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No.

§. 相対論的プロパゲータ

相対論（ディラック方程式）→ $E>0$ の解と $E<0$ の解

$E>0$ ↕ 電子　　$E<0$ ↕ 陽電子

シュテュッケルベルク－ファインマンの解釈：
時間に逆行して伝搬する $E<0$ の電子
↔ 時間に順行して伝搬する $E>0$ の陽電子

四 プロパゲータ

時間発展 → ディラック方程式に従う

$$\left[i\hbar \gamma^{\mu} \left(\frac{\partial}{\partial x_{\mu}} + \frac{ie}{\hbar} A_{\mu} \right) - mc \right] \psi = 0$$

$$\sum_{\gamma=1}^{4} \left[\gamma_{\mu} \left\{ i\hbar \frac{\partial}{\partial x'_{\mu}} - e A^{\mu}(x') \right\} - mc \right]_{\alpha\gamma} \left(S_F(x', x) \right)_{\gamma\beta} = \hbar \delta_{\alpha,\beta} \underbrace{\delta^{(4)}(x - x')}_{\| \; \delta(x_0 - x_0') \delta(\boldsymbol{r} - \boldsymbol{r}')}$$

※スピノルの添字を省略すると

$$\left[\gamma_{\mu} \left\{ i\hbar \frac{\partial}{\partial x_{\mu}'} - e A^{\mu}(x') \right\} - mc \right] S_F(x', x) = \hbar \delta^{(4)}(x - x')$$

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・自由粒子の場合

$$\left(i\hbar \gamma_\mu \frac{\partial}{\partial x'_\mu} - mc \right) S_F^{(0)}(x', x) = \hbar \delta^{(4)}(x - x')$$

これをフーリエ変換を使って解く.

$$S_F^{(0)}(x', x) = \int \frac{d^4p}{(2\pi\hbar)^4} e^{-\frac{i}{\hbar} p(x'-x)} \tilde{S}_F^{(0)}(p)$$

とする。 $[\, p(x'-x) = p_0 \cdot c(t'-t) - \boldsymbol{p} \cdot (\boldsymbol{x}'-\boldsymbol{x}) \,]$

↓

$$\hbar \delta^{(4)}(x-x') = \left(i\hbar \gamma_\mu \frac{\partial}{\partial x'_\mu} - mc \right) S_F^{(0)}(x', x)$$

$$= \int \frac{d^4p}{(2\pi\hbar)^4} \left(p^\mu \gamma_\mu - mc \right) e^{-\frac{i}{\hbar} p(x'-x)} \tilde{S}_F^{(0)}(p)$$

$p^\mu \gamma_\mu$ ||| $\not{p}$

$$= \hbar \int \frac{d^4p}{(2\pi\hbar)^4} e^{-\frac{i}{\hbar} p(x'-x)}$$

↓

$$\tilde{S}_F^{(0)}(p) = \frac{\hbar}{\not{p} - mc} = \hbar \frac{\not{p} + mc}{p^2 - m^2c^2} \qquad (p^2 + m^2c^2)$$

↑ "off-shell の状態"

特異点 $p^2 = m^2c^2$

$$\rightarrow p_0 c = \pm \sqrt{\boldsymbol{p}^2c^2 + m^2c^4} = \pm E$$

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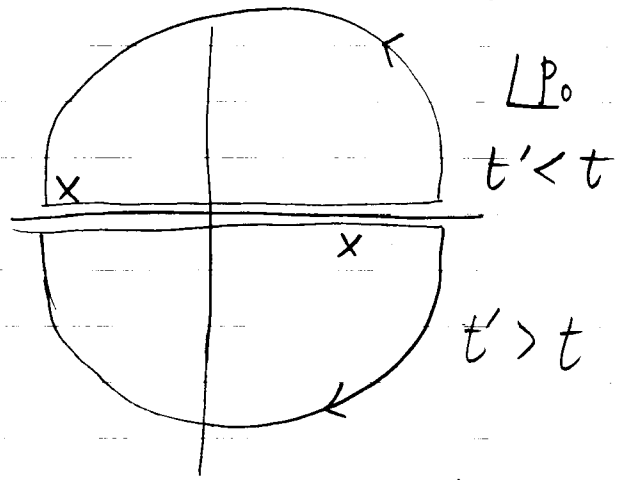
時間に順行する粒子 → $E>0$
逆行 $E<0$



$$S_F^{(0)}(x',x)\big|_{t'>t} = \int \frac{d\mathbb{P}}{(2\pi\hbar)^3} e^{\frac{i}{\hbar}\mathbb{P}\cdot(\mathbb{r}'-\mathbb{r})} \times \int \frac{dP_0}{2\pi\hbar} e^{-\frac{i}{\hbar}P_0c(t'-t)} \cdot \hbar \frac{\not{P}+mc}{P^2-m^2c^2}$$

$$\parallel$$

$$(P_0-E/c)(P_0+E/c)$$

$$= -\int \frac{d\mathbb{P}}{(2\pi\hbar)^3} e^{\frac{i}{\hbar}\mathbb{P}\cdot(\mathbb{r}'-\mathbb{r})} \cdot 2\pi i \frac{1}{2\pi\hbar}\cdot\hbar \cdot \frac{E\gamma_0 - c\mathbb{P}\cdot\vec{\gamma}+mc^2}{2E} \times e^{-\frac{i}{\hbar}E(t'-t)}$$

$$= -i\int \frac{d\mathbb{P}}{(2\pi\hbar)^3} e^{\frac{i}{\hbar}[\mathbb{P}\cdot(\mathbb{r}'-\mathbb{r})-E(t'-t)]} \frac{E\gamma_0 - c\mathbb{P}\cdot\vec{\gamma}+mc^2}{2E}$$

$$S_F^{(0)}(x',x)\big|_{t'<t} = \int \frac{d\mathbb{P}}{(2\pi\hbar)^3} e^{\frac{i}{\hbar}\mathbb{P}\cdot(\mathbb{r}'-\mathbb{r})} 2\pi i \frac{1}{2\pi\hbar}\hbar \cdot \frac{-E\gamma_0 - c\mathbb{P}\cdot\vec{\gamma}+mc^2}{-2E} \times e^{\frac{i}{\hbar}E(t'-t)}$$

$$= -i\int \frac{d\mathbb{P}}{(2\pi\hbar)^3} e^{\frac{i}{\hbar}(-\mathbb{P}\cdot(\mathbb{r}'-\mathbb{r})+E(t'-t))} \frac{-E\gamma_0 + c\mathbb{P}\cdot\vec{\gamma}+mc^2}{2E}$$

↑
$\mathbb{P}\to-\mathbb{P}$.

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$$(p_\mu \gamma^\mu - mc + i\varepsilon)(p_\mu \gamma^\mu + mc - i\varepsilon) = p^2 - mc^2 + i\varepsilon$$

No.

相対論的フロパゲーター

$$\left[\gamma_\mu \left\{ i\hbar \frac{\partial}{\partial x'_\mu} - qA^\mu(x') \right\} - mc \right] S_F(x', x) = \hbar \delta^{(4)}(x - x')$$

・自由粒子の場合

$$\left(i\hbar \gamma_\mu \frac{\partial}{\partial x'_\mu} - mc \right) S_F^{(0)}(x', x) = \hbar \delta^{(4)}(x - x')$$

フーリエ変換： $S_F^{(0)}(x', x) = \int \frac{d^4p}{(2\pi\hbar)^4} e^{-\frac{i}{\hbar} p(x'-x)} \tilde{S}_F^{(0)}(p)$

$$\rightarrow \tilde{S}_F^{(0)}(p) = \frac{\hbar}{p^\mu \gamma_\mu - mc} = \hbar \frac{\not{p} + mc}{p^2 - m^2c^2}$$

$p^\mu \gamma_\mu = \not{p}$

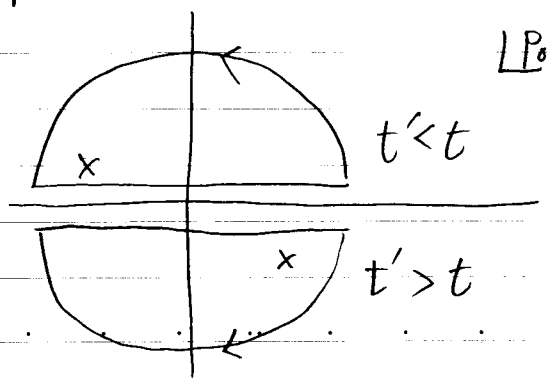
$p^2 = p_0^2 - \mathbf{p}^2$

特異点， $p^2 = p_0^2 - \mathbf{p}^2 = mc^2$

$$\rightarrow p_0 c = \pm \sqrt{\mathbf{p}^2 c^2 + m^2 c^4} = \pm E$$

時間に順行する粒子 $\rightarrow E > 0$
逆行 $E < 0$ } となるように極をずらす

$\int dp_0 \, e^{-\frac{i}{\hbar} p_0 c (t' - t)} \cdots$



$$S_F^{(0)}(x', x) = \int \frac{d^4p}{(2\pi\hbar)^4} e^{-\frac{i}{\hbar} p(x'-x)} \times \frac{\hbar(\not{p} + mc)}{p^2 - m^2c^2 + i\varepsilon}$$

とするのと等しい

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規格化された平面波解

$$\psi_{ps}^{(+)}(x) = \sqrt{\frac{E+mc^2}{2E}} \begin{pmatrix} \chi_s \\ \frac{c(\boldsymbol{\sigma}\cdot\boldsymbol{p})}{E+mc^2}\chi_s \end{pmatrix} \frac{e^{-ipx/\hbar}}{(2\pi\hbar)^{3/2}}$$

$$\psi_{ps}^{(-)}(x) = \sqrt{\frac{-E+mc^2}{2E}} \begin{pmatrix} \frac{-c(\boldsymbol{\sigma}\cdot\boldsymbol{p})}{E-mc^2}\chi_s \\ \chi_s \end{pmatrix} \frac{e^{ipx/\hbar}}{(2\pi\hbar)^{3/2}}$$

を用いて；　$\rightarrow \int d^4x\, \psi_{ps}^{(\pm)}(x)^\dagger \psi_{ps'}^{(\pm)}(x) = 2\pi\hbar\, \delta_{ss'}\delta(p-p')$

$$S_F^{(0)}(x',x) = -i\theta(t'-t)\int dp \sum_s \psi_{ps}^{(+)}(x')\,\bar{\psi}_{ps}^{(+)}(x) + i\theta(t-t')\int dp \sum_s \psi_{ps}^{(-)}(x')\,\bar{\psi}_{ps}^{(-)}(x)$$

と書くこともできる。

$$\rightarrow\quad \theta(t'-t)\,\psi_{ps}^{(+)}(x') = i\int d^4x\, S_F^{(0)}(x',x)\,\gamma_0\,\psi_{ps}^{(+)}(x)$$

$$\theta(t-t')\,\psi_{ps}^{(-)}(x') = -i\int d^4x\, S_F^{(0)}(x',x)\,\gamma_0\,\psi_{ps}^{(-)}(x)$$

「ファインマン伝搬関数」

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• $g \neq 0$ の場合

$$(i\hbar\gamma_\mu \frac{\partial}{\partial x'_\mu} - g\gamma_\mu A^\mu(x') - mc) S_F(x', x) = \hbar \delta^{(4)}(x'-x)$$

↓

$$S_F(x', x) = S_F^{(0)}(x', x) + \frac{g}{\hbar}\int d^4y\, S_F^{(0)}(x', y) A(y) S_F(y, x)$$

(note)

$$(i\hbar\gamma_\mu \frac{\partial}{\partial x'_\mu} - mc) S_F(x', x)$$

$$= (i\hbar\gamma_\mu \frac{\partial}{\partial x'_\mu} - mc) S_F^{(0)}(x', x)$$

$$+ \frac{g}{\hbar}\int d^4y \underbrace{(i\hbar\gamma_\mu \frac{\partial}{\partial x'_\mu} - mc) S_F^{(0)}(x', y)}_{\| \; \hbar\delta^{(4)}(x'-y)} A(y) S_F(y, x)$$

$$= \hbar\delta^{(4)}(x'-x) + gA(x') S_F(x', x)$$

$$\rightarrow (i\hbar\gamma_\mu \frac{\partial}{\partial x'_\mu} - gA(x') - mc) S_F(x', x) = \delta^{(4)}(x'-x)$$

同様に,

$$\Psi(x) = \underbrace{\psi(x)}_{\text{自由粒子}} + \frac{g}{\hbar}\int d^4y\, S_F^{(0)}(x, y) A(y) \Psi(y)$$

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(note)

$$\Xi(x) - \Psi(x) = \frac{g}{\hbar}\int d^4y\, S_F^{(0)}(x,y)\not{A}(y)\Xi(y)$$

$$\begin{cases} \xrightarrow{t\to\infty} \int dP \sum_s \Psi_{ps}^{(+)}(x)\left[-i\frac{g}{\hbar}\int d^4y\, \bar{\Psi}_{ps}^{(+)}(y)\not{A}(y)\Xi(y)\right] \\ \qquad : \text{時間に順行する正エネルギー解} \\ \xrightarrow{t\to-\infty} \int dP \sum_s \Psi_{ps}^{(-)}(x)\left[i\frac{g}{\hbar}\int d^4y\, \bar{\Psi}_{ps}^{(-)}(y)\not{A}(y)\Xi(y)\right] \\ \qquad : \text{時間に逆行する負エネルギー解} \end{cases}$$

$$S_{fi} = \langle \Psi_f | \Xi \rangle = \delta_{f,i} \mp i\frac{g}{\hbar}\int d^4y\, \bar{\Psi}_f(y)\not{A}(y)\Xi_i(y)$$

$-$: $E>0$, $+$: $E<0$

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四 クーロン散乱

静止した Ze の電荷がつくる静電ポテンシャルによる電子の散乱

$$A_0(x) = \frac{1}{4\pi\varepsilon_0}\cdot\frac{1}{c}\frac{Ze}{r}, \quad A(x)=0$$

↓

$$S_{fi} = i\frac{e}{\hbar}\int d^4x\, \bar{\psi}_f(x) \cancel{A}(x)\, \psi_i(x) \qquad (i\to f,\ e\to -e)$$

↓ 摂動の最低次

$\psi_i(x)$

(平面波解)

$$\psi_i(x) = \psi^{(+)}_{p_i s_i}(x) = \sqrt{\frac{mc^2}{E_i}}\cdot\underbrace{\sqrt{\frac{E_i+mc^2}{2mc^2}}\begin{pmatrix}\chi_s\\ \frac{c\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E_i+mc^2}\chi_s\end{pmatrix}}_{\parallel\ u(p_i,s_i)}\frac{1}{\sqrt{V}}e^{-ip_i x/\hbar}$$

$$\bar{\psi}_f(x) = \bar{\psi}^{(+)}_{p_f s_f}(x) = \sqrt{\frac{mc^2}{E_f V}}\,\bar{u}(p_f,s_f)\,e^{+ip_f x/\hbar}$$

↓

$$S_{fi} = i\frac{e}{\hbar c}\frac{Ze}{4\pi\varepsilon_0}\cdot\frac{1}{V}\frac{mc^2}{\sqrt{E_i E_f}}\,\bar{u}(p_f s_f)\,\gamma_0\,u(p_i s_i)$$

$$\times\underbrace{\int d^4x\,\frac{1}{r}e^{\frac{i}{\hbar}(p_f-p_i)x}}$$

$$\int_{-\infty}^{\infty} e^{ikx} dx = 2\pi \delta(k)$$

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(note) $\int dx_0\, e^{\frac{i}{\hbar}(P_f^0 - P_i^0)x_0} = c\int_{-\infty}^{\infty} dt\; e^{\frac{i}{\hbar}(E_f - E_i)t}$

$= 2\pi\hbar c\, \delta(E_i - E_f)$

$\int d\boldsymbol{r}\, \frac{1}{r} e^{-\frac{i}{\hbar}(\boldsymbol{P}_f - \boldsymbol{P}_i)\cdot \boldsymbol{r}} \cdot \underbrace{e^{-r/a}}_{\text{スクリーニング}} = \frac{4\pi\hbar^2}{|\boldsymbol{P}_f - \boldsymbol{P}_i|^2 + \frac{\hbar^2}{a^2}} \xrightarrow{a\to\infty} \frac{4\pi\hbar^2}{|\boldsymbol{P}_f - \boldsymbol{P}_i|^2}$

↓

$$S_{fi} = (\cdots) \times \frac{\bar{u}(P_f S_f)\gamma_0 u(P_i S_i)}{\underbrace{|\boldsymbol{P}_f - \boldsymbol{P}_i|^2}_{\| \; q^2}} 2\pi\hbar\, \delta(E_f - E_i)$$

◀ 遷移確率

$$P_{fi} = |S_{fi}|^2 = (\cdots) \frac{|\bar{u}_f \gamma_0 u_i|^2}{q^4} \cdot \underbrace{(2\pi\hbar\, \delta(E_f - E_i))^2}_{\| \; T\to\infty} \times \frac{V d\boldsymbol{P}_f}{(2\pi\hbar)^3}$$

$$2\pi\hbar\, \delta(E_f - E_i) \int_{-T/2}^{T/2} e^{\frac{i}{\hbar}(E_f - E_i)t} \times dt$$

$$= 2\pi\hbar\, \delta(E_f - E_i) \int_{-T/2}^{T/2} dt$$

$$= 2\pi\hbar\, T\, \delta(E_f - E_i)$$

↓ 単位時間当たりの遷移確率：

$$R_{fi} = \frac{P_{fi}}{T} = (\cdots) \frac{|\bar{u}_f \gamma_0 u_i|^2}{q^4} d\boldsymbol{P}_f\, \delta(E_f - E_i)$$

散乱断面積 $d\sigma = \frac{R_{fi}}{(J)}$ フラックス

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• フラックス $\vec{j} = c\psi^\dagger \vec{\alpha} \psi$

$$= c\frac{1}{V}\cdot\frac{E+mc^2}{2E}\left(\chi_s^\dagger \quad \chi_s^\dagger \frac{c\sigma\cdot p}{E+mc^2}\right)\begin{pmatrix}0 & \sigma\\ \sigma & 0\end{pmatrix}\begin{pmatrix}\chi_s\\ \frac{c\sigma\cdot p}{E+mc^2}\chi_s\end{pmatrix}$$

$$= c\cdot\frac{1}{V}\frac{E+mc^2}{2E}\left(\chi_s^\dagger \sigma \frac{c\sigma\cdot p}{E+mc^2}\chi_s + \chi_s^\dagger \frac{c\sigma\cdot p}{E+mc^2}\sigma\chi_s\right)$$

$p = p e_z$ とおくと

$$\vec{j} = c\frac{1}{V}\cdot\frac{E+mc^2}{2E}\cdot\frac{2cp}{E+mc^2}e_z = \frac{pc^2}{E}\cdot\frac{1}{V}e_z$$

↓

$$\frac{d\sigma}{d\Omega} = 4\cdot\frac{\hbar^2 Z^2\alpha^2 m^2c^2}{q^4}\left|\bar{u}_f \gamma_0 u_i\right|^2$$

• スピンを区別しないとき（Moll 断面積）

$$\left(\frac{1}{2}\sum_{s_i}\right)\sum_{s_f}\left|\bar{u}(p_f s_f)\gamma_0 u(p_i s_i)\right|^2$$

$$= \cdots = \frac{E^2}{m^2c^4}\left(1-\beta^2\sin^2\frac{\theta}{2}\right);\ \beta \equiv \frac{pc}{E}$$

θ は p_i と p_f の間の角

↓ $\frac{d\sigma}{d\Omega} = \frac{1}{4}\cdot\frac{\hbar^2 Z^2\alpha^2}{p^2\beta^2\sin^4\frac{\theta}{2}}\left(1-\beta^2\sin^2\frac{\theta}{2}\right)$ ← $\begin{cases} q = 2|p_i|\sin\frac{\theta}{2} \\ \beta = |p_i|c/E \end{cases}$

(note) NR の極限

$$p \to mv_0,\ \beta = \frac{pc}{E} \to \frac{mv_0 c}{mc^2} = \frac{v_0}{c},\ 1-\beta^2\sin^2\frac{\theta}{2} \to 1$$

↓ $\frac{d\sigma}{d\Omega} = \frac{1}{4}\left(\frac{Z\alpha c\hbar}{mv_0^2}\right)^2\frac{1}{\sin^4\frac{\theta}{2}}$ （ラザフォード散乱と一致）