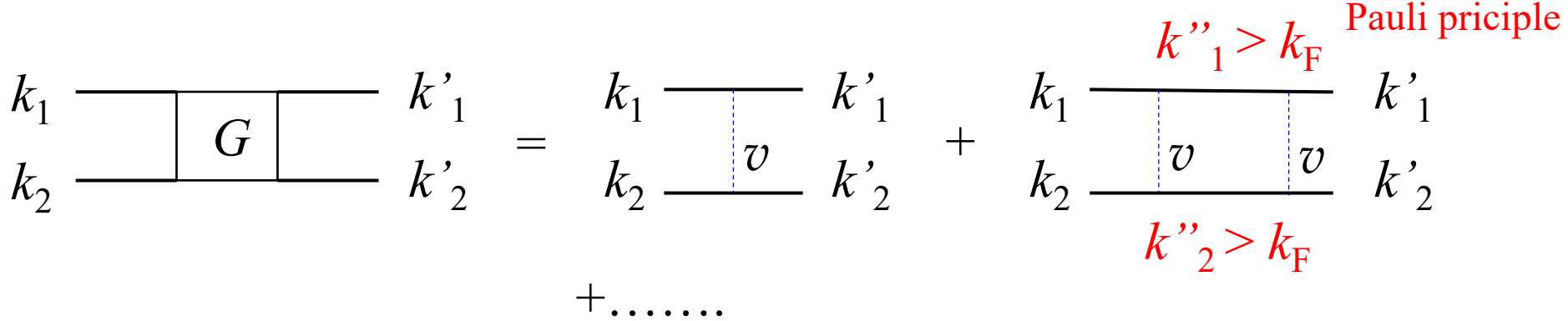
⇒
$$\psi = \phi - \frac{1}{H_0 - E} V \psi \quad H_0 = -\frac{\hbar^2}{2m} \nabla^2, \quad (H_0 - E) \phi = 0$$

⇒
$$V \psi = V \phi - V \frac{1}{H_0 - E} V \psi \quad \Rightarrow \quad T = V - V \frac{1}{H_0 - E} T$$

(Vψ = Tφ)

核内における核子間相互作用(媒質効果)

two-body (multiple) scattering *in medium*



Bethe-Goldstone equation

$$G = v + v \frac{Q_F}{E - H_0} G$$

*scattering: suppressed
 because intermediate states have to have
 $k > k_F \rightarrow$ independent particle picture

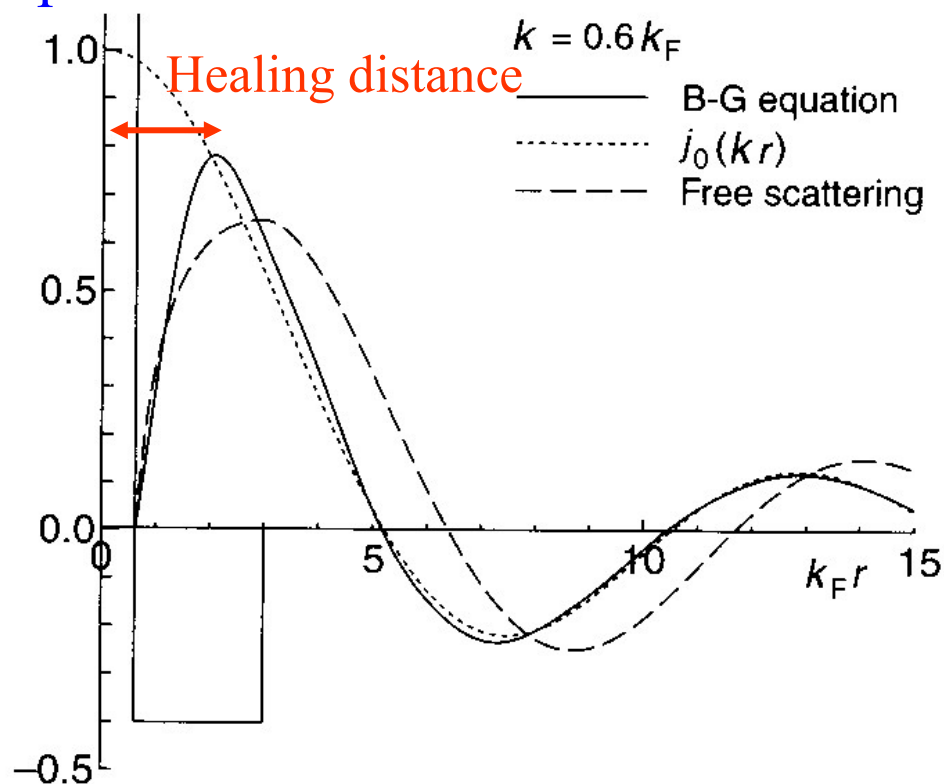
◆ Hard core

$$G = v + v \frac{Q_F}{E - H_0} G \quad \longleftrightarrow \quad G = \frac{v}{1 - v \frac{Q_F}{E - H_0}}$$

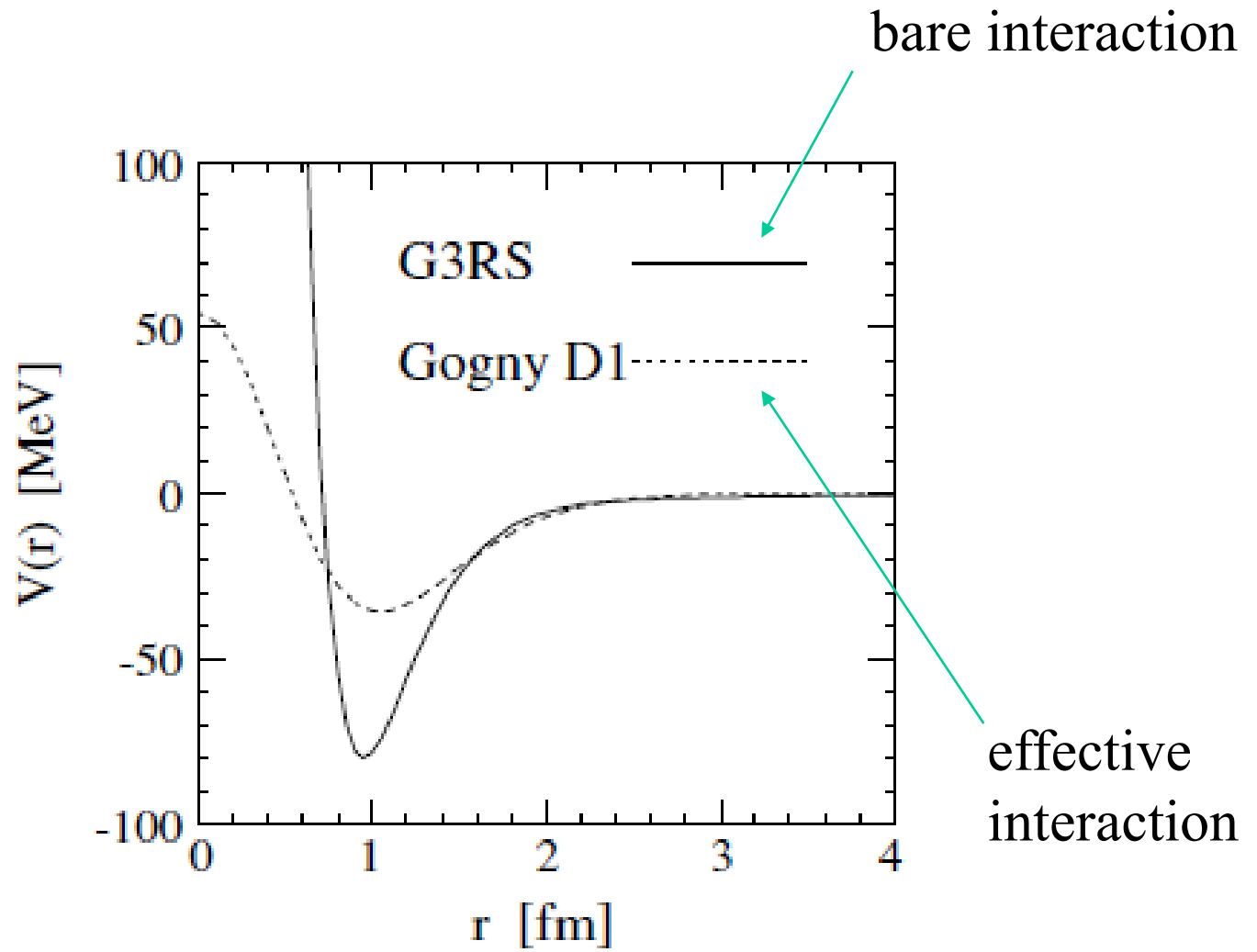


Even if v tends to infinity, G may stay finite.

◆ Independent particle motion



→ use G instead of v in mean-field calculations



M. Matsuo, Phys. Rev. C73('06)044309

Phenomenological effective interactions

G-matrix

- ab initio
- but, cumbersome to compute (especially for finite nuclei)
- qualitatively good, but quantitatively not successful



HF calculations with a phenomenological effective interaction

Philosophy: take the functional form of G , but determine the parameters phenomenologically

- Skyrme interaction (non-rel., zero range)
- Gogny interaction (non-rel., finite range)
- Relativistic mean-field model (relativistic, “meson exchanges”)

Skyrme interaction density dependent zero-range interaction

$$\begin{aligned}
 v(\mathbf{r}, \mathbf{r}') &= t_0(1 + x_0\hat{P}_\sigma)\delta(\mathbf{r} - \mathbf{r}') \\
 &+ \frac{1}{2}t_1(1 + x_1\hat{P}_\sigma)(\mathbf{k}^2\delta(\mathbf{r} - \mathbf{r}') + \delta(\mathbf{r} - \mathbf{r}')\mathbf{k}^2) \\
 &+ t_2(1 + x_2\hat{P}_\sigma)\mathbf{k}\delta(\mathbf{r} - \mathbf{r}')\mathbf{k} \\
 &+ \frac{1}{6}t_3(1 + x_3\hat{P}_\sigma)\delta(\mathbf{r} - \mathbf{r}')\rho^\alpha((\mathbf{r} + \mathbf{r}')/2) \\
 &+ iW_0(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2) \cdot \mathbf{k} \times \delta(\mathbf{r} - \mathbf{r}')\mathbf{k}
 \end{aligned}$$

if $x_i=0, t_1=t_2=0$:

$$\mathbf{k} = (\nabla_1 - \nabla_2)/2i$$

$$v(\mathbf{r}, \mathbf{r}') = \underbrace{t_0\delta(\mathbf{r} - \mathbf{r}')}_{\text{short-range attraction}} + \underbrace{\frac{1}{6}t_3\delta(\mathbf{r} - \mathbf{r}')\rho^\alpha(\mathbf{r})}_{\text{repulsion to avoid collapse}}$$

$$\underbrace{+iW_0(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2) \cdot \mathbf{k} \times \delta(\mathbf{r} - \mathbf{r}')\mathbf{k}}_{\text{spin-orbit interaction}}$$

spin-orbit interaction

Skyrme interaction density dependent zero-range interaction

$$\begin{aligned}v(\mathbf{r}, \mathbf{r}') &= t_0(1 + x_0\hat{P}_\sigma)\delta(\mathbf{r} - \mathbf{r}') \\ &+ \frac{1}{2}t_1(1 + x_1\hat{P}_\sigma)(\mathbf{k}^2\delta(\mathbf{r} - \mathbf{r}') + \delta(\mathbf{r} - \mathbf{r}')\mathbf{k}^2) \\ &+ t_2(1 + x_2\hat{P}_\sigma)\mathbf{k}\delta(\mathbf{r} - \mathbf{r}')\mathbf{k} \\ &+ \frac{1}{6}t_3(1 + x_3\hat{P}_\sigma)\delta(\mathbf{r} - \mathbf{r}')\rho^\alpha((\mathbf{r} + \mathbf{r}')/2) \\ &+ iW_0(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2) \cdot \mathbf{k} \times \delta(\mathbf{r} - \mathbf{r}')\mathbf{k}\end{aligned}$$

$$\mathbf{k} = (\nabla_1 - \nabla_2)/2i$$

(note) finite range effect \longleftrightarrow momentum dependence

$$\begin{aligned}\langle \mathbf{p} | V | \mathbf{p}' \rangle &= \frac{1}{(2\pi\hbar)^3} \int d\mathbf{r} e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{r}/\hbar} V(\mathbf{r}) \\ &\sim V_0 + V_1(\mathbf{p}^2 + \mathbf{p}'^2) + V_2\mathbf{p}\mathbf{p}' + \dots \\ &\rightarrow V_0\delta(\mathbf{r}) + V_1(\hat{\mathbf{p}}^2\delta(\mathbf{r}) + \delta(\mathbf{r})\hat{\mathbf{p}}^2) + V_2\hat{\mathbf{p}}\delta(\mathbf{r})\hat{\mathbf{p}}\end{aligned}$$

Skyrme interaction density dependent zero-range interaction

$$\begin{aligned}v(\mathbf{r}, \mathbf{r}') &= t_0(1 + x_0\hat{P}_\sigma)\delta(\mathbf{r} - \mathbf{r}') \\ &+ \frac{1}{2}t_1(1 + x_1\hat{P}_\sigma)(\mathbf{k}^2\delta(\mathbf{r} - \mathbf{r}') + \delta(\mathbf{r} - \mathbf{r}')\mathbf{k}^2) \\ &+ t_2(1 + x_2\hat{P}_\sigma)\mathbf{k}\delta(\mathbf{r} - \mathbf{r}')\mathbf{k} \\ &+ \frac{1}{6}t_3(1 + x_3\hat{P}_\sigma)\delta(\mathbf{r} - \mathbf{r}')\rho^\alpha((\mathbf{r} + \mathbf{r}')/2) \\ &+ iW_0(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2) \cdot \mathbf{k} \times \delta(\mathbf{r} - \mathbf{r}')\mathbf{k}\end{aligned}$$

$$\mathbf{k} = (\nabla_1 - \nabla_2)/2i$$

the exchange potential \longrightarrow local

$$\begin{aligned}0 &= \left[-\frac{\hbar^2}{2m}\nabla^2 + \int v(\mathbf{r} - \mathbf{r}') \left(\sum_j |\psi_j(\mathbf{r}')|^2 \right) d\mathbf{r}' - \epsilon_i \right] \psi_i(\mathbf{r}) \\ &- \int v(\mathbf{r} - \mathbf{r}') \left(\sum_j \psi_j^*(\mathbf{r}')\psi_i(\mathbf{r}') \right) d\mathbf{r}'\psi_j(\mathbf{r})\end{aligned}$$

Skyrme interactions: 10 adjustable parameters

$$\begin{aligned}v(\mathbf{r}, \mathbf{r}') &= t_0(1 + x_0\hat{P}_\sigma)\delta(\mathbf{r} - \mathbf{r}') \\ &+ \frac{1}{2}t_1(1 + x_1\hat{P}_\sigma)(\mathbf{k}^2\delta(\mathbf{r} - \mathbf{r}') + \delta(\mathbf{r} - \mathbf{r}')\mathbf{k}^2) \\ &+ t_2(1 + x_2\hat{P}_\sigma)\mathbf{k}\delta(\mathbf{r} - \mathbf{r}')\mathbf{k} \\ &+ \frac{1}{6}t_3(1 + x_3\hat{P}_\sigma)\delta(\mathbf{r} - \mathbf{r}')\rho^\alpha((\mathbf{r} + \mathbf{r}')/2) \\ &+ iW_0(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2) \cdot \mathbf{k} \times \delta(\mathbf{r} - \mathbf{r}')\mathbf{k}\end{aligned}$$

A fitting strategy:

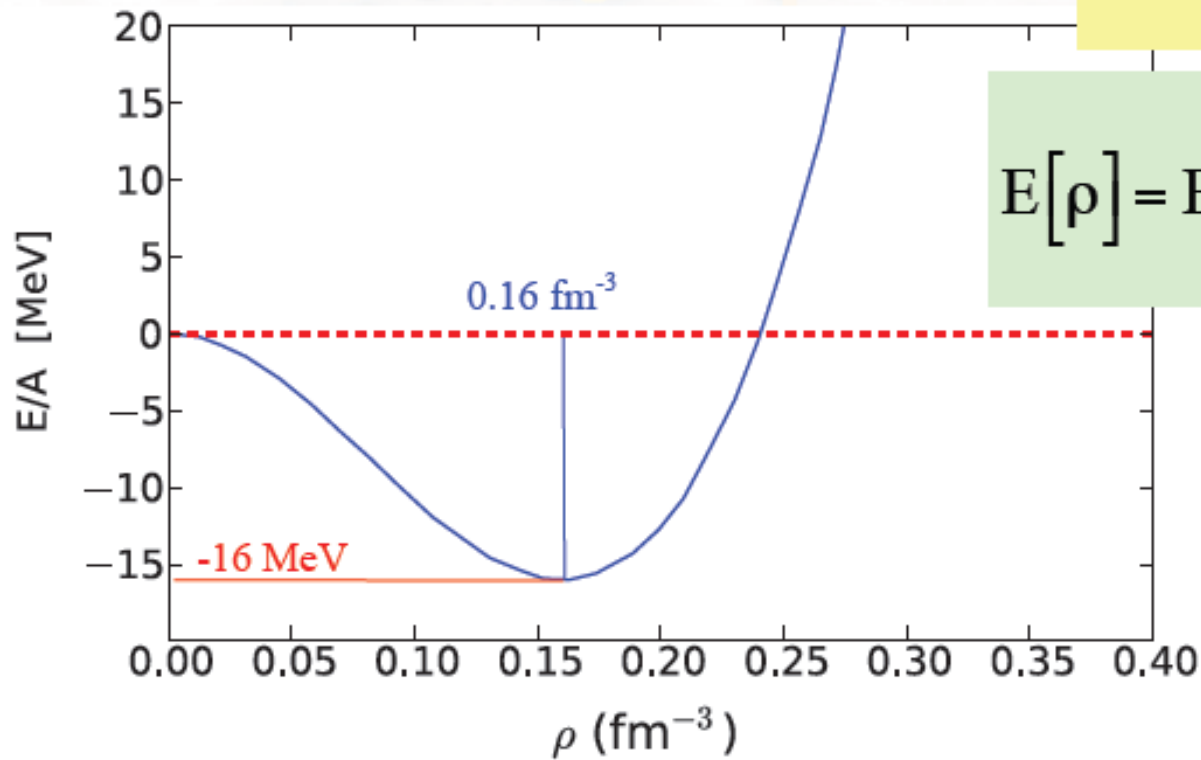
B.E. and r_{rms} : ^{16}O , ^{40}Ca , ^{48}Ca , ^{56}Ni , ^{90}Zr , ^{208}Pb ,.....

Infinite nuclear matter: E/A , ρ_{eq} ,.....

Parameter sets:

SIII, SkM*, SGII, SLy4,.....

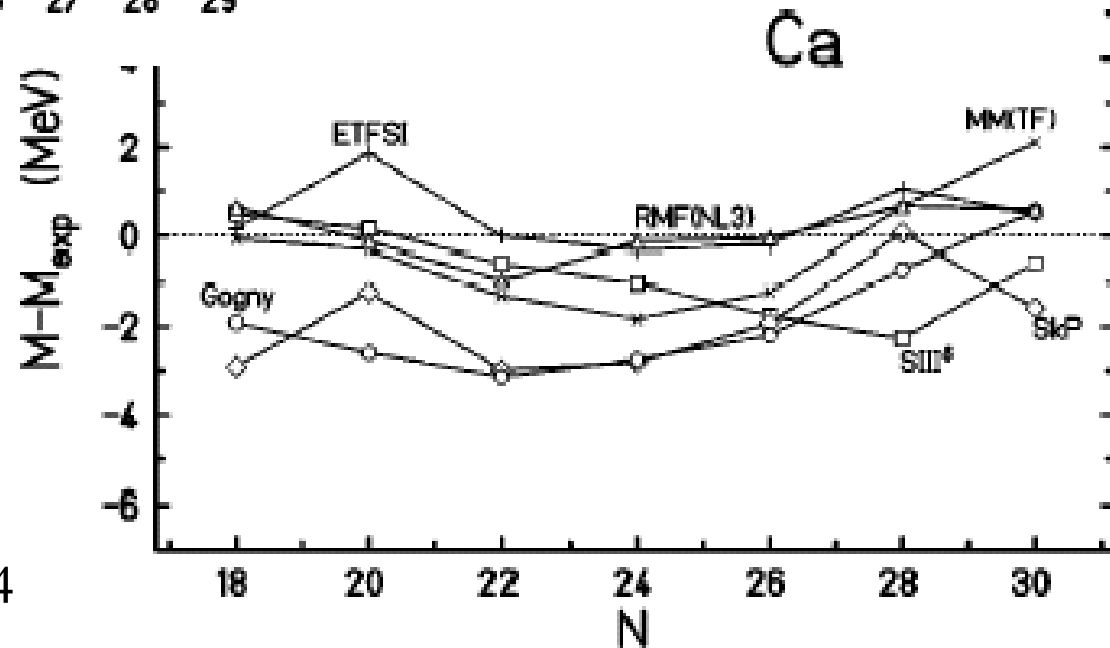
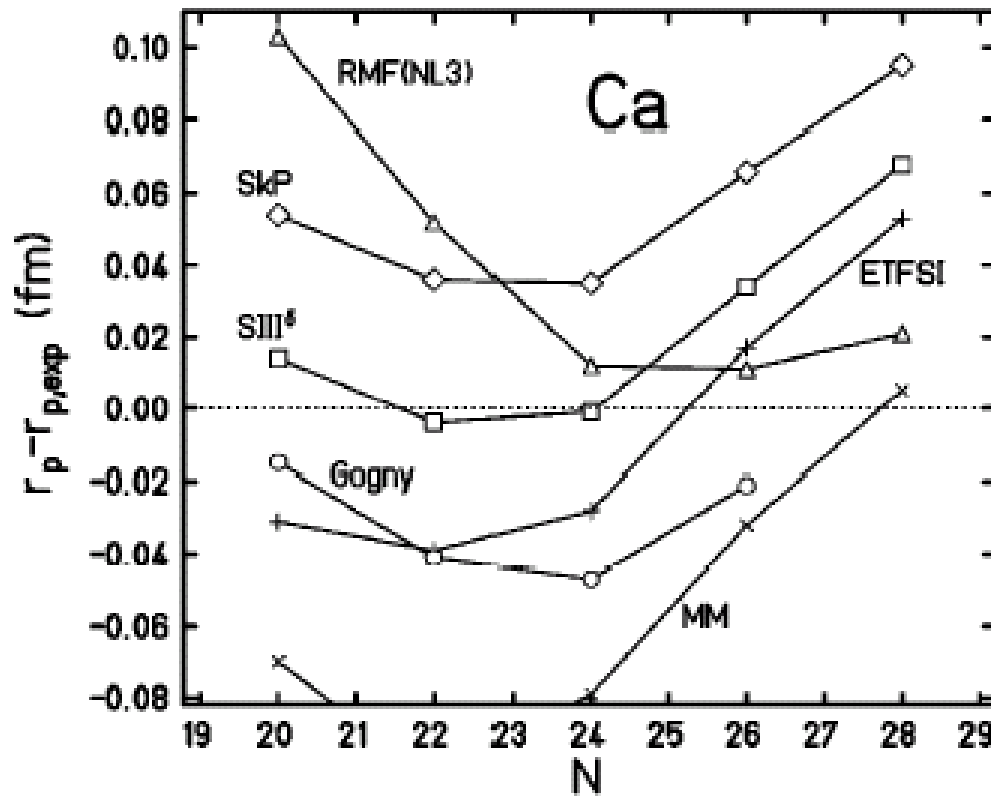
EOS of infinite nuclear matter



$$K_{\infty} = 9\rho^2 \left. \frac{d^2[E(\rho)/\rho]}{d\rho^2} \right|_{\rho_0}$$

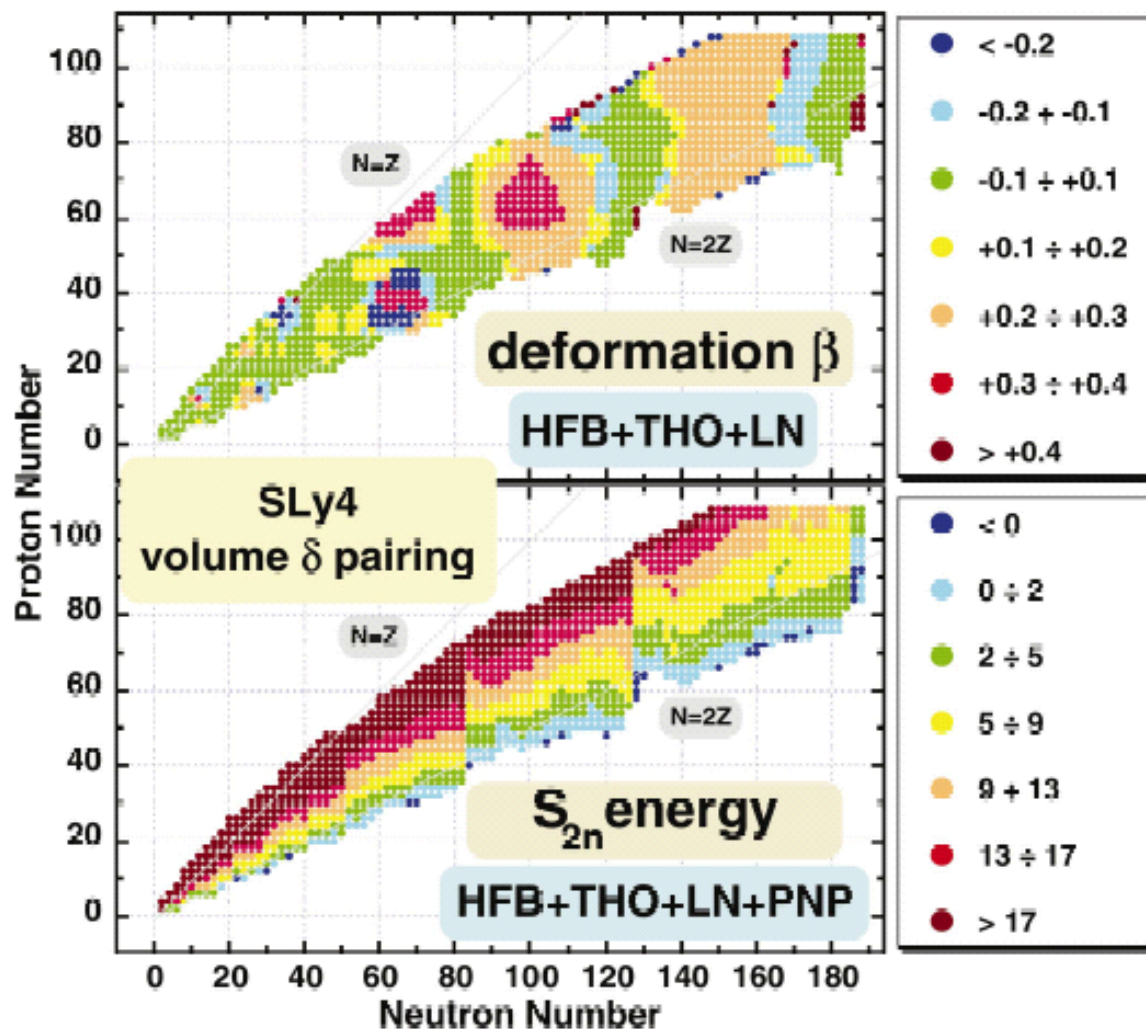
$$E[\rho] = E[\rho_0] + \frac{1}{18} K_{\infty} \left(\frac{\rho - \rho_0}{\rho_0} \right)^2$$

Examples of HF calculations
for masses and radii



Z. Patyk et al.,
PRC59('99)704

deformation and two-neutron separation energy

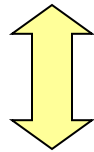


Density Functional Theory

With Skyrme interaction:

$$\begin{aligned}\langle \Psi | H | \Psi \rangle &= E[\rho, \tau, J] \\ &= \int d\mathbf{r} \left(\frac{\hbar^2}{2m} \tau + \frac{1}{2} t_0 \left(1 + \frac{1}{2} x_0 \right) \rho^2 \right. \\ &\quad \left. - \frac{1}{2} t_0 \left(x_0 + \frac{1}{2} \right) \sum_q \rho_q^2 \cdots \right)\end{aligned}$$

Energy functional in terms of local densities



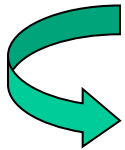
Close analog to the Density Functional Theory (DFT)

密度汎関数法

i) Hohenberg-Kohn Theorem

$$H = H_0 + V_{\text{ext}}$$

Lemma : $\rho(\mathbf{r}) \rightarrow V_{\text{ext}}(\mathbf{r})$ (unique)



Density: the basic variable

(密度が分かれば原理的に全て分かる)

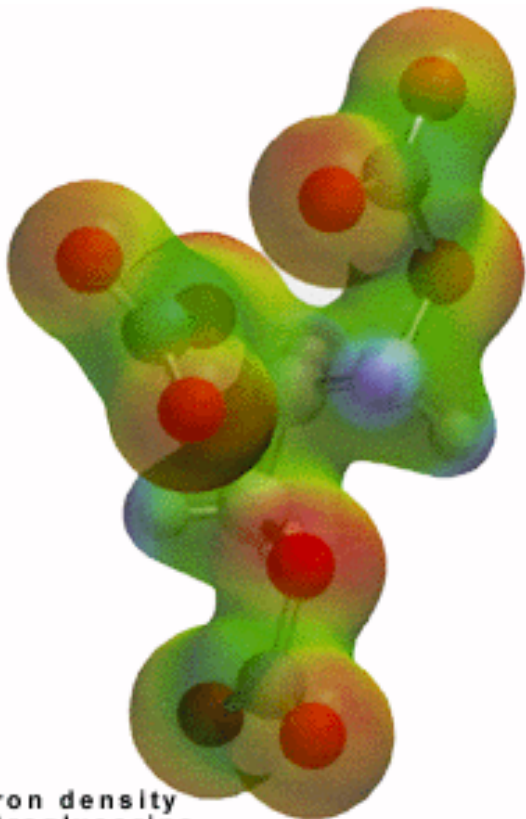
ii) Hohenberg-Kohn variational principle

The existence of a functional $E[\rho]$, which gives the exact g.s. energy for a given g.s. density

$$\longrightarrow E[\rho] \geq E_{gs}$$

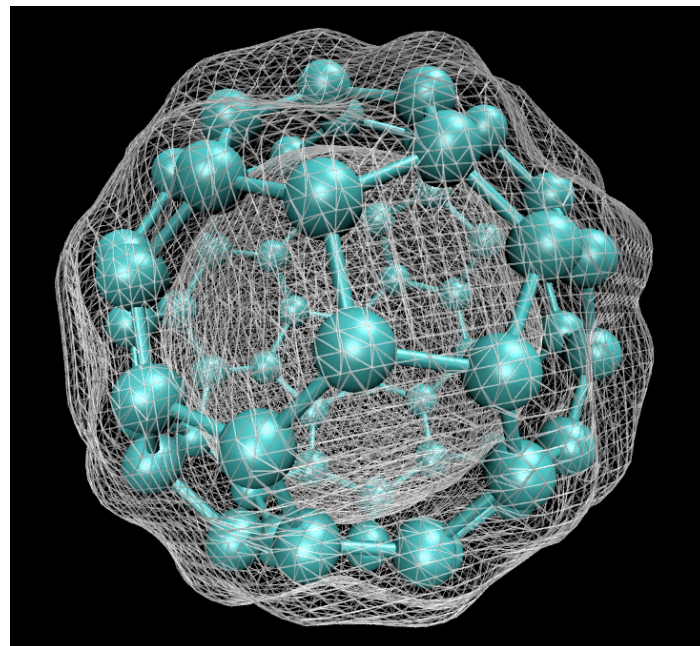
うまい方法で $E[\rho]$ を作れば、それを使って多体計算が簡単に行える。

$$E[\rho] = E_{\text{HF}}[\rho] + E_{\text{corr}}[\rho]$$



The electron density
of nitroglycerine

ニトログリセリンの電子密度
(Nobelprize.org より)



C_{60} の電子密度
(Wikipedia より)

Non-local potentials

$$\left[-\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{r}) - E \right] \psi(\mathbf{r}) + \int d\mathbf{r}' V_{\text{NL}}(\mathbf{r}, \mathbf{r}') \psi(\mathbf{r}') = 0$$

➤ Local equivalent potential

$$\left[-\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{r}) - E \right] \psi(\mathbf{r}) + \left[\frac{1}{\psi(\mathbf{r})} \int d\mathbf{r}' V_{\text{NL}}(\mathbf{r}, \mathbf{r}') \psi(\mathbf{r}') \right] \psi(\mathbf{r}) = 0$$

E-dep. potential

➤ Wigner 変換

$$V_W(\mathbf{r}, \mathbf{p}) = \int V_{\text{NL}}(\mathbf{r} - \mathbf{s}/2, \mathbf{r} + \mathbf{s}/2) e^{i\mathbf{p} \cdot \mathbf{s}/\hbar} d\mathbf{s}$$

✓ momentum expansion

✓ effective mass approximation

cf. Perrey-Buck 型

$$V_{\text{NL}}(\mathbf{r}, \mathbf{r}') = U \left(\frac{1}{2} |\mathbf{r} + \mathbf{r}'| \right) \exp \left[- \left(\frac{\mathbf{r} - \mathbf{r}'}{\beta} \right)^2 \right]$$