§. カレントのローレンツ変換性とディラック共役

$$\rho = \psi^\dagger \psi, \quad \vec{j} = c\psi^\dagger \vec{\alpha} \psi$$

$$\rightarrow \quad j^\mu = (c\rho, \vec{j})$$

$$\partial_\mu j^\mu = 0$$

ローレンツ変換 ： $x'^\mu = \Lambda^\mu{}_\nu x^\nu$

$$\left(g_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = g_{\alpha\beta}, \ \Lambda^0{}_0 \geq 1, \ \det \Lambda = 1 \right)$$

2つの慣性系で同じ形の方程式が成り立つはず。

$$\begin{cases} \left(i\hbar \gamma^\mu \dfrac{\partial}{\partial x^\mu} - mc \right) \psi(x) = 0 \\ \left(i\hbar \gamma'^\mu \dfrac{\partial}{\partial x'^\mu} - mc \right) \tilde{\psi}(x') = 0 \end{cases}$$

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

$$\{\gamma'^\mu, \gamma'^\nu\} = 2g^{\mu\nu}$$

$$\downarrow$$

γ^μ と $\gamma^{\mu\prime}$ はユニタリー変換でつながっている。

$$\gamma'^\mu = U^\dagger \gamma^\mu U \quad \rightarrow \quad U\gamma'^\mu = \gamma^\mu U$$

$$\downarrow$$

$$\left(i\hbar \underbrace{U \gamma'^\mu \frac{\partial}{\partial x'^\mu}}_{\parallel} - Umc \right) \tilde{\psi}(x') = 0$$

$$\gamma^\mu U \frac{\partial}{\partial x'^\mu} = \gamma^\mu \frac{\partial}{\partial x'^\mu} U$$

$U\tilde{\Psi}(x') \equiv \psi'(x')$ とすると

$$\left(i\hbar\gamma^\mu \frac{\partial}{\partial x'^\mu} - mc\right)\psi'(x') = 0$$

$\psi'(x') = S(\Lambda)\psi(x)$ を仮定.

$$\rightarrow \psi(x) = S^{-1}(\Lambda)\psi'(x') = S^{-1}(\Lambda)\psi'(\Lambda x)$$

(note) $x^\mu = (\Lambda^{-1})^\mu{}_\nu x'^\nu$

$$\rightarrow \psi(x) = S(\Lambda^{-1})\psi'(x') = S(\Lambda^{-1})\psi'(\Lambda x)$$

(note) $\left(i\hbar\gamma^\mu \frac{\partial}{\partial x^\mu} - mc\right)\psi(x) = 0$

$$\rightarrow \left[i\hbar S(\Lambda)\gamma^\mu S^{-1}(\Lambda)\underline{S(\Lambda)}\underbrace{\frac{\partial}{\partial x^\mu}}_{=\frac{\partial x'^\nu}{\partial x^\mu}\frac{\partial}{\partial x'^\nu} = \Lambda^\nu{}_\mu \frac{\partial}{\partial x'^\nu}} - mc\,\underline{S(\Lambda)}\right]\underline{\psi(x)} = 0$$

$$\Downarrow$$

$$\left[i\hbar S(\Lambda)\gamma^\mu S^{-1}(\Lambda)\Lambda^\nu{}_\mu \frac{\partial}{\partial x'^\nu} - mc\right]\psi'(x') = 0$$

$$\leftrightarrow \left(i\hbar\gamma^\mu \frac{\partial}{\partial x'^\mu} - mc\right)\psi'(x') = 0$$

$$\downarrow$$

$$S(\Lambda)\gamma^\mu \underbrace{S^{-1}(\Lambda)\Lambda^\nu{}_\mu}_{=\Lambda^\nu{}_\mu S^{-1}(\Lambda)} = \gamma^\nu$$

$$\Updownarrow$$

$$\Lambda^\nu{}_\mu\gamma^\mu = S^{-1}(\Lambda)\gamma^\nu S(\Lambda)$$

無限小ローレンツ変換： $\Lambda^{\nu}{}_{\mu} = \delta^{\nu}_{\mu} + \Delta\omega^{\nu}{}_{\mu}$

$$\Delta\omega^{\mu\nu} = g^{\mu\lambda}\Delta\omega^{\nu}{}_{\lambda} \underset{\uparrow}{=} -\Delta\omega^{\nu\mu}$$

$$\begin{aligned} g_{\alpha\beta} &= g_{\mu\nu}\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta} \\ &= g_{\mu\nu}(\delta^{\mu}_{\alpha} + \Delta\omega^{\mu}{}_{\alpha})(\delta^{\nu}_{\beta} + \Delta\omega^{\nu}{}_{\beta}) \\ &= g_{\alpha\beta} + (g_{\mu\beta}\Delta\omega^{\mu}{}_{\alpha} + g_{\alpha\nu}\Delta\omega^{\nu}{}_{\beta}) \\ &= g_{\alpha\beta} + \underbrace{(\Delta\omega_{\beta\alpha} + \Delta\omega_{\alpha\beta})}_{=0} \end{aligned}$$

$$\left.\begin{aligned} S(\Lambda) &= 1 - \frac{i}{4}\Delta\omega^{\mu\nu}\sigma_{\mu\nu} \\ S^{-1}(\Lambda) &= 1 + \frac{i}{4}\Delta\omega^{\mu\nu}\sigma_{\mu\nu} \end{aligned}\right\} \text{とおける.}$$

$$\Lambda^{\nu}{}_{\mu}\gamma^{\mu} = (\delta^{\nu}_{\mu} + \Delta\omega^{\nu}{}_{\mu})\gamma^{\mu} = \gamma^{\nu} + \Delta\omega^{\nu}{}_{\mu}\gamma^{\mu}$$

$$\begin{aligned} S^{-1}(\Lambda)\gamma^{\nu}S(\Lambda) &= \left(1 + \frac{i}{4}\Delta\omega^{\alpha\beta}\sigma_{\alpha\beta}\right)\gamma^{\nu}\left(1 - \frac{i}{4}\Delta\omega^{\alpha\beta}\sigma_{\alpha\beta}\right) \\ &\sim \gamma^{\nu} + \frac{i}{4}\Delta\omega^{\alpha\beta}\sigma_{\alpha\beta}\gamma^{\nu} - \frac{i}{4}\gamma^{\nu}\Delta\omega^{\alpha\beta}\sigma_{\alpha\beta} \\ &= \gamma^{\nu} + \frac{i}{4}\Delta\omega^{\alpha\beta}[\sigma_{\alpha\beta}, \gamma^{\nu}] \end{aligned}$$

$$\Lambda^{\nu}{}_{\mu}\gamma^{\mu} = S^{-1}(\Lambda)\gamma^{\nu}S(\Lambda)$$

$$\longrightarrow \Delta\omega^{\nu}{}_{\mu}\gamma^{\mu} = \frac{i}{4}\Delta\omega^{\alpha\beta}[\sigma_{\alpha\beta}, \gamma^{\nu}]$$

この式は
$$\sigma_{\alpha\beta} = \frac{i}{2}[\gamma_\alpha, \gamma_\beta] = \frac{i}{2}(\gamma_\alpha\gamma_\beta - \underbrace{\gamma_\beta\gamma_\alpha}_{= 2g_{\alpha\beta} - \gamma_\alpha\gamma_\beta})$$
$$= i(\gamma_\alpha\gamma_\beta - g_{\alpha\beta})$$
とおくと満たされる。

(note) $\frac{i}{4}\Delta\omega^{\alpha\beta}[\sigma_{\alpha\beta}, \gamma^\nu] = -\frac{1}{4}\Delta\omega^{\alpha\beta}[\gamma_\alpha\gamma_\beta, \gamma^\nu]$
$$= \cdots = \Delta\omega^\nu{}_\mu \gamma^\mu$$

(note) $x'^\mu = \Lambda^\mu{}_\nu x^\nu$

$$\Lambda^\mu{}_\nu = \lim_{N\to\infty}(\delta^\mu{}_{\mu_1} + \Delta\omega^\mu{}_{\mu_1})(\delta^{\mu_1}{}_{\mu_2} + \Delta\omega^{\mu_1}{}_{\mu_2}) \times \cdots (\delta^{\mu_{N-1}}{}_\nu + \Delta\omega^{\mu_{N-1}}{}_\nu)$$
$$= \lim_{N\to\infty}\left[\left(1+\frac{\omega}{N}\right)^N\right]^\mu{}_\nu = (e^\omega)^\mu{}_\nu$$
$$(\omega^\mu{}_\nu = N\Delta\omega^\mu{}_\nu)$$

$$\psi'(x') = S(\Lambda)\psi(x)$$
$$S(\Lambda) = \lim_{N\to\infty}\left(1 - \frac{i}{4}\Delta\omega^{\mu\nu}\sigma_{\mu\nu}\right)^N$$
$$\sim e^{-\frac{i}{4}\omega^{\mu\nu}\sigma_{\mu\nu}}$$
$$S^{-1}(\Lambda) = e^{\frac{i}{4}\omega^{\mu\nu}\sigma_{\mu\nu}} = S(\Lambda^{-1})$$

ディラック共役： $\bar{\psi}(x) \equiv \psi^{\dagger}(x)\gamma_0$

$$\bar{\psi}'(x') = \psi'^{\dagger}(x')\gamma_0 = \psi^{\dagger}(x)S(\Lambda)^{\dagger}\gamma_0$$

$$= \underbrace{\psi^{\dagger}(x)\gamma_0}_{=\bar{\psi}(x)}\underbrace{\gamma_0 S(\Lambda)^{\dagger}\gamma_0}_{}$$

$$\gamma_0 S(\Lambda)^{\dagger}\gamma_0 = \gamma_0 e^{\frac{i}{4}\omega^{\mu\nu}\sigma^{\dagger}_{\mu\nu}}\gamma_0$$

$$= e^{\frac{i}{4}\omega^{\mu\nu}\gamma_0\sigma^{\dagger}_{\mu\nu}\gamma_0} \quad \leftarrow \gamma_0^2 = 1$$

$$= e^{\frac{i}{4}\omega^{\mu\nu}\sigma_{\mu\nu}} \quad \leftarrow \gamma_0\sigma_{0i}\gamma_0 = -\sigma_{0i}, \quad \gamma_0\sigma_{ij}\gamma_0 = \sigma_{ij}$$

$$= S^{-1}(\Lambda)$$

より

$$\bar{\psi}'(x')\psi'(x') = \bar{\psi}(x)S^{-1}(\Lambda)S(\Lambda)\psi(x) = \bar{\psi}(x)\psi(x)$$

→ ローレンツ・スカラー

$$\bar{\psi}'(x')\gamma^{\mu}\psi'(x') = \bar{\psi}(x)\underline{S^{-1}(\Lambda)\gamma^{\mu}S(\Lambda)}\psi(x)$$

$$= \bar{\psi}(x)\Lambda^{\mu}{}_{\nu}\gamma^{\nu}\psi(x)$$

$$= \Lambda^{\mu}{}_{\nu}\bar{\psi}(x)\gamma^{\nu}\psi(x)$$

→ ローレンツ・ベクター

(note) $\alpha^i = \gamma^0 \gamma^i$

↓

$$\bar{\psi} \gamma^\mu \psi = (\psi^\dagger \gamma_0 \gamma^0 \psi, \ \psi^\dagger \gamma_0 \gamma^i \psi)$$

$$= (\psi^\dagger \psi, \ \psi^\dagger \alpha^i \psi)$$

$$= \frac{1}{c} j^\mu$$

(note) $\bar{\psi}\psi$ は空間反転（パリティ変換）に対しても不変

(note) 双一次形式

$\bar{\psi}\psi$ ：スカラー

$\bar{\psi}\gamma^\mu\psi$ ：ベクター

$\bar{\psi}\sigma^{\mu\nu}\psi$ ：2階のテンソル

$\bar{\psi}\gamma_5\psi$ ：擬スカラー　($\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$)

ローレンツ・スカラー，パリティ・マイナス

$\bar{\psi}\gamma_5\gamma^\mu\psi$ ：擬ベクトル

空間反転に対しベクトル（$\bar{\psi}\gamma^\mu\psi$）と反対の変換性

§. Dirac方程式の平面波解

・静止した粒子の解 : $\mathbb{P}\psi = \frac{\hbar}{i}\nabla\psi = 0$

$$i\hbar\frac{\partial}{\partial t}\psi = \beta mc^2\psi = mc^2\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\psi$$

線形独立な解

$$\psi^{(+)}_{\mathbb{P}=0,s} = e^{-imc^2t/\hbar}\begin{pmatrix}\chi_s\\ 0\end{pmatrix} \rightarrow E = mc^2$$

$$\psi^{(-)}_{\mathbb{P}=0,s} = e^{+imc^2t/\hbar}\begin{pmatrix}0\\ \chi_s\end{pmatrix} \rightarrow E = -mc^2$$

$$\chi_\uparrow = \begin{pmatrix}1\\ 0\end{pmatrix}, \quad \chi_\downarrow = \begin{pmatrix}0\\ 1\end{pmatrix}$$

スピノル
(スピン波動関数)

(note)
$$S_z \equiv \frac{\hbar}{2}\Sigma_3 = \frac{\hbar}{2}\begin{pmatrix}\sigma_3 & 0\\ 0 & \sigma_3\end{pmatrix}$$

$$\begin{cases} S_z\psi^{(\pm)}_{\mathbb{P}=0,\uparrow} = +\frac{\hbar}{2}\psi^{(\pm)}_{\mathbb{P}=0,\uparrow} \\ S_z\psi^{(\pm)}_{\mathbb{P}=0,\downarrow} = -\frac{\hbar}{2}\psi^{(\pm)}_{\mathbb{P}=0,\downarrow} \end{cases}$$

・一般の平面波解

$$\psi = \begin{pmatrix}\psi_A\\ \psi_B\end{pmatrix} = \begin{pmatrix}u_A(\mathbb{P})\\ u_B(\mathbb{P})\end{pmatrix}\underbrace{e^{i\mathbb{P}\cdot\mathbb{X}/\hbar - iEt/\hbar}}_{= e^{-ip_\mu x^\mu/\hbar}}$$

$$p_\mu = \left(\frac{E}{c}, -\mathbb{P}\right)$$
$$x^\mu = (ct, +\mathbb{X})$$

$$i\hbar\frac{\partial}{\partial t}\psi = \left(-i\hbar c\vec{\alpha}\cdot\vec{\nabla} + \beta mc^2\right)\psi$$

$\downarrow$

$$E\begin{pmatrix} u_A \\ u_B \end{pmatrix} = \begin{pmatrix} mc^2 & c\vec{\sigma}\cdot\vec{p} \\ c\vec{\sigma}\cdot\vec{p} & -mc^2 \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

$\Downarrow$

$$\begin{cases} (E-mc^2)u_A = c\,\vec{\sigma}\cdot\vec{p}\,u_B \\ (E+mc^2)u_B = c\,\vec{\sigma}\cdot\vec{p}\,u_A \end{cases}$$

or

$$u_A = \frac{c}{E-mc^2}(\vec{\sigma}\cdot\vec{p})u_B$$

$$u_B = \frac{c}{E+mc^2}(\vec{\sigma}\cdot\vec{p})u_A$$

$E>0$ に対し

$$u_A = \chi_s \rightarrow u_B = \frac{c}{E+mc^2}(\vec{\sigma}\cdot\vec{p})\chi_s$$

$\Downarrow$

$$u^{(+)}(p,s) = N\begin{pmatrix} \chi_s \\ \frac{c}{E+mc^2}(\vec{\sigma}\cdot\vec{p})\chi_s \end{pmatrix}$$

$E<0$ に対し

$$u_B = \chi_s \rightarrow u_A = \frac{c}{E-mc^2}(\vec{\sigma}\cdot\vec{p})\chi_s$$

$\Downarrow$

$$u^{(-)}(p,s) = N\begin{pmatrix} \frac{c}{E-mc^2}(\vec{\sigma}\cdot\vec{p})\chi_s \\ \chi_s \end{pmatrix}$$

・規格化

$$u^{(+)}(\mathbb{P}, S) = \sqrt{\frac{E+mc^2}{2mc^2}} \begin{pmatrix} \chi_S \\ \frac{c}{E+mc^2}(\vec{\sigma}\cdot\vec{p})\chi_S \end{pmatrix}$$

ととると

$$\bar{u}^{(+)}(\mathbb{P}, S)u^{(+)}(\mathbb{P}, S) = \frac{E+mc^2}{2mc^2}\left(\chi_S^\dagger \quad \frac{c(\sigma\cdot\mathbb{P})}{E+mc^2}\chi_S^\dagger\right)\underbrace{\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{=\gamma_0}\begin{pmatrix} \chi_S \\ \frac{c\sigma\cdot\mathbb{P}\chi_S}{E+mc^2} \end{pmatrix}}_{=\begin{pmatrix} \chi_S \\ -\frac{c\sigma\cdot\mathbb{P}}{E+mc^2}\chi_S \end{pmatrix}}$$

$$= \frac{E+mc^2}{2mc^2}\left(1 - \left(\frac{c}{E+mc^2}\right)^2 \underbrace{(\sigma\cdot\mathbb{P})^2}_{=p^2}\right)$$

$$= \frac{\cancel{E+mc^2}}{2mc^2}\cdot\frac{(E+mc^2)^2 - c^2p^2}{(E+mc^2)^{\cancel{2}}} = \frac{E^2 + 2mc^2E + m^2c^4 - \cancel{c^2p^2}}{2mc^2(E+mc^2)}$$

$$E^2 \to \cancel{p^2c^2} + m^2c^4$$

$$= \frac{2mc^2(E+mc^2)}{2mc^2(E+mc^2)} = 1.$$

同様に

$$u^{(-)}(\mathbb{P}, S) = \sqrt{\frac{-E+mc^2}{2mc^2}} \begin{pmatrix} \frac{c\sigma\cdot\mathbb{P}}{E-mc^2}\chi_S \\ \chi_S \end{pmatrix}$$

とすると

$$\bar{u}^{(-)}(\mathbb{P}, S)u^{(-)}(\mathbb{P}, S) = -1.$$

$$u^{(+)}(\mathbb{P},S) = \sqrt{\frac{mc^2}{E}} \cdot \sqrt{\frac{E+mc^2}{2mc^2}} \begin{pmatrix} \chi_S \\ \frac{c}{E+mc^2}(\boldsymbol{\sigma}\cdot\mathbb{P})\chi_S \end{pmatrix}$$

$$= \sqrt{\frac{E+mc^2}{2E}} \begin{pmatrix} \chi_S \\ \frac{c}{E+mc^2}(\boldsymbol{\sigma}\cdot\mathbb{P})\chi_S \end{pmatrix}$$

ととると

$$u^{(+)\dagger}(\mathbb{P},S)\, u^{+}(\mathbb{P},S) = \frac{E+mc^2}{2E}\left(1+\left(\frac{c\,\boldsymbol{\sigma}\cdot\mathbb{P}}{E+mc^2}\right)^2\right)$$

$$= \frac{\cancel{E+mc^2}}{2E} \cdot \frac{E^2+2mc^2E+m^2c^4+p^2c^2}{(E+mc^2)^{\cancel{2}}}$$

$$= \frac{2m^2c^4+2p^2c^2+2mc^2E}{2E(E+mc^2)} = \frac{2E(E+mc^2)}{2E(E+mc^2)} = 1,$$

同様に

$$u^{(-)}(\mathbb{P},S) = \underbrace{\sqrt{\frac{mc^2}{E}} \cdot \sqrt{\frac{-E+mc^2}{2mc^2}}}_{\displaystyle \sqrt{\frac{-E+mc^2}{2E}}} \begin{pmatrix} \frac{c\,\boldsymbol{\sigma}\cdot\mathbb{P}}{E-mc^2}\chi_S \\ \chi_S \end{pmatrix}$$ ととると

$$u^{(-)\dagger}(\mathbb{P},S)\, u^{(-)}(\mathbb{P},S) = \frac{-E+mc^2}{2E}\left(\frac{c^2p^2}{(E-mc^2)^2}+1\right)$$

$$= \frac{-E+mc^2}{2E} \cdot \frac{c^2p^2+E^2-2mc^2E+m^2c^4}{(E-mc^2)^2} = \frac{-E+mc^2}{2E} \cdot \frac{2E(E-mc^2)}{(E-mc^2)^2}$$

$$= -1.$$

(note) 非相対論極限

$$E = \sqrt{m^2c^4 + p^2c^2} \sim mc^2$$

$$\downarrow$$

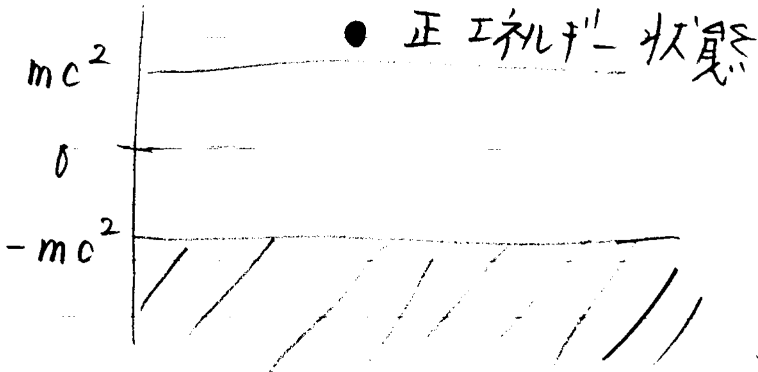
$$u^{(+)}(p,s) \sim N \begin{pmatrix} \chi_s \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{2mc}\chi_s \end{pmatrix}$$

$$\sim N \begin{pmatrix} \chi_s \\ \frac{1}{2}\vec{\sigma}\cdot\frac{\vec{v}}{c}\chi_s \end{pmatrix} \begin{matrix} \} \text{ large 成分} \\ \} \text{ small 成分} \end{matrix}$$

▨ 空孔理論

負エネルギー解 (E<0) → 正エネルギー解が不安定

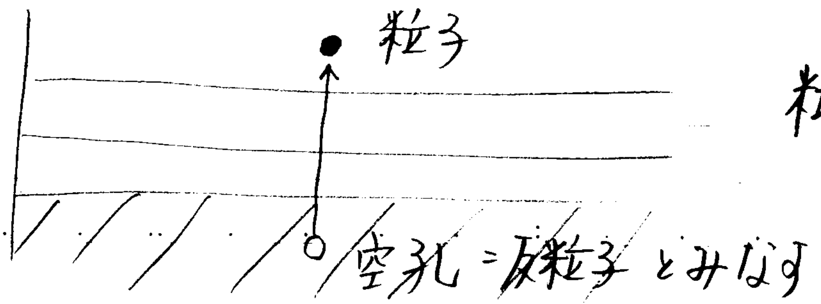
Dirac：負エネルギー解はすべて粒子が詰まっている



→ パウリ原理のため負エネルギー状態に遷移することはない。

} ディラックの海
=状態には粒子がつまっている

外からエネルギーが与えられると



粒子・反粒子の対生成

※完全な理解には場の理論が必要