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§. Klein-Gordon 方程式

(復習) $E = \frac{\mathbb{P}^2}{2m} + V(\mathbb{r})$

$E \to i\hbar \frac{\partial}{\partial t}$, $\mathbb{P} \to \frac{\hbar}{i}\nabla$

$\downarrow$ $i\hbar\frac{\partial}{\partial t}\psi = \left(-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbb{r})\right)\psi$

相対論では： $E^2 = m^2c^4 + \mathbb{P}^2c^2$

$\downarrow$ $-\hbar^2\frac{\partial^2}{\partial t^2}\psi = (m^2c^4 - \hbar^2c^2\nabla^2)\psi$

or $\left(-\frac{\partial^2}{\partial (ct)^2} + \nabla^2\right)\psi = \frac{m^2c^2}{\hbar^2}\psi$

$\Box \equiv \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$ (ダランベルシアン)

$\downarrow$

$$\boxed{\left(\Box + \frac{m^2c^2}{\hbar^2}\right)\phi = 0}$$ クライン・ゴルドン方程式

(note) $x^\mu = (ct, \mathbb{r})$
$x_\mu = (ct, -\mathbb{r})$

$p^\mu = i\hbar\frac{\partial}{\partial x_\mu} = i\hbar\left(\frac{1}{c}\frac{\partial}{\partial t}, -\nabla\right) \equiv i\hbar\partial^\mu$

$p_\mu = i\hbar\frac{\partial}{\partial x^\mu} = i\hbar\left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right) \equiv i\hbar\partial_\mu$

$\downarrow$

$\Box = \partial^\mu\partial_\mu$

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・Klein-Gordon方程式の問題点：確率解釈

$$\begin{cases} \left(\Box + \left(\frac{mc}{\hbar}\right)^2\right)\phi = 0 \\ \left(\Box + \left(\frac{mc}{\hbar}\right)^2\right)\phi^* = 0 \end{cases}$$

↓

$$\phi^*\left(\Box + \cancel{\left(\frac{mc}{\hbar}\right)^2}\right)\phi - \phi\left(\Box + \cancel{\left(\frac{mc}{\hbar}\right)^2}\right)\phi^* = 0$$

$$\rightarrow \quad \partial^\mu \underbrace{\left(\phi^* \partial_\mu \phi - \phi\, \partial_\mu \phi^*\right)}_{\| \\ \frac{2m}{i\hbar} j_\mu} = 0$$

$\frac{2m}{i\hbar} j_\mu$ とすると $\partial^\mu j_\mu = 0$ であるが、

$$\rho \equiv \frac{1}{c} j_0 = \frac{i\hbar}{2mc^2}\left(\phi^* \frac{\partial}{\partial t}\phi - \phi \frac{\partial}{\partial t}\phi^*\right)$$

は負になる場合がある。→ 確率密度とは解釈できない。

(note) 非相対論

$$\rho = |\psi|^2, \quad \boldsymbol{j} = \frac{\hbar}{2im}\left(\psi^* \nabla \psi - \psi \nabla \psi^*\right)$$

として

$$\frac{\partial}{\partial t}\rho + \nabla \cdot \boldsymbol{j} = 0$$

$\rho = |\psi|^2$ は必ず正.

§. Dirac 方程式

KG 方程式の困難：t に関して 2階

→ t に関して 1階の方程式を作りたい。

→ 相対論的共変性（ローレンツ変換に対する共変性）から空間に対しても 1階

∴ $$\left(\frac{1}{c}\frac{\partial}{\partial t}+\vec{\alpha}\cdot\vec{\nabla}+i\beta\frac{mc}{\hbar}\right)\psi(x)=0$$ を仮定。

（$\vec{\alpha}, \beta$ は x に依らない）

この方程式の解が KG 方程式の解になっていれば $E^2=m^2c^4+p^2c^2$ が満たされることになる。

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2+\left(\frac{mc}{\hbar}\right)^2\right)\psi=0$$

∴ $$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2+\left(\frac{mc}{\hbar^2}\right)^2\right)=\left(\frac{1}{c}\frac{\partial}{\partial t}-\vec{\alpha}\cdot\vec{\nabla}-i\beta\frac{mc}{\hbar}\right)\times\left(\frac{1}{c}\frac{\partial}{\partial t}+\vec{\alpha}\cdot\vec{\nabla}+i\beta\frac{mc}{\hbar}\right)$$

となっていればよい。

$$\left(\frac{1}{c}\partial_t-\alpha_i\partial_i-i\beta\frac{mc}{\hbar}\right)\left(\frac{1}{c}\partial_t+\alpha_j\partial_j+i\beta\frac{mc}{\hbar}\right)$$

$$=\frac{1}{c^2}(\partial_t)^2\underline{-\alpha_i\alpha_j\partial_i\partial_j}\underline{-i\alpha_i\beta\,\partial_i\frac{mc}{\hbar}-i\beta\alpha_j\frac{mc}{\hbar}\partial_j}\underline{+\beta^2\left(\frac{mc}{\hbar}\right)^2}$$

∴ $$\boxed{\beta^2=1,\quad \alpha_i\alpha_j+\alpha_j\alpha_i=2\delta_{i,j},\quad \alpha_i\beta+\beta\alpha_i=0}$$

→ α_i, β は行列。波動関数 ψ も ベクトル（多成分）

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$(\alpha_i)^2 = 1, \quad \beta^2 = 1$ → それぞれの行列の固有値は±1.

(note)

$$\alpha_i\beta + \beta\alpha_i = 0$$

$$\rightarrow \quad \underbrace{\alpha_i^2\beta}_{=\beta} + \alpha_i\beta\alpha_i = 0$$

$$\beta\alpha_i\beta + \underbrace{\beta^2\alpha_i}_{=\alpha_i} = 0$$

$$\Downarrow \quad Tr(\alpha_i) = -Tr(\beta\alpha_i\beta) = -Tr(\beta^2\alpha_i) = -Tr(\alpha_i)$$

$$\Downarrow \quad Tr(\alpha_i) = 0$$

同様に $Tr(\beta) = 0$

$$\beta = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

α_i, βの固有値は±1 なので 成分の数は偶数に限る

2×2 エルミート行列で反可換で独立な行列

→ パウリ行列 σ_i の3つだけ

→ いまは α_i, β の4つ必要なので
この可能性は排除.

→ 4×4 行列が最小元の行列.

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$$\beta^2 = 1,\quad \alpha_i\alpha_j + \alpha_j\alpha_i = 2\delta_{i,j},\quad \alpha_i\beta + \beta\alpha_i = 0$$

を満たすものの例として

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix},\quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

がある（ディラック表示）。

(note)
$$\alpha_i\beta + \beta\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} = 0$$

$$\alpha_i\alpha_j = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} = \begin{pmatrix} \sigma_i\sigma_j & 0 \\ 0 & \sigma_i\sigma_j \end{pmatrix}$$

$$\alpha_i\alpha_j + \alpha_j\alpha_i = 2\delta_{i,j}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \leftarrow \quad \sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}$$

これより

$$\boxed{i\hbar\frac{\partial}{\partial t}\psi = \left(-i\hbar c\vec{\alpha}\cdot\vec{\nabla} + \beta mc^2\right)\psi}$$

Dirac 方程式

$$= H\psi$$

$$H = -i\hbar c\vec{\alpha}\cdot\vec{\nabla} + \beta mc^2 = c\vec{\alpha}\cdot\vec{p} + \beta mc^2$$

確率解釈：

$$\begin{cases} i\hbar \dfrac{\partial \psi}{\partial t} = (-i\hbar c \vec{\alpha}\cdot\vec{\nabla} + \beta mc^2)\psi \\ -i\hbar \psi^\dagger \overleftarrow{\partial_t} = \psi^\dagger (i\hbar c\vec{\alpha}\cdot\overleftarrow{\nabla} + \beta mc^2) \end{cases}$$

$\downarrow$

$$i\hbar \psi^\dagger \frac{\partial \psi}{\partial t} + i\hbar \frac{\partial \psi^\dagger}{\partial t}\psi = -i\hbar c\, \psi^\dagger \alpha_i \partial_i \psi - i\hbar c\, (\partial_i \psi^\dagger)\alpha_i \psi$$

$\Updownarrow$

$$i\hbar \frac{\partial}{\partial t}(\psi^\dagger \psi) + i\hbar c\, \partial_i (\psi^\dagger \alpha_i \psi) = 0$$

$\Updownarrow$

$$\rho = \psi^\dagger\psi = |\psi|^2,\quad \boldsymbol{j} = c\,\psi^\dagger \vec{\alpha}\psi$$

として $\dfrac{\partial \rho}{\partial t} + \nabla\cdot\boldsymbol{j} = 0$ （連続の方程式）

(note) $j^\mu = (c\rho, \boldsymbol{j}) = (c\psi^\dagger\psi,\ c\psi^\dagger\vec{\alpha}\psi)$

$\longrightarrow \partial_\mu j^\mu = 0$

・電磁場中でのディラック方程式

$$\begin{cases} i\hbar \dfrac{\partial}{\partial t} \to i\hbar\dfrac{\partial}{\partial t} - q\varphi = i\hbar\left(\dfrac{\partial}{\partial t} + \dfrac{i}{\hbar}q\varphi\right) \\ \dfrac{\hbar}{i}\nabla \to \dfrac{\hbar}{i}\nabla - qA = \dfrac{\hbar}{i}\left(\nabla - \dfrac{iq}{\hbar}A\right) \end{cases}$$

$\downarrow$

$$i\hbar\left(\frac{\partial}{\partial t} + \frac{i}{\hbar}q\varphi\right)\psi = \left[-i\hbar c\,\vec{\alpha}\cdot\left(\nabla - \frac{iq}{\hbar}A\right) + \beta mc^2\right]\psi$$

• γ 行列

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \vec{\alpha}\cdot\vec{\nabla} + i\beta\frac{mc}{\hbar}\right)\psi = 0$$

$$\rightarrow \left(i\hbar\,\beta\frac{\partial}{\partial(ct)} + i\hbar\,\beta\,\vec{\alpha}\cdot\vec{\nabla} - \underbrace{\beta^2}_{=1} mc\right)\psi = 0$$

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right)$$

$$\gamma^0 = \beta$$

$$\gamma^i = \beta\alpha_i$$

$$\downarrow \quad \boxed{(i\hbar\,\gamma^\mu\partial_\mu - mc)\psi = 0}$$

(note) $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\,1$

$$g^{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

(note) $(\gamma^0)^\dagger = \beta^\dagger = \gamma^0$

$$(\gamma^i)^\dagger = \alpha_i^\dagger\beta^\dagger = \alpha_i\beta = -\beta\alpha_i = -\gamma^i$$

↑
反エルミート行列

ディラック表示で

$$\gamma^0 = \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

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共変微分:

$$i\hbar\frac{\partial}{\partial t} \to i\hbar\left(\frac{\partial}{\partial t} + \frac{i}{\hbar} q \varphi\right)$$

$$-i\hbar\nabla \to -i\hbar\left(\nabla - \frac{iq}{\hbar} A\right)$$

$\Downarrow$

$$i\hbar\frac{\partial}{\partial x^\mu} \to i\hbar\frac{\partial}{\partial x^\mu} - qA_\mu = i\hbar\left(\frac{\partial}{\partial x^\mu} + \frac{iq}{\hbar}A_\mu\right)$$

$$A_\mu = \left(\frac{\varphi}{c}, -A\right)$$

$$\frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial (ct)}, \nabla\right)$$

$\Downarrow$

$$\left[i\hbar\gamma^\mu \underbrace{\left(\frac{\partial}{\partial x^\mu} + \frac{iq}{\hbar}A_\mu\right)}_{\equiv D_\mu} - mc\right]\psi = 0$$

§. カレントのローレンツ変換性とディラック共役

$$\rho = \psi^\dagger \psi, \quad \vec{j} = c\,\psi^\dagger \vec{\alpha}\, \psi$$

$$\rightarrow \quad j^\mu = (c\rho, \vec{j})$$

$$\partial_\mu j^\mu = 0$$

ローレンツ変換 : $x'^\mu = \Lambda^\mu{}_\nu x^\nu$

$$(g_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = g_{\alpha\beta}, \ \Lambda^0{}_0 \geq 1, \ \det \Lambda = 1 \ )$$

2つの慣性系で同じ形の方程式が成り立つはず。

$$\begin{cases} \left(i\hbar \gamma^\mu \dfrac{\partial}{\partial x^\mu} - mc\right)\psi(x) = 0 \\ \left(i\hbar \gamma'^\mu \dfrac{\partial}{\partial x'^\mu} - mc\right)\tilde{\psi}(x') = 0 \end{cases}$$

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$
$$\{\gamma'^\mu, \gamma'^\nu\} = 2g^{\mu\nu}$$

$$\downarrow$$

γ^μ と $\gamma^{\mu\prime}$ はユニタリー変換でつながっている。

$$\gamma'^\mu = U^\dagger \gamma^\mu U \quad \rightarrow \quad U\gamma'^\mu = \gamma^\mu U$$

$$\downarrow \quad \left(i\hbar\, \underbrace{U\gamma'^\mu \frac{\partial}{\partial x'^\mu}}_{} - Umc\right)\tilde{\psi}(x') = 0$$

$$\| $$

$$\gamma^\mu U \frac{\partial}{\partial x'^\mu} = \gamma^\mu \frac{\partial}{\partial x'^\mu} U$$

$U\tilde{\Psi}(x') \equiv \psi'(x')$ とすると

$$\left(i\hbar\gamma^\mu \frac{\partial}{\partial x'^\mu} - mc\right)\psi'(x') = 0$$

$\psi'(x') = S(\Lambda)\psi(x)$ を仮定.

$$\rightarrow \psi(x) = S^{-1}(\Lambda)\psi'(x') = S^{-1}(\Lambda)\psi'(\Lambda x)$$

(note) $x^\mu = (\Lambda^{-1})^\mu{}_\nu x'^\nu$

$$\rightarrow \psi(x) = S(\Lambda^{-1})\psi'(x') = S(\Lambda^{-1})\psi'(\Lambda x)$$

(note) $\left(i\hbar\gamma^\mu \frac{\partial}{\partial x^\mu} - mc\right)\psi(x) = 0$

$$\rightarrow \left[i\hbar S(\Lambda)\gamma^\mu S^{-1}(\Lambda) S(\Lambda)\underbrace{\frac{\partial}{\partial x^\mu}}_{= \frac{\partial x'^\nu}{\partial x^\mu}\frac{\partial}{\partial x'^\nu} = \Lambda^\nu{}_\mu \frac{\partial}{\partial x'^\nu}} - mc\, S(\Lambda)\right]\psi(x) = 0$$

$\Downarrow$

$$\left[i\hbar S(\Lambda)\gamma^\mu S^{-1}(\Lambda)\Lambda^\nu{}_\mu \frac{\partial}{\partial x'^\nu} - mc\right]\psi'(x') = 0$$

$$\leftrightarrow \left(i\hbar\gamma^\mu \frac{\partial}{\partial x'^\mu} - mc\right)\psi'(x') = 0$$

$\downarrow$

$$S(\Lambda)\gamma^\mu \underbrace{S^{-1}(\Lambda)\Lambda^\nu{}_\mu}_{= \Lambda^\nu{}_\mu S^{-1}(\Lambda)} = \gamma^\nu$$

$\Downarrow$

$$\Lambda^\nu{}_\mu \gamma^\mu = S^{-1}(\Lambda)\gamma^\nu S(\Lambda)$$

無限小ローレンツ変換： $\Lambda^\nu{}_\mu = \delta^\nu_\mu + \Delta\omega^\nu{}_\mu$

$$\Delta\omega^{\mu\nu} = g^{\mu\lambda}\Delta\omega^\nu{}_\lambda \underset{\uparrow}{=} -\Delta\omega^{\nu\mu}$$

$$\begin{aligned}
g_{\alpha\beta} &= g_{\mu\nu}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta \\
&= g_{\mu\nu}(\delta^\mu_\alpha + \Delta\omega^\mu{}_\alpha)(\delta^\nu_\beta + \Delta\omega^\nu{}_\beta) \\
&= g_{\alpha\beta} + (g_{\mu\beta}\Delta\omega^\mu{}_\alpha + g_{\alpha\nu}\Delta\omega^\nu{}_\beta) \\
&= g_{\alpha\beta} + \underbrace{(\Delta\omega_{\beta\alpha} + \Delta\omega_{\alpha\beta})}_{=0}
\end{aligned}$$

$$\left.\begin{aligned}
S(\Lambda) &= 1 - \frac{i}{4}\Delta\omega^{\mu\nu}\sigma_{\mu\nu} \\
S^{-1}(\Lambda) &= 1 + \frac{i}{4}\Delta\omega^{\mu\nu}\sigma_{\mu\nu}
\end{aligned}\right\} \text{とおける.}$$

$$\Lambda^\nu{}_\mu\gamma^\mu = (\delta^\nu_\mu + \Delta\omega^\nu{}_\mu)\gamma^\mu = \gamma^\nu + \Delta\omega^\nu{}_\mu\gamma^\mu$$

$$\begin{aligned}
S^{-1}(\Lambda)\gamma^\nu S(\Lambda) &= \left(1 + \frac{i}{4}\Delta\omega^{\alpha\beta}\sigma_{\alpha\beta}\right)\gamma^\nu\left(1 - \frac{i}{4}\Delta\omega^{\alpha\beta}\sigma_{\alpha\beta}\right) \\
&\sim \gamma^\nu + \frac{i}{4}\Delta\omega^{\alpha\beta}\sigma_{\alpha\beta}\gamma^\nu - \frac{i}{4}\gamma^\nu\Delta\omega^{\alpha\beta}\sigma_{\alpha\beta} \\
&= \gamma^\nu + \frac{i}{4}\Delta\omega^{\alpha\beta}[\sigma_{\alpha\beta}, \gamma^\nu]
\end{aligned}$$

$$\Lambda^\nu{}_\mu\gamma^\mu = S^{-1}(\Lambda)\gamma^\nu S(\Lambda)$$

$$\longrightarrow \Delta\omega^\nu{}_\mu\gamma^\mu = \frac{i}{4}\Delta\omega^{\alpha\beta}[\sigma_{\alpha\beta}, \gamma^\nu]$$

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No.

この式は
$$\sigma_{\alpha\beta} = \frac{i}{2}[\gamma_\alpha, \gamma_\beta] = \frac{i}{2}(\gamma_\alpha\gamma_\beta - \underbrace{\gamma_\beta\gamma_\alpha}_{=\,2g_{\alpha\beta} - \gamma_\alpha\gamma_\beta})$$
$$= i(\gamma_\alpha\gamma_\beta - g_{\alpha\beta})$$
とおくと満たされる。

(note) $\frac{i}{4}\Delta\omega^{\alpha\beta}[\sigma_{\alpha\beta}, \gamma^\nu] = -\frac{1}{4}\Delta\omega^{\alpha\beta}[\gamma_\alpha\gamma_\beta, \gamma^\nu]$
$$= \cdots = \Delta\omega^\nu{}_\mu\gamma^\mu$$

(note) $x'^\mu = \Lambda^\mu{}_\nu x^\nu$
$$\Lambda^\mu{}_\nu = \lim_{N\to\infty}(\delta^\mu{}_{\mu_1} + \Delta\omega^\mu{}_{\mu_1})(\delta^{\mu_1}{}_{\mu_2} + \Delta\omega^{\mu_1}{}_{\mu_2}) \times \cdots (\delta^{\mu_{N-1}}{}_\nu + \Delta\omega^{\mu_{N-1}}{}_\nu)$$
$$= \lim_{N\to\infty}\left[\left(1 + \frac{\omega}{N}\right)^N\right]^\mu{}_\nu = (e^\omega)^\mu{}_\nu$$
$$(\omega^\mu{}_\nu = N\Delta\omega^\mu{}_\nu)$$

$$\psi'(x') = S(\Lambda)\psi(x)$$
$$S(\Lambda) = \lim_{N\to\infty}\left(1 - \frac{i}{4}\Delta\omega^{\mu\nu}\sigma_{\mu\nu}\right)^N$$
$$\sim e^{-\frac{i}{4}\omega^{\mu\nu}\sigma_{\mu\nu}}$$
$$S^{-1}(\Lambda) = e^{\frac{i}{4}\omega^{\mu\nu}\sigma_{\mu\nu}} = S(\Lambda^{-1})$$

ディラック共役： $\bar{\psi}(x) \equiv \psi^\dagger(x)\gamma_0$

$$\bar{\psi}'(x') = \psi'^\dagger(x')\gamma_0 = \psi^\dagger(x)S(\Lambda)^\dagger\gamma_0$$

$$= \underbrace{\psi^\dagger(x)\gamma_0}_{=\bar{\psi}(x)}\underbrace{\gamma_0 S(\Lambda)^\dagger\gamma_0}_{=\gamma_0 e^{\frac{i}{4}\omega^{\mu\nu}\sigma^\dagger_{\mu\nu}}\gamma_0}$$

$$\gamma_0 e^{\frac{i}{4}\omega^{\mu\nu}\sigma^\dagger_{\mu\nu}}\gamma_0 \quad \leftarrow \gamma_0^2 = 1$$

$$= e^{\frac{i}{4}\omega^{\mu\nu}\gamma_0\sigma^\dagger_{\mu\nu}\gamma_0} \quad \leftarrow \gamma_0\sigma_{0i}\gamma_0 = -\sigma_{0i}, \ \gamma_0\sigma_{ij}\gamma_0 = \sigma_{ij}$$

$$= e^{\frac{i}{4}\omega^{\mu\nu}\sigma_{\mu\nu}}$$

$$= S^{-1}(\Lambda)$$

より

$$\bar{\psi}'(x')\psi'(x') = \bar{\psi}(x)S^{-1}(\Lambda)S(\Lambda)\psi(x) = \bar{\psi}(x)\psi(x)$$

→ ローレンツ・スカラー

$$\bar{\psi}'(x')\gamma^\mu\psi'(x') = \bar{\psi}(x)\underline{S^{-1}(\Lambda)\gamma^\mu S(\Lambda)}\psi(x)$$

$$= \bar{\psi}(x)\Lambda^\mu{}_\nu\gamma^\nu\psi(x)$$

$$= \Lambda^\mu{}_\nu\bar{\psi}(x)\gamma^\nu\psi(x)$$

→ ローレンツ・ベクター

(note) $\alpha^i = \gamma^0 \gamma^i$

↓

$$\bar{\psi}\gamma^\mu\psi = (\psi^\dagger \gamma_0 \gamma^0 \psi, \psi^\dagger \gamma_0 \gamma^i \psi)$$
$$= (\psi^\dagger \psi, \psi^\dagger \alpha^i \psi)$$
$$= \frac{1}{c} j^\mu$$

(note) $\bar{\psi}\psi$ は空間反転(パリティ変換)に対しても不変

(note) 双一次形式

$\bar{\psi}\psi$: スカラー

$\bar{\psi}\gamma^\mu\psi$: ベクター

$\bar{\psi}\sigma^{\mu\nu}\psi$: 2階のテンソル

$\bar{\psi}\gamma_5\psi$: 擬スカラー ($\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$)

ローレンツ・スカラー, パリティ・マイナス

$\bar{\psi}\gamma_5\gamma^\mu\psi$: 擬ベクトル

空間反転に対し ベクトル($\bar{\psi}\gamma^\mu\psi$)と反対の変換性