§. 相対論的散乱問題の準備：非相対論的プロパゲータ

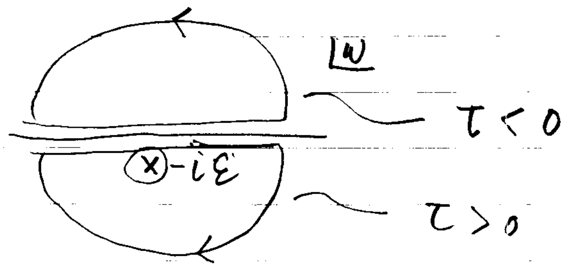
時刻 t での波動関数 $\psi(x) = \psi(\mathbf{r}, t)$

→ 時刻 t' $(>t)$ への時間発展：

$$\theta(t'-t)\,\psi(x') = i\int d^3x\ G(x';x)\,\psi(x)$$

(note)

$$\theta(\tau) = \lim_{\varepsilon\to+0} \frac{-1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{-i\omega\tau}}{\omega+i\varepsilon}\,d\omega$$



↓

$$\frac{d\theta(\tau)}{d\tau} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega\tau}\,d\omega = \delta(\tau)$$

$G(x';x)$ ：プロパゲータ（伝搬関数），グリーン関数

(note)

$$\left(i\hbar\frac{\partial}{\partial t'} - H(x')\right)\theta(t'-t)\,\psi(x')$$

$$= i\hbar\,\delta(t'-t)\,\psi(x') + \theta(t'-t)\underbrace{\left(i\hbar\frac{\partial}{\partial t'} - H(x')\right)\psi(x')}_{=0}$$

$$= i\hbar\,\delta(t'-t)\,\psi(x')$$

$$= i\int d\mathbf{r}\,\left[\left(i\hbar\frac{\partial}{\partial t'} - H(x')\right)G(x',x)\right]\psi(x)$$

$$\Downarrow\quad \boxed{\left[i\hbar\frac{\partial}{\partial t'} - H(x')\right]G(x';x) = \hbar\,\delta(t'-t)\,\delta(\mathbf{r}'-\mathbf{r})}$$

(note) 完全系： $\sum_n \psi_n(\mathbf{r}')\psi_n^*(\mathbf{r}) = \sum_n \langle \mathbf{r}'|\psi_n\rangle\langle\psi_n|\mathbf{r}\rangle$
$= \delta(\mathbf{r}-\mathbf{r}')$

$$\Downarrow$$

$$G(x', x) = -i\,\theta(t'-t)\sum_n \psi_n(x')\psi_n^*(x)$$

(note) $\left(i\hbar\frac{\partial}{\partial t'} - H(x')\right)G(x', x)$ $\qquad\left(i\hbar\frac{\partial}{\partial t'} - H(x')\right)\psi_n(x') = 0$

$$= \hbar\,\delta(t'-t)\sum_n \psi_n(x')\psi_n^*(x)$$

$$= \hbar\,\delta(t'-t)\sum_n \psi_n(\mathbf{r}', t)\psi_n^*(\mathbf{r}, t)$$
$$= \hbar\,\delta(t'-t)\,\delta(\mathbf{r}'-\mathbf{r})$$

・自由粒子の場合

$H = H_0 = -\frac{\hbar^2}{2m}\nabla^2 \longrightarrow$ 固有関数 $\varphi_p(x) = \dfrac{e^{\frac{i}{\hbar}\left(\mathbf{p}\cdot\mathbf{r} - \frac{p^2}{2m}t\right)}}{(2\pi\hbar)^{3/2}}$

(note) $\int d\mathbf{p}\ \varphi_p(\mathbf{r}', t)\varphi_p^*(\mathbf{r}, t) = \delta(\mathbf{r}'-\mathbf{r})$

$$\downarrow$$

$$G_0(x', x) = -i\,\theta(t'-t)\int d\mathbf{p}\ \varphi_p(x')\varphi_p^*(x)$$

$$= -i\,\theta(t'-t)\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\exp\left[\frac{i}{\hbar}\left(\mathbf{p}\cdot(\mathbf{r}'-\mathbf{r}) - \frac{p^2}{2m}(t'-t)\right)\right]$$

$$\parallel$$

$$-\frac{t'-t}{2m}\left(p^2 - \frac{2m}{t'-t}\,\mathbf{p}\cdot(\mathbf{r}'-\mathbf{r})\right)$$

$$\parallel$$

$$-\frac{t'-t}{2m}\left(\mathbf{p} - \frac{m(\mathbf{r}'-\mathbf{r})}{t'-t}\right)^2 + \frac{m(\mathbf{r}'-\mathbf{r})^2}{2(t'-t)}$$

$$= -i\,\theta(t'-t)\cdot\frac{1}{(2\pi\hbar)^3}\left(\frac{\pi\cdot 2m\hbar}{+i(t'-t)}\right)^{\frac{3}{2}} e^{\frac{im(\mathbf{r}'-\mathbf{r})^2}{2\hbar(t'-t)}}$$

$$= -i\,\theta(t'-t)\left(\frac{m}{2\pi i\hbar(t'-t)}\right)^{\frac{3}{2}}\exp\left(\frac{im(\mathbf{r}'-\mathbf{r})^2}{2\hbar(t'-t)}\right)$$

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GRADUATE SCHOOL OF SCIENCE
KYOTO UNIVERSITY

$$e^{-\frac{i}{\hbar}(H_0+V)\Delta t} \sim 1-\frac{i}{\hbar}(H_0+V)\Delta t+\frac{1}{2}\left(-\frac{i}{\hbar}\right)^2 \times (H_0+V)^2(\Delta t)^2 + \cdots$$

$$\sim H_0^2(\Delta t)^2 + \cancel{2H_0V(\Delta t)^2} \sim 0$$

• 摂動展開

時刻 t_1 と $t_1+\Delta t_1$ の間に ポテンシャル $V(x_1)=V(\mathbf{r}_1,t_1)$ が加わったとする。

$$\left(i\hbar\frac{\partial}{\partial t_1}-H_0\right)\psi(x_1) = V(x_1)\psi(x_1)$$

$$\to \psi(\mathbf{r}_1,t_1+\Delta t_1) = \exp\left[-\frac{i}{\hbar}\left(H_0+V(x_1)\right)\Delta t_1\right]\varphi(\mathbf{r}_1,t_1)$$

$$\sim \exp\left(-\frac{i}{\hbar}H_0\Delta t_1\right)\varphi(x_1) - \frac{i}{\hbar}V(x_1)\Delta t_1\varphi(x_1)$$

$$= \varphi(\mathbf{r}_1,t_1+\Delta t_1) - \frac{i}{\hbar}V(\mathbf{r}_1,t_1)\Delta t_1\varphi(\mathbf{r}_1,t_1)$$

$$\Delta\psi(x_1) \equiv \psi(\mathbf{r}_1,t_1+\Delta t) - \varphi(\mathbf{r}_1,t_1+\Delta t_1)$$

$$= -\frac{i}{\hbar}V(\mathbf{r}_1,t_1)\Delta t_1\varphi(\mathbf{r}_1,t_1) = -\frac{i}{\hbar}V(x_1)\varphi(x_1)\times\Delta t_1$$

$\Delta\psi(x_1)$ の時間発展：

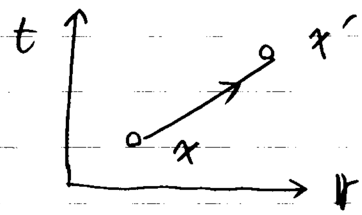
$$\theta(t'-t_1)\Delta\psi(x') = i\int d\mathbf{r}_1\, G_0(x',x_1)\Delta\psi(x_1)$$

$$= \frac{1}{\hbar}\int d\mathbf{r}_1\, G_0(x',x_1)V(x_1)\varphi(x_1)\Delta t_1$$

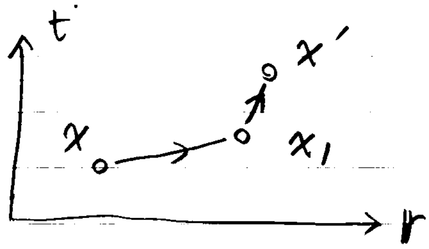
$$\theta(t'-t)\psi(x') = \theta(t'-t)\varphi(x') + \theta(t'-t_1)\theta(t_1-t)\Delta\psi(x')$$

$$= i\int d\mathbf{r}\left[G_0(x',x)+\frac{\Delta t_1}{\hbar}\int d\mathbf{r}_1\, G_0(x',x_1)V(x_1)G_0(x_1,x)\right]\times\varphi(x)$$

$$\boxed{G(x',x) = G_0(x',x) + \frac{\Delta t_1}{\hbar}\int d\mathbf{r}_1\, G_0(x',x_1)V(x_1)G_0(x_1,x)}$$



第1項



第2項

t_1の積分 ＋高次項

$$G(x',x) = G_0(x',x) + \frac{1}{\hbar}\int d\mathbf{r}_1 dt_1\, G_0(x',x_1)V(x_1)G_0(x_1,x)$$

$$+\left(\frac{1}{\hbar}\right)^2 \int d\mathbf{r}_1 dt_1 \int d\mathbf{r}_2 dt_2\, G_0(x',x_2)V(x_2)G_0(x_2,x_1)$$

$$\times V(x_1)G_0(x_1,x)$$

$$+ \cdots$$

$$= G_0(x',x) + \frac{1}{\hbar}\int d\mathbf{r}_1 dt_1\, G_0(x',x_1)V(x_1)$$

$$\times \left[G_0(x_1,x) + \frac{1}{\hbar}\int d\mathbf{r}_2 dt_2\, G_0(x_1,x_2)V(x_2)G_0(x_2,x) + \cdots \right]$$

$$= G_0(x',x) + \frac{1}{\hbar}\int d\mathbf{r}_1 dt_1\, G_0(x',x_1)V(x_1)G(x_1,x)$$

（時間に依存する）リップマン－シュウィンガー方程式

・S行列

始状態 $\varphi_i(x) \longrightarrow$ 終状態 $\varphi_f(x)$

$$\psi_i(x') = \lim_{t\to-\infty} i\int d\mathbf{r}\, G(x',x)\,\varphi_i(x)$$

$$= \lim_{t\to-\infty} i\int d\mathbf{r}\left[G_0(x',x) + \frac{i}{\hbar}\int d\mathbf{r}_1 dt_1\, G_0(x',x_1)V(x_1) \times \underline{G(x_1,x)}\right]\underline{\varphi_i(x)}$$

$$= \varphi_i(x') + \frac{i}{\hbar}\int d\mathbf{r}_1 dt_1\, G_0(x',x_1)V(x_1)\,\psi_i(x_1)$$

◀ S行列 $S_{fi} = \langle \varphi_f | \psi_i \rangle$

$$= \lim_{t'\to\infty}\int d\mathbf{r}'\, \varphi_f^*(x')\,\psi_i(x')$$

$$= \lim_{t'\to\infty}\int d\mathbf{r}'\, \varphi_f^*(x')\left[\varphi_i(x') + \frac{i}{\hbar}\int d\mathbf{r}_1 dt_1\, G_0(x',x_1)V(x_1)\psi_i(x_1)\right]$$

$$= \delta_{f,i} + \lim_{t'\to+\infty}\frac{i}{\hbar}\int d\mathbf{r}'\int d\mathbf{r}_1 dt_1\, \varphi_f^*(x') \times G_0(x',x_1)V(x_1)\,\psi_i(x_1)$$

(note) $\langle \psi_i | \psi_j \rangle = \langle \varphi_i | \varphi_j \rangle = \delta_{i,j}$

$$S_{fi} = \langle \varphi_f | \psi_i \rangle \rightarrow \sum_f \varphi_f(x)\, S_{fi} = \lim_{t\to\infty}\psi_i(x)$$

$$\delta_{i,j} = \lim_{t\to\infty}\langle \psi_i | \psi_j \rangle = \sum_n \lim_{t\to\infty}\langle \psi_i | \varphi_n \rangle\langle \varphi_n | \psi_j \rangle$$

$$= \sum_n S_{ni}^* S_{nj}$$ （S行列のユニタリー性）

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(補足)

時間に依存する摂動量

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = (H_0 + V)|\psi(t)\rangle$$

相互作用表示： $|\psi(t)\rangle_I = e^{iH_0 t/\hbar} |\psi(t)\rangle$

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle_I = \cancel{-H_0 |\psi(t)\rangle_I} + e^{iH_0 t/\hbar} \cdot \underbrace{i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle}_{= (\cancel{H_0} + V)|\psi(t)\rangle}$$

$$= \underbrace{e^{iH_0 t/\hbar} V e^{-iH_0 t/\hbar}}_{= V_I(t)} \underbrace{e^{iH_0 t/\hbar} |\psi(t)\rangle}_{= |\psi(t)\rangle_I}$$

$$|\psi(t)\rangle_I = |\psi(t_0)\rangle_I - \frac{i}{\hbar}\int_{t_0}^{t} dt' \, V_I(t') |\psi(t')\rangle_I$$

$$= |\psi(t_0)\rangle_I - \frac{i}{\hbar}\int_{t_0}^{t} dt' \, V_I(t') |\psi(t_0)\rangle_I$$

$$+ \left(-\frac{i}{\hbar}\right)^2 \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' \, V_I(t') V_I(t'') |\psi(t_0)\rangle_I$$

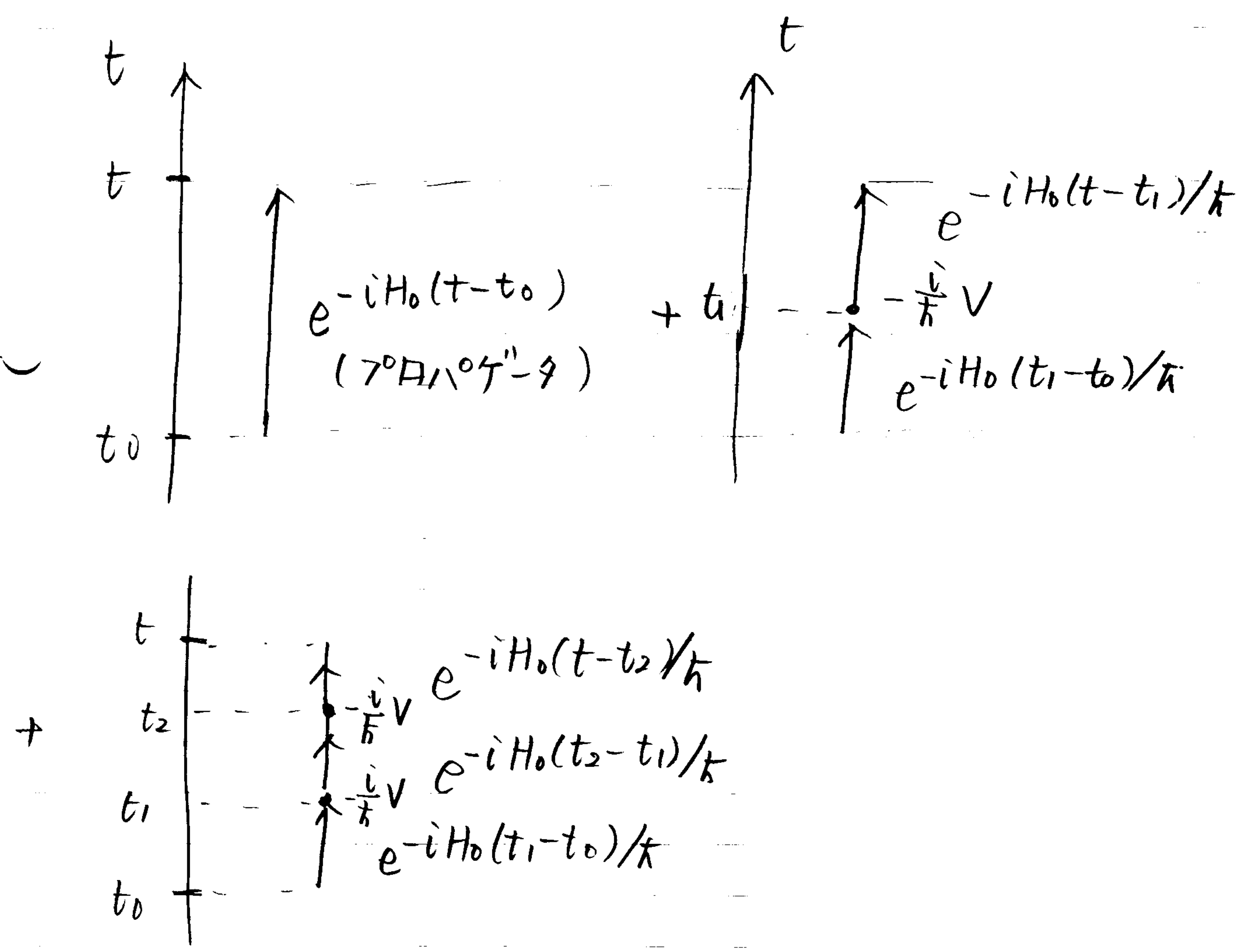
$$+ \cdots$$

$$= U_I(t, t_0) |\psi(t_0)\rangle_I$$

S行列： $S_{fi} = \langle \psi_f | U_I(\infty, -\infty) | \psi_i \rangle$

$$
\begin{aligned}
U(t,t_0) &= e^{-iH_0 t/\hbar}\, U_I(t,t_0)\, e^{+iH_0 t_0/\hbar} \\
&= e^{-iH_0(t-t_0)/\hbar} - \frac{i}{\hbar}\int_{t_0}^{t} dt_1\, e^{-iH_0(t-t_1)/\hbar}\, V\, e^{-iH_0(t_1-t_0)/\hbar} \\
&\quad + \left(-\frac{i}{\hbar}\right)^2 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_2} dt_2\, e^{-iH_0(t-t_1)/\hbar}\, V\, e^{-iH_0(t_1-t_2)/\hbar} \\
&\qquad \times V\, e^{-iH_0(t_2-t_0)/\hbar}
\end{aligned}
$$

$+ \cdots$



§. 相対論的プロパゲータ

相対論(ディラック方程式) $\longrightarrow$ $E>0$ の解と $E<0$ の解

$E>0$ $\updownarrow$ 電子

$E<0$ $\updownarrow$ 陽電子

シュテュッケルベルク－ファインマンの解釈：
時間に逆行して伝搬する $E<0$ の電子
$\longleftrightarrow$ 時間に順行して伝搬する $E>0$ の陽電子

▣ プロパゲータ

時間発展 → ディラック方程式に従う

$$\left[i\hbar \gamma^\mu \left(\frac{\partial}{\partial x^\mu} + \frac{ig}{\hbar} A_\mu \right) - mc \right] \psi = 0$$

$$\downarrow$$

$$\sum_{\gamma=1}^{4} \left[\gamma_\mu \left\{ i\hbar \frac{\partial}{\partial x'_\mu} - g A^\mu(x') \right\} - mc \right]_{\alpha\gamma} \left(S_F(x', x) \right)_{\gamma\beta} = \hbar \delta_{\alpha,\beta} \delta^{(4)}(x - x')$$

$$\delta^{(4)}(x-x') = \delta(x_0 - x_0')\,\delta(\mathbf{r} - \mathbf{r}')$$

※スピノルの添字を省略すると

$$\left[\gamma_\mu \left\{ i\hbar \frac{\partial}{\partial x'_\mu} - g A^\mu(x') \right\} - mc \right] S_F(x', x) = \hbar \delta^{(4)}(x - x')$$

・自由粒子の場合

$$\left(i\hbar\gamma_\mu \frac{\partial}{\partial x'_\mu} - mc\right) S_F^{(0)}(x', x) = \hbar\delta^{(4)}(x - x')$$

これをフーリエ変換を使って解く。

$$S_F^{(0)}(x', x) = \int \frac{d^4p}{(2\pi\hbar)^4} e^{-\frac{i}{\hbar}p(x'-x)} \tilde{S}_F^{(0)}(p)$$

とする。 $[\, p(x'-x) = p_0 \cdot c(t'-t) - \mathbb{p}\cdot(\mathbb{r}'-\mathbb{r}) \,]$

↓

$$\underbrace{\hbar\delta^{(4)}(x - x')} = \left(i\hbar\gamma_\mu \frac{\partial}{\partial x'_\mu} - mc\right) S_F^{(0)}(x', x)$$

$$= \int \frac{d^4p}{(2\pi\hbar)^4} (\underbrace{p^\mu\gamma_\mu}_{\equiv\, \not{p}} - mc)\, e^{-\frac{i}{\hbar}p(x'-x)} \tilde{S}_F^{(0)}(p)$$

$$= \hbar \int \frac{d^4p}{(2\pi\hbar)^4} e^{-\frac{i}{\hbar}p(x'-x)}$$

↓

$$\tilde{S}_F^{(0)}(p) = \frac{\hbar}{\not{p} - mc} = \hbar \frac{\not{p} + mc}{p^2 - m^2c^2} \quad (p^2 \neq m^2c^2)$$

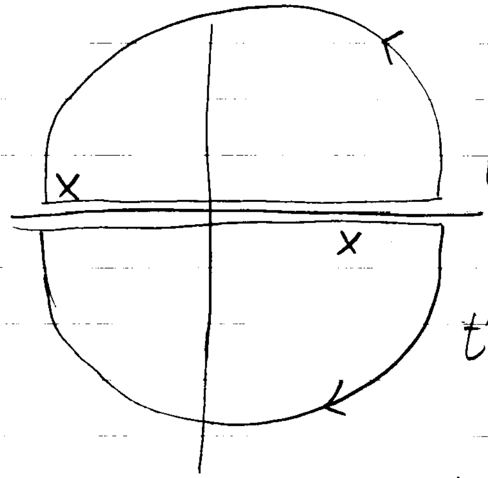
↑

"off-shell の状態"

特異点 $p^2 = m^2c^2$

$$\rightarrow p_0 c = \pm\sqrt{\mathbb{p}^2c^2 + m^2c^4} = \pm E$$

時間に順行する粒子 $\longrightarrow E>0$
逆行 $E<0$



$$S_F^{(0)}(x',x)\Big|_{t'>t}=\int\frac{d\mathbb{P}}{(2\pi\hbar)^3}e^{\frac{i}{\hbar}\mathbb{P}\cdot(\mathbb{r}'-\mathbb{r})}$$

$$\times\int\frac{dp_0}{2\pi\hbar}e^{-\frac{i}{\hbar}p_0c(t'-t)}\cdot\hbar\frac{\not{p}+mc}{\underset{(p_0-E/c)(p_0+E/c)}{\underset{\parallel}{(p^2-m^2c^2)}}}$$

$$=-\int\frac{d\mathbb{P}}{(2\pi\hbar)^3}e^{\frac{i}{\hbar}\mathbb{P}\cdot(\mathbb{r}'-\mathbb{r})}\cdot\cancel{2\pi}i\frac{1}{\cancel{2\pi\hbar}}\cdot\cancel{\hbar}\frac{E\gamma_0-c\mathbb{P}\cdot\vec{\gamma}+mc^2}{2E}$$

$$\times e^{-\frac{i}{\hbar}E(t'-t)}$$

$$=-i\int\frac{d\mathbb{P}}{(2\pi\hbar)^3}e^{\frac{i}{\hbar}[\mathbb{P}\cdot(\mathbb{r}'-\mathbb{r})-E(t'-t)]}\frac{E\gamma_0-c\mathbb{P}\cdot\vec{\gamma}+mc^2}{2E}$$

$$S_F^{(0)}(x',x)\Big|_{t'<t}=\int\frac{d\mathbb{P}}{(2\pi\hbar)^3}e^{\frac{i}{\hbar}\mathbb{P}\cdot(\mathbb{r}'-\mathbb{r})}2\pi i\cdot\frac{1}{2\pi\hbar}\hbar\cdot\frac{-E\gamma_0-c\mathbb{P}\cdot\vec{\gamma}+mc^2}{-2E}$$

$$\times e^{\frac{i}{\hbar}E(t'-t)}$$

$$=-i\int\frac{d\mathbb{P}}{(2\pi\hbar)^3}e^{\frac{i}{\hbar}(-\mathbb{P}\cdot(\mathbb{r}'-\mathbb{r})+E(t'-t))}\frac{-E\gamma_0+c\mathbb{P}\cdot\vec{\gamma}+mc^2}{2E}$$

$\uparrow$

$\mathbb{P}\to-\mathbb{P}$.