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No.

§. Dirac方程式の平面波解

・静止した粒子の解 : $\mathbb{P}\psi = \frac{\hbar}{i}\nabla\psi = 0$

↳ $i\hbar\frac{\partial}{\partial t}\psi = \beta mc^2\psi = mc^2\begin{pmatrix}1&0\\0&-1\end{pmatrix}\psi$

線形独立な解

$$\psi^{(+)}_{\mathbb{P}=0,s} = e^{-imc^2t/\hbar}\begin{pmatrix}\chi_s\\0\end{pmatrix} \quad \to E = mc^2$$

$$\psi^{(-)}_{\mathbb{P}=0,s} = e^{+imc^2t/\hbar}\begin{pmatrix}0\\\chi_s\end{pmatrix} \quad \to E = -mc^2$$

$$\chi_\uparrow = \begin{pmatrix}1\\0\end{pmatrix},\quad \chi_\downarrow = \begin{pmatrix}0\\1\end{pmatrix}$$

スピノル
(スピン波動関数)

(note)
$$S_z \equiv \frac{\hbar}{2}\Sigma_3 = \frac{\hbar}{2}\begin{pmatrix}\sigma_3&0\\0&\sigma_3\end{pmatrix}$$

$$\begin{cases} S_z\psi^{(\pm)}_{\mathbb{P}=0,\uparrow} = +\frac{\hbar}{2}\psi^{(\pm)}_{\mathbb{P}=0,\uparrow} \\ S_z\psi^{(\pm)}_{\mathbb{P}=0,\downarrow} = -\frac{\hbar}{2}\psi^{(\pm)}_{\mathbb{P}=0,\downarrow} \end{cases}$$

・一般の平面波解

$$\psi = \begin{pmatrix}\psi_A\\\psi_B\end{pmatrix} = \begin{pmatrix}u_A(\mathbb{P})\\u_B(\mathbb{P})\end{pmatrix}\underbrace{e^{i\mathbb{P}\cdot\mathbb{X}/\hbar - iEt/\hbar}}_{\parallel}$$

$$e^{-iP_\mu x^\mu/\hbar}$$

$P_\mu = \left(\frac{E}{c}, -\mathbb{P}\right)$

$x^\mu = (ct, +\mathbb{X})$

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$$i\hbar\frac{\partial}{\partial t}\psi = \left(-i\hbar c\vec{\alpha}\cdot\vec{\nabla} + \beta mc^2\right)\psi$$

$$\downarrow$$

$$E\begin{pmatrix} u_A \\ u_B \end{pmatrix} = \begin{pmatrix} mc^2 & c\vec{\sigma}\cdot\vec{p} \\ c\vec{\sigma}\cdot\vec{p} & -mc^2 \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

$$\downarrow$$

$$\begin{cases} (E - mc^2)u_A = c\,\vec{\sigma}\cdot\vec{p}\,u_B \\ (E + mc^2)u_B = c\,\vec{\sigma}\cdot\vec{p}\,u_A \end{cases}$$

or

$$u_A = \frac{c}{E - mc^2}(\vec{\sigma}\cdot\vec{p})u_B$$

$$u_B = \frac{c}{E + mc^2}(\vec{\sigma}\cdot\vec{p})u_A$$

$E > 0$ に対し

$$u_A = \chi_S \rightarrow u_B = \frac{c}{E + mc^2}(\vec{\sigma}\cdot\vec{p})\chi_S$$

$$\downarrow \quad u^{(+)}(p,s) = N\begin{pmatrix} \chi_S \\ \frac{c}{E+mc^2}(\vec{\sigma}\cdot\vec{p})\chi_S \end{pmatrix}$$

$E < 0$ に対し

$$u_B = \chi_S \rightarrow u_A = \frac{c}{E - mc^2}(\vec{\sigma}\cdot\vec{p})\chi_S$$

$$\downarrow \quad u^{(-)}(p,s) = N\begin{pmatrix} \frac{c}{E-mc^2}(\vec{\sigma}\cdot\vec{p})\chi_S \\ \chi_S \end{pmatrix}$$

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・規格化

$$u^{(+)}(\mathbf{p},s)=\sqrt{\frac{E+mc^2}{2mc^2}}\begin{pmatrix}\chi_s\\ \dfrac{c}{E+mc^2}(\vec{\sigma}\cdot\vec{p})\chi_s\end{pmatrix}$$

ととると

$$\bar{u}^{(+)}(\mathbf{p},s)u^{(+)}(\mathbf{p},s)=\frac{E+mc^2}{2mc^2}\left(\chi_s^{\dagger}\quad\frac{c(\boldsymbol{\sigma}\cdot\mathbf{p})}{E+mc^2}\chi_s^{\dagger}\right)\underbrace{\begin{pmatrix}1&0\\0&-1\end{pmatrix}}_{=\gamma_0}\begin{pmatrix}\chi_s\\ \dfrac{c\boldsymbol{\sigma}\cdot\mathbf{p}}{E+mc^2}\chi_s\end{pmatrix}$$

$$\left(\chi_s\quad -\frac{c\boldsymbol{\sigma}\cdot\mathbf{p}}{E+mc^2}\chi_s\right)$$

$$=\frac{E+mc^2}{2mc^2}\left(1-\left(\frac{c}{E+mc^2}\right)^2\underbrace{(\boldsymbol{\sigma}\cdot\mathbf{p})^2}_{=p^2}\right)$$

$$=\frac{E+mc^2}{2mc^2}\cdot\frac{(E+mc^2)^2-c^2p^2}{(E+mc^2)^2}=\frac{\overbrace{E^2}^{\to p^2c^2+m^2c^4}+2mc^2E+m^2c^4-c^2p^2}{2mc^2(E+mc^2)}$$

$$=\frac{2mc^2(E+mc^2)}{2mc^2(E+mc^2)}=1.$$

同様に

$$u^{(-)}(\mathbf{p},s)=\sqrt{\frac{-E+mc^2}{2mc^2}}\begin{pmatrix}\dfrac{c\boldsymbol{\sigma}\cdot\mathbf{p}}{E-mc^2}\chi_s\\ \chi_s\end{pmatrix}$$ とすると

$$\bar{u}^{(-)}(\mathbf{p},s)u^{(-)}(\mathbf{p},s)=-1.$$

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$$u^{(+)}(\mathbf{p},s)=\sqrt{\frac{mc^2}{E}}\cdot\sqrt{\frac{E+mc^2}{2mc^2}}\begin{pmatrix}\chi_s\\ \frac{c}{E+mc^2}(\boldsymbol{\sigma}\cdot\mathbf{p})\chi_s\end{pmatrix}$$

$$=\sqrt{\frac{E+mc^2}{2E}}\begin{pmatrix}\chi_s\\ \frac{c}{E+mc^2}(\boldsymbol{\sigma}\cdot\mathbf{p})\chi_s\end{pmatrix}$$

ととると

$$u^{(+)\dagger}(\mathbf{p},s)\,u^{+}(\mathbf{p},s)=\frac{E+mc^2}{2E}\left(1+\left(\frac{c\boldsymbol{\sigma}\cdot\mathbf{p}}{E+mc^2}\right)^2\right)$$

$$=\frac{E+mc^2}{2E}\cdot\frac{E^2+2mc^2E+m^2c^4+p^2c^2}{(E+mc^2)^2}$$

$$=\frac{2m^2c^4+2p^2c^2+2mc^2E}{2E(E+mc^2)}=\frac{2E(E+mc^2)}{2E(E+mc^2)}=1,$$

◀

同様に

$$u^{(-)}(\mathbf{p},s)=\underbrace{\sqrt{\frac{mc^2}{E}}\cdot\sqrt{\frac{-E+mc^2}{2mc^2}}}_{\parallel}\begin{pmatrix}\frac{c\boldsymbol{\sigma}\cdot\mathbf{p}}{E-mc^2}\chi_s\\ \chi_s\end{pmatrix}$$

$$\sqrt{\frac{-E+mc^2}{2E}}$$

ととると

$$u^{(-)\dagger}(\mathbf{p},s)\,u^{(-)}(\mathbf{p},s)=\frac{-E+mc^2}{2E}\left(\frac{c^2p^2}{(E-mc^2)^2}+1\right)$$

$$=\frac{-E+mc^2}{2E}\cdot\frac{c^2p^2+E^2-2mc^2E+m^2c^4}{(E-mc^2)^2}=\frac{-E+mc^2}{2E}\cdot\frac{2E(E-mc^2)}{(E-mc^2)^2}$$

$$=-1.$$

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(note) 非相対論極限

$$E = \sqrt{m^2c^4 + p^2c^2} \sim mc^2$$

↓

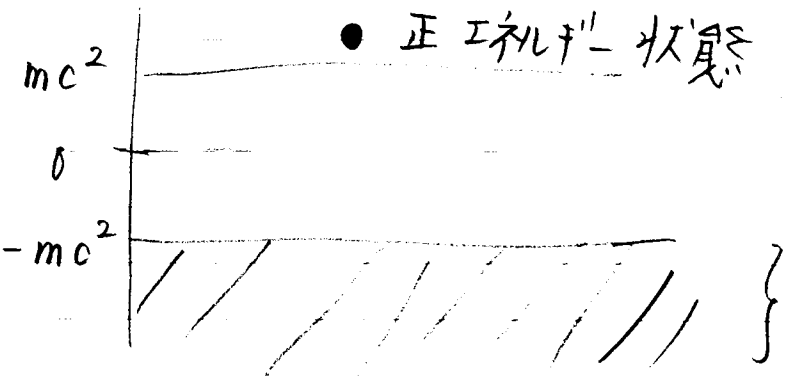
$$u^{(+)}(p,s) \sim N \begin{pmatrix} \chi_s \\ \frac{\vec{\sigma}\cdot\vec{p}}{2mc}\chi_s \end{pmatrix}$$

$$\sim N \begin{pmatrix} \chi_s \\ \frac{1}{2}\vec{\sigma}\cdot\frac{\vec{v}}{c}\chi_s \end{pmatrix} \begin{matrix} \} \text{large 成分} \\ \} \text{small 成分} \end{matrix}$$

空孔理論

負エネルギー解 ($E<0$) → 正エネルギー解が不安定

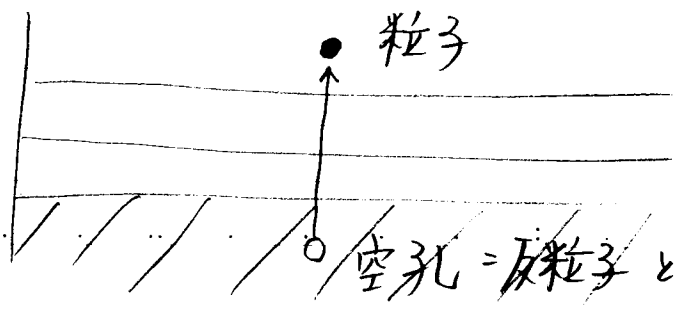
Dirac：負エネルギー解はすべて粒子が詰まっている



● 正エネルギー状態 → パウリ原理の為 負エネルギー状態に遷移することはない

} ディラックの海 = 状態には粒子がつまっている

外からエネルギーが与えられると



粒子・反粒子の対生成

※完全な理解には場の理論が必要

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共変微分： $i\hbar\frac{\partial}{\partial t} \to i\hbar\left(\frac{\partial}{\partial t} + \frac{i}{\hbar} q\varphi\right)$

$-i\hbar\nabla \to -i\hbar\left(\nabla - \frac{iq}{\hbar}A\right)$

$\downarrow$

$i\hbar\frac{\partial}{\partial x^\mu} \to i\hbar\frac{\partial}{\partial x^\mu} - qA_\mu = i\hbar\left(\frac{\partial}{\partial x^\mu} + \frac{iq}{\hbar}A_\mu\right)$

$A_\mu = \left(\frac{\varphi}{c}, -A\right)$

$\frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial(ct)}, \nabla\right)$

$\downarrow$

$\left[i\hbar\gamma^\mu\underbrace{\left(\frac{\partial}{\partial x^\mu} + \frac{iq}{\hbar}A_\mu\right)}_{\|\|\ D_\mu} - mc\right]\psi = 0$

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§. Dirac方程式の非相対論的極限

$$i\hbar\left(\frac{\partial}{\partial t}+\frac{i}{\hbar}q\varphi\right)\psi=\left[-i\hbar c\vec{\alpha}\cdot\left(\nabla-\frac{iq}{\hbar}A\right)+\beta mc^2\right]\psi$$

$A_\mu=\left(\frac{\varphi}{c},-A\right)$ が時間に依らないとして

$$\psi=\psi(x,t=0)\,e^{-iEt/\hbar}$$

$$=\begin{pmatrix}\psi_A\\ \psi_B\end{pmatrix}e^{-iEt/\hbar}$$

とおく。

$$(E-q\varphi)\begin{pmatrix}\psi_A\\ \psi_B\end{pmatrix}=\begin{pmatrix}mc^2 & c\vec{\sigma}\cdot(\mathbb{P}-qA)\\ c\vec{\sigma}\cdot(\mathbb{P}-qA) & -mc^2\end{pmatrix}\begin{pmatrix}\psi_A\\ \psi_B\end{pmatrix}$$

$$\psi_B=\frac{c}{E-q\varphi+mc^2}\,\vec{\sigma}\cdot(\mathbb{P}-qA)\,\psi_A$$

$$\left[\vec{\sigma}\cdot(P-qA)\right]\frac{c^2}{E-q\varphi+mc^2}\left[\vec{\sigma}\cdot(\mathbb{P}-qA)\right]\psi_A$$

$$=(E-q\varphi-mc^2)\,\psi_A$$

非相対論近似： $E\sim mc^2$, $|q\varphi|\ll mc^2$

$$E_{NR}\equiv E-mc^2$$

$$\frac{c^2}{E-q\varphi+mc^2}=\frac{c^2}{2mc^2+E_{NR}-q\varphi}=\frac{c^2}{2mc^2}\left(1-\frac{E_{NR}-q\varphi}{2mc^2}+\cdots\right)$$

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$$(\nabla \times A + A \times \nabla)_k = \varepsilon_{ijk}(\partial_i A_j + A_i \partial_j)$$
$$= \varepsilon_{ijk}(\partial_i A_j) + A_j \partial_i + A_i \partial_j)$$
$$= \varepsilon_{ijk}(\partial_i A_j)$$

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- leading order

$$\frac{c^2}{E - q\varphi + mc^2} \sim \frac{1}{2m}$$

$$\downarrow$$

$$\frac{1}{2m}\left[\vec{\sigma}\cdot(P - qA)\right]^2 \psi_A = (E_{NR} - q\varphi)\psi_A$$

$$\smile$$

(note) $(\vec{\sigma}\cdot\vec{A})(\vec{\sigma}\cdot\vec{B}) = \vec{A}\cdot\vec{B} + i\vec{\sigma}\cdot(\vec{A}\times\vec{B})$

$$\downarrow$$

$$\left[\vec{\sigma}\cdot(P - qA)\right]^2 = (P - qA)^2 + i\vec{\sigma}\cdot\underbrace{(P - qA)\times(P - qA)}$$

$$\parallel$$

$$\frac{\hbar}{i}\nabla \times (-qA)$$

$$-qA \times P$$

$$\parallel$$

$$-\frac{\hbar q}{i}(\nabla \times A)$$

$$\parallel$$

$$-\frac{\hbar q}{i} B$$

$$= (P - qA)^2 - \hbar q\,\vec{\sigma}\cdot\vec{B}$$

$$\updownarrow$$

$$\left[\frac{1}{2m}(P - qA)^2 - \frac{\hbar q}{2m}\vec{\sigma}\cdot\vec{B} + q\varphi\right]\psi_A = E_{NR}\psi_A$$

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・the next order （簡単のため $A=0$ とする）

$$(\vec{\sigma}\cdot\vec{p})\frac{1}{2m}\left(1-\frac{E_{NR}-q\varphi}{2mc^2}\right)(\vec{\sigma}\cdot\vec{p})\psi_A=(E_{NR}-q\varphi)\psi_A$$

・ψ_A のみでは規格化されていない

$\leftarrow \int dr\,(\psi_A^+\psi_A+\psi_B^+\psi_B)=1$

・左辺に E_{NR} があって $H_{NR}(E_{NR})\psi_A=E_{NR}\psi_A$ の形になっている

・$i\hbar\nabla$ のような項が現われる（非エルミート）

(note) $\psi_B\sim\frac{\vec{\sigma}\cdot\vec{p}}{2mc}\psi_A$ なので規格化は

$$\int dr\;\psi_A^+\left(1+\frac{p^2}{4m^2c^2}\right)\psi_A\sim 1$$

↓ ψ_A の代わりに $\Psi\equiv\underbrace{\left(1+\frac{p^2}{8m^2c^2}\right)}_{\Omega}\psi_A$ を考える。

$$H_A\psi_A=E_{NR}\psi_A$$

$$\rightarrow\quad \Omega^{-1}H_A\Omega^{-1}\Psi=E_{NR}\Omega^{-1}\psi_A=E_{NR}\Omega^{-2}\Psi$$

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(note)

$$\Omega^{-1} \sim 1 - \frac{\mathbf{p}^2}{8m^2c^2}$$

$$\Omega^{-2} \sim 1 - \frac{\mathbf{p}^2}{4m^2c^2}$$

$$\left(1-\frac{\mathbf{p}^2}{8m^2c^2}\right)\left[(\vec{\sigma}\cdot\vec{p})\frac{1}{2m}\left(1-\frac{E_{NR}-q\varphi}{2mc^2}\right)(\vec{\sigma}\cdot\vec{p})+q\varphi\right]\left(1-\frac{\mathbf{p}^2}{8m^2c^2}\right)\Psi$$

$$= E_{NR}\left(1-\frac{\mathbf{p}^2}{4m^2c^2}\right)\Psi$$

$$\left[\frac{(\vec{\sigma}\cdot\vec{p})^2}{2m}+q\varphi-\frac{\mathbf{p}^2}{8m^2c^2}\left(\frac{(\vec{\sigma}\cdot\vec{p})^2}{2m}+q\varphi\right)-\left(\frac{(\vec{\sigma}\cdot\vec{p})^2}{2m}+q\varphi\right)\frac{\mathbf{p}^2}{8m^2c^2}\right.$$

$$\left.-\frac{\vec{\sigma}\cdot\vec{p}}{2m}\left(\frac{E_{NR}-q\varphi}{2mc^2}\right)(\vec{\sigma}\cdot\vec{p})\right]\Psi \sim E_{NR}\left(1-\frac{\mathbf{p}^2}{4m^2c^2}\right)\Psi$$

(note) $(\vec{\sigma}\cdot\vec{p})^2 = \vec{p}^{\,2}$

$$E_{NR}\,\mathbf{p}^2 = \frac{1}{2}\{E_{NR}, \mathbf{p}^2\}$$

$$\left[\frac{\mathbf{p}^2}{2m}+q\varphi-\frac{\mathbf{p}^4}{8m^3c^2}+\frac{1}{8m^2c^2}\left(\{\mathbf{p}^2, E_{NR}-q\varphi\}\right.\right.$$

$$\left.\left.-2(\vec{\sigma}\cdot\vec{p})(E_{NR}-q\varphi)(\vec{\sigma}\cdot\vec{p})\right)\right]\Psi$$

$$= E_{NR}\Psi.$$

(note)

$$\{A^2, B\} - 2ABA = [A, [A, B]]$$

$$A(AB-BA)-(AB-BA)A$$
$$= A^2B + BA^2 - 2ABA$$

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$y = (1+x)^{1/2}$
$y' = \frac{1}{2}(1+x)^{-\frac{1}{2}}$
$y'' = -\frac{1}{4}(1+x)^{-\frac{3}{2}}$

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$$\{\mathbb{P}^2, E_{NR} - q\varphi\} - 2(\vec{\sigma}\cdot\vec{\mathbb{P}})(E_{NR} - q\varphi)(\vec{\sigma}\cdot\vec{\mathbb{P}})$$

$$= [(\vec{\sigma}\cdot\vec{\mathbb{P}}), \underbrace{[\vec{\sigma}\cdot\vec{\mathbb{P}}, E_{NR} - q\varphi]}]$$

$$\| \quad -\frac{\hbar}{i} q\, \vec{\sigma}\cdot\nabla\varphi \quad = -i\hbar q\, \vec{\sigma}\cdot\vec{E}$$

$$\| \quad -\vec{E}$$

$$= -i\hbar q\,(\vec{\sigma}\cdot\vec{\mathbb{P}})(\vec{\sigma}\cdot\vec{E}) + i\hbar q\,(\vec{\sigma}\cdot\vec{E})(\vec{\sigma}\cdot\vec{\mathbb{P}})$$

$$= -i\hbar q\,(\vec{\mathbb{P}}\cdot\vec{E}) + 2i\hbar q\cdot i\vec{\sigma}\cdot(\vec{E}\times\vec{\mathbb{P}})$$

$$\uparrow$$

$$\nabla\times\mathbb{E} = 0$$

$$= -q\hbar^2(\nabla\cdot\mathbb{E}) - 2q\hbar\,\vec{\sigma}\cdot(\vec{E}\times\vec{\mathbb{P}})$$

$$H_{NR} = \frac{\mathbb{P}^2}{2m} + q\varphi - \frac{\mathbb{P}^4}{8m^3c^2} - \frac{q\hbar\,\vec{\sigma}\cdot(\vec{E}\times\vec{\mathbb{P}})}{4m^2c^2} \underbrace{- \frac{q\hbar^2}{8m^2c^2}\nabla\cdot\mathbb{E}}$$

ダーウィン項

$$\uparrow$$

$$\sqrt{\mathbb{P}^2c^2 + m^2c^4} - mc^2$$

$$= \sqrt{m^2c^4\left(1 + \frac{\mathbb{P}^2}{m^2c^2}\right)} - mc^2$$

$$= mc^2\left(1 + \frac{\mathbb{P}^2}{2m^2c^2} - \frac{1}{8}\frac{\mathbb{P}^4}{m^4c^4} + \cdots\right) - mc^2$$

$$= \frac{\mathbb{P}^2}{2m} - \frac{\mathbb{P}^4}{8m^3c^2} + \cdots$$

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中心ポテンシャル $q\varphi(r) = V(r)$ に対して

$$\mathbb{F} = -\nabla\varphi = -\frac{\mathbb{r}}{r}\frac{dV}{dr}\cdot\frac{1}{q}$$

$$\downarrow$$

$$-\frac{qh}{4m^2c^2}\vec{\sigma}\cdot(\mathbb{F}\times\mathbb{p}) = \frac{h}{4m^2c^2}\left(\frac{1}{r}\frac{dV}{dr}\right)\vec{\sigma}\cdot(\mathbb{r}\times\mathbb{p})$$

$$= \frac{1}{2m^2c^2}\left(\frac{1}{r}\frac{dV}{dr}\right)\vec{L}\cdot\vec{S}$$

スピン軌道力