（復習）ディラック方程式の平面波解：

$$\psi = \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = \begin{pmatrix} u_A(\mathbf{p}) \\ u_B(\mathbf{p}) \end{pmatrix} \underbrace{e^{i\mathbf{p}\cdot\mathbf{r}/\hbar - iEt/\hbar}}_{= e^{-ip_\mu x^\mu/\hbar}}$$

$$i\hbar\frac{\partial}{\partial t}\psi = (-i\hbar c\vec{\alpha}\cdot\nabla + \beta mc^2)\psi$$

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\downarrow$

$$E\begin{pmatrix} u_A(\mathbf{p}) \\ u_B(\mathbf{p}) \end{pmatrix} = \begin{pmatrix} mc^2 & c\vec{\sigma}\cdot\vec{p} \\ c\vec{\sigma}\cdot\vec{p} & -mc^2 \end{pmatrix}\begin{pmatrix} u_A(\mathbf{p}) \\ u_B(\mathbf{p}) \end{pmatrix}$$

$\downarrow$

$$\begin{cases} (E-mc^2)u_A = c\,\vec{\sigma}\cdot\vec{p}\,u_B \\ (E+mc^2)u_B = c\,\vec{\sigma}\cdot\vec{p}\,u_A \end{cases}$$

(note) $(E+mc^2)(E-mc^2)u_A = c\vec{\sigma}\cdot\vec{p}\,(E+mc^2)u_B$

$$= c^2(\vec{\sigma}\cdot\vec{p})^2 u_A$$

$$= p^2c^2 u_A$$

$$\rightarrow E^2 = m^2c^4 + p^2c^2$$

$$\rightarrow E = \pm\sqrt{m^2c^4 + p^2c^2}$$

$E>0$ に対し
$$u_s^{(+)}(\mathbf{p}) \propto \begin{pmatrix} \chi_s \\ \dfrac{c\vec{\sigma}\cdot\vec{p}}{E+mc^2}\chi_s \end{pmatrix};$$

$E<0$ に対し
$$u_s^{(-)}(\mathbf{p}) \propto \begin{pmatrix} \dfrac{c\vec{\sigma}\cdot\vec{p}}{E-mc^2}\chi_s \\ \chi_s \end{pmatrix}$$

§. 相対論的散乱問題の準備：非相対論的プロパゲータ

時刻 t での波動関数 $\psi(x) = \psi(\mathbf{r}, t)$

→ 時刻 $t'\,(>t)$ への時間発展：

$$\theta(t'-t)\,\psi(\mathbf{r}', t') = i\int d\mathbf{r}\; G(\mathbf{r}', t'; \mathbf{r}, t)\,\psi(\mathbf{r}, t)$$

と書く。

(note) $$|\psi(t')\rangle = e^{-iH(t'-t)/\hbar}\,|\psi(t)\rangle$$

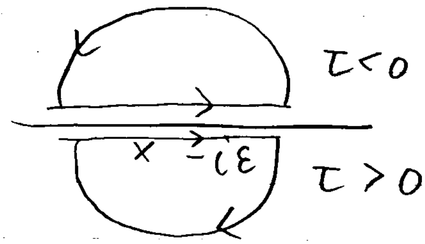
$$\rightarrow \quad \theta(t'-t)\langle \mathbf{r}'|\psi(t')\rangle$$

$$= \theta(t'-t)\int d\mathbf{r}\; \langle \mathbf{r}'|e^{-iH(t'-t)/\hbar}|\mathbf{r}\rangle\langle \mathbf{r}|\psi(t)\rangle$$

↓

$$i\,G(\mathbf{r}', t'; \mathbf{r}, t) = \theta(t'-t)\langle \mathbf{r}'|e^{-iH(t'-t)/\hbar}|\mathbf{r}\rangle$$

(note) $$\theta(\tau) = \lim_{\varepsilon\to 0}\frac{-1}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{-i\omega\tau}}{\omega + i\varepsilon}\,d\omega$$



↓

$$\frac{d\theta}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\; e^{-i\omega\tau} = \delta(\tau)$$

$G(\mathbf{r}', t'; \mathbf{r}, t)$ ：プロパゲータ(伝搬関数)
，グリーン関数

(note)

$$\left(i\hbar\frac{\partial}{\partial t'} - H(x')\right)\theta(t'-t)\psi(x')$$

$$= i\hbar\,\delta(t'-t)\psi(x') + \theta(t'-t)\underbrace{\left(i\hbar\frac{\partial}{\partial t'}-H(x')\right)\psi(x')}_{=0}$$

$$= i\hbar\,\delta(t'-t)\psi(x') = i\hbar\,\delta(t'-t)\int d\mathbf{r}\,\delta(\mathbf{r}-\mathbf{r}')\psi(\mathbf{r},t)$$

$$= i\int d\mathbf{r}\left[\left(i\hbar\frac{\partial}{\partial t'}-H(x')\right)G(\mathbf{r}',t';\mathbf{r},t)\right]\psi(x)$$

$$\rightarrow \boxed{\left(i\hbar\frac{\partial}{\partial t'}-H(x')\right)G(\mathbf{r}',t';\mathbf{r},t) = \hbar\,\delta(t'-t)\,\delta(\mathbf{r}-\mathbf{r}')}$$

← グリーン関数

(note) 完全系

$$G(x',x) = -i\,\theta(t'-t)\langle\mathbf{r}'|e^{-iH(t'-t)/\hbar}\;|\mathbf{r}\rangle$$

$$1 = \sum_n |\psi_n\rangle\langle\psi_n|$$

$$= -i\sum_n \theta(t'-t)\langle\mathbf{r}'|e^{-iH(t'-t)/\hbar}|\psi_n(t)\rangle\langle\psi_n(t)|\mathbf{r}\rangle$$

$$= -i\sum_n \theta(t'-t)\langle\mathbf{r}'|\psi_n(t')\rangle\langle\psi_n(t)|\mathbf{r}\rangle$$

$$= -i\,\theta(t'-t)\sum_n \psi_n(\mathbf{r}',t')\,\psi_n^*(\mathbf{r},t)$$

四 摂動展開

$$
\begin{cases}
\left(i\hbar\frac{\partial}{\partial t'} - \underbrace{H(x')}_{H_0(x')+V(x')}\right) G(x',x) = \hbar\,\delta(t'-t)\,\delta(\mathbf{r}'-\mathbf{r}) \\[2ex]
\left(i\hbar\frac{\partial}{\partial t'} - H_0(x')\right) G_0(x',x) = \hbar\,\delta(t'-t)\,\delta(\mathbf{r}'-\mathbf{r})
\end{cases}
$$

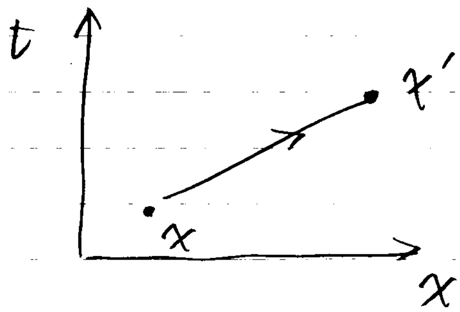
$$
\rightarrow\ G(x',x) = G_0(x',x) + \frac{1}{\hbar}\int d\mathbf{r}_1 dt_1\, G_0(x',x_1)V(x_1)G(x_1,x)
$$

「時間に依存するリップマン・シュウィンガー方程式」

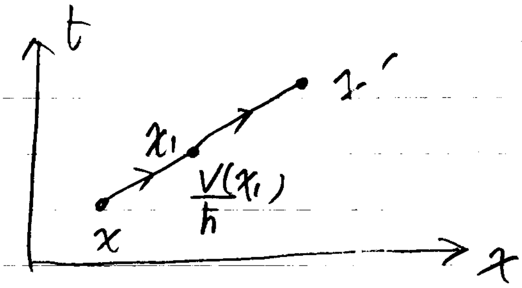
(note)
$$
\left(i\hbar\frac{\partial}{\partial t'} - H(x')\right)G(x',x)
$$
$$
= \left(i\hbar\frac{\partial}{\partial t'} - H_0(x') - V(x')\right)G_0(x',x)
$$
$$
+\frac{1}{\hbar}\int dx_1 \left(\underbrace{i\hbar\frac{\partial}{\partial t'} - H_0(x')} - V(x')\right)G_0(x',x_1)V(x_1) \times G(x_1,x)
$$
$$
= \hbar\delta(t'-t)\delta(\mathbf{r}'-\mathbf{r}) - V(x')G_0(x',x)
$$
$$
+ V(x')G(x',x) - \frac{1}{\hbar}\int dx_1\, V(x')G_0(x',x_1)V(x_1)G(x_1,x)
$$
$$
= \hbar\delta(t'-t)\delta(\mathbf{r}'-\mathbf{r}) + V(x')\cancel{\left(G(x',x) - G_0(x',x) - \frac{1}{\hbar}\int dx_1\, G_0(x',x_1)V(x_1)G(x_1,x)\right)}
$$
$$
= \hbar\delta(t'-t)\delta(\mathbf{r}'-\mathbf{r})
$$

逐次近似：

$$G(x', x) \sim G_0(x', x) + \frac{1}{\hbar}\int dx_1\, G_0(x', x_1)\, V(x_1)\, G_0(x_1, x) + \cdots$$



$+$



$+ \cdots$

・S行列

始状態 $\varphi_i(x) \longrightarrow$ 終状態 $\varphi_f(x)$

$$\psi_i(x') = \lim_{t\to-\infty} i\int d\mathbf{r}\, G(x',x)\varphi_i(x)$$

$$= \lim_{t\to-\infty} i\int d\mathbf{r}\left[G_0(x',x) + \frac{i}{\hbar}\int d\mathbf{r}_1 dt_1\, G_0(x',x_1)V(x_1) \times G(x_1,x)\right]\varphi_i(x)$$

$$= \varphi_i(x') + \frac{i}{\hbar}\int d\mathbf{r}_1 dt_1\, G_0(x',x_1)V(x_1)\psi_i(x_1)$$

◀ S行列 $S_{fi} = \langle \varphi_f | \psi_i \rangle$

$$= \lim_{t'\to\infty}\int d\mathbf{r}'\, \varphi_f^*(x')\psi_i(x')$$

$$= \lim_{t'\to\infty}\int d\mathbf{r}'\, \varphi_f^*(x')\left[\varphi_i(x') + \frac{i}{\hbar}\int d\mathbf{r}_1 dt_1\, G_0(x',x_1)V(x_1)\psi_i(x_1)\right]$$

$$= \delta_{f,i} + \lim_{t'\to+\infty}\frac{1}{\hbar}\int d\mathbf{r}'\int d\mathbf{r}_1 dt_1\, \varphi_f^*(x') \times G_0(x',x_1)V(x_1)\psi_i(x_1)$$

(note) $\langle \psi_i | \psi_j \rangle = \langle \varphi_i | \varphi_j \rangle = \delta_{i,j}$

$$S_{fi} = \langle \varphi_f | \psi_i \rangle \to \sum_f \varphi_f(x) S_{fi} = \lim_{t\to\infty}\psi_i(x)$$

$$\delta_{i,j} = \lim_{t\to\infty}\langle \psi_i | \psi_j \rangle = \sum_n \lim_{t\to\infty}\langle \psi_i | \varphi_n \rangle\langle \varphi_n | \psi_j \rangle$$

$$= \sum_n S_{ni}^* S_{nj}$$ (S行列のユニタリー性)