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§. リップマン－シュウィンガー方程式

$$\left(-\frac{\hbar^2}{2\mu}\nabla^2 + V\right)\psi = E\psi$$

$$\rightarrow \left(\underbrace{-\frac{\hbar^2}{2\mu}\nabla^2}_{\hat{H}_0} - E\right)\psi = -V\psi$$

形式解

$$\boxed{\psi = \phi - \frac{1}{\hat{H}_0 - E - i\eta} V\psi}$$

リップマン－シュウィンガー方程式

$(\hat{H}_0 - E)\phi = 0$

η：正の微小量（波動関数の境界条件を与える。）

・グリーン関数

$$\hat{G}_0^{(+)} = -\frac{1}{\hat{H}_0 - E - i\eta} \rightarrow \psi(\mathbf{r}) = \phi(\mathbf{r}) + \int d\mathbf{r}'\, G_0^{(+)}(\mathbf{r}, \mathbf{r}')V(\mathbf{r}') \times \psi(\mathbf{r}')$$

座標表示では

$$G_0^{(+)}(\mathbf{r}, \mathbf{r}') = -\langle \mathbf{r} | \frac{1}{\hat{H}_0 - \underset{\frac{k^2\hbar^2}{2\mu}}{E} - i\eta} | \mathbf{r}' \rangle$$

$$= -\int d\mathbf{k}' \langle \mathbf{r} | \mathbf{k}' \rangle \cdot \frac{1}{\frac{k'^2\hbar^2}{2\mu} - \frac{k^2\hbar^2}{2\mu} - i\eta} \langle \mathbf{k}' | \mathbf{r}' \rangle$$

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$$\int \frac{d\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} = \delta(\mathbf{r}-\mathbf{r}')$$

$$= -\int \frac{d\mathbf{k}'}{(2\pi)^3} e^{i\mathbf{k}'\cdot\mathbf{r}} \cdot \frac{2\mu}{\hbar^2} \frac{1}{k'^2 - k^2 - i\eta'} e^{-i\mathbf{k}'\cdot\mathbf{r}'}$$

$$\left(\eta' = \frac{2\mu}{\hbar^2}\eta\right)$$

$$= -\frac{2\mu}{\hbar^2} \cdot \frac{1}{(2\pi)^3} \int k'^2 dk' d\hat{\mathbf{k}}' \; e^{ik's\cos\theta} \frac{1}{k'^2 - k^2 - i\eta'}$$

$$(\mathbf{s} \equiv \mathbf{r} - \mathbf{r}')$$

$$= \frac{-1}{(2\pi)^3} \cdot \frac{2\mu}{\hbar^2} \int_0^\infty k'^2 dk' \cdot \underbrace{2\pi \int_{-1}^{1} d(\cos\theta) \frac{e^{ik's\cos\theta}}{k'^2 - k^2 - i\eta'}}_{= \frac{2\pi}{k'^2 - k^2 - i\eta'} \frac{1}{ik's}\left(e^{ik's} - e^{-ik's}\right)}$$

$$= \frac{-1}{i} \cdot \frac{1}{(2\pi)^2} \cdot \frac{2\mu}{\hbar^2} \underbrace{\int_0^\infty dk' \frac{k' dk'}{k'^2 - k^2 - i\eta'} \left(\frac{e^{ik's}}{s} - \frac{e^{-ik's}}{s}\right)}_{= \int_{-\infty}^{\infty} dk' \frac{k' dk'}{k'^2 - k^2 - i\eta'} \cdot \frac{e^{ik's}}{s}}$$

$$= \int_{-\infty}^{\infty} dk' \frac{k' dk'}{(k' + k + i\eta'')(k' - k - i\eta'')} \cdot \frac{e^{ik's}}{s}$$

$$\left(\eta'' = \frac{\eta'}{2k}\right)$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} dk' \left(\frac{1}{k' + k + i\eta''} + \frac{1}{k' - k - i\eta''}\right) \times \frac{e^{ik's}}{s}$$

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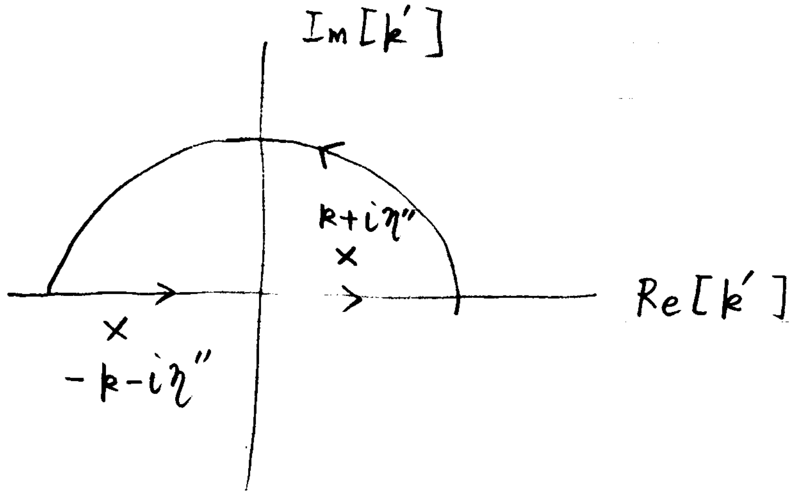
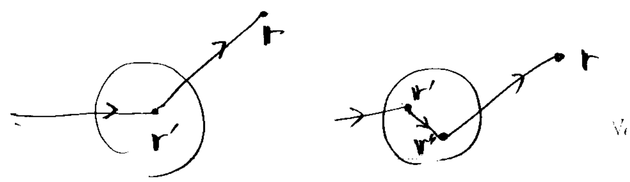


$$G^{(+)}(\mathbf{r},\mathbf{r}') = \frac{-1}{\cancel{i}}\,\frac{1}{(2\pi)^2}\cdot\frac{2\mu}{\hbar^2}\cdot\frac{1}{2}\cdot 2\pi\cancel{i}\,\frac{e^{iks}}{s}$$

$$= \frac{-\mu}{2\pi\hbar^2}\cdot\frac{e^{iks}}{s}$$

$$= -\frac{2\mu}{\hbar^2}\cdot\frac{1}{4\pi}\,\frac{e^{+ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}$$ (外向き波)



$$\psi(\mathbf{r}) = \phi(\mathbf{r}) + \int d\mathbf{r}'\, G^{(+)}(\mathbf{r},\mathbf{r}')\, V(\mathbf{r}')\,\psi(\mathbf{r}')$$

$$= e^{i\mathbf{k}\cdot\mathbf{r}} - \frac{2\mu}{\hbar^2}\cdot\frac{1}{4\pi}\int d\mathbf{r}'\,\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\,V(\mathbf{r}')\,\psi(\mathbf{r}')$$

(note) $r\to\infty$ では

$$k|\mathbf{r}-\mathbf{r}'| = k\sqrt{r^2 - 2\mathbf{r}\cdot\mathbf{r}' + r'^2}$$

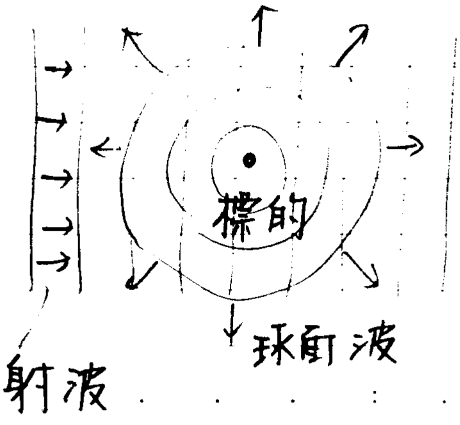
$$\sim kr - \underbrace{k\frac{\mathbf{r}}{r}}_{=\ \mathbf{k}'}\cdot\mathbf{r}'$$

$$\psi(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} \underbrace{- \frac{\mu}{2\pi\hbar^2}\int d\mathbf{r}'\, e^{-i\mathbf{k}'\cdot\mathbf{r}'}\, V(\mathbf{r}')\,\psi(\mathbf{r}')}_{=\ f(\theta)} \cdot \frac{e^{ikr}}{r}$$

$|\mathbf{r}-\mathbf{r}'| \sim r$

$f(\theta)$ 散乱振幅

$$= \underbrace{e^{i\mathbf{k}\cdot\mathbf{r}}}_{\text{入射波}} + \underbrace{f(\theta)\cdot\frac{e^{ikr}}{r}}_{\text{散乱波(球面波)}}$$



cf. ボルン近似

$$\frac{d\sigma}{d\Omega} = \frac{\mu^2}{4\pi\hbar^4}\,|\tilde{V}(q)|^2$$

§. 散乱振幅と散乱断面積

散乱波 $\psi_{sc}(\boldsymbol{r}) = f(\theta)\cdot\dfrac{e^{ikr}}{r}$ に対するフラックス

$$\boldsymbol{j}_{sc} = \frac{\hbar}{2i\mu}\left(\psi_{sc}^{*}\nabla\psi_{sc} - \psi_{sc}\nabla\psi_{sc}^{*}\right)$$

を計算する。

(note) $\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$ $\qquad r = \sqrt{x^2+y^2+z^2}$

$$\downarrow \quad \frac{\partial r}{\partial x} = \frac{x}{r}$$

$\dfrac{\partial z}{\partial x}$

$$z = r\cos\theta \quad\rightarrow\quad 0 = \frac{\partial r}{\partial x}\cos\theta - r\sin\theta\cdot\frac{\partial\theta}{\partial x}$$

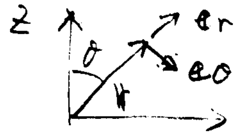
$$\downarrow\quad \frac{\partial\theta}{\partial x} = \frac{x\cos\theta}{r^2\sin\theta} = \frac{1}{r}\cos\theta\cos\varphi$$

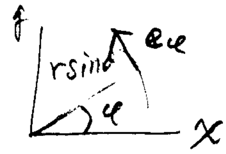
$$y = r\sin\theta\sin\varphi \quad\rightarrow\quad 0 = \frac{x}{r}\sin\theta\sin\varphi + r\cos\theta\cdot\frac{1}{r}\cos\theta\cos\varphi \times\sin\varphi + r\sin\theta\cos\varphi\cdot\frac{\partial\varphi}{\partial x}$$

$$= \sin^2\theta\,\underline{\sin\varphi\cos\varphi} + \cos^2\theta\,\underline{\cos\varphi\sin\varphi} + r\sin\theta\cos\varphi\frac{\partial\varphi}{\partial x}$$

$$= \sin\varphi\cos\varphi + r\sin\theta\cos\varphi\frac{\partial\varphi}{\partial x}$$

$$\downarrow\quad \frac{\partial\varphi}{\partial x} = -\frac{\sin\varphi}{r\sin\theta}$$





$$\partial_x = \frac{\partial r}{\partial x}\partial_r + \frac{\partial \theta}{\partial x}\partial_\theta + \frac{\partial \varphi}{\partial x}\partial_\varphi$$

$$= \sin\theta\cos\varphi\,\partial_r + \frac{1}{r}\cos\theta\cos\varphi\,\partial_\theta - \frac{\sin\varphi}{r\sin\theta}\partial_\varphi$$

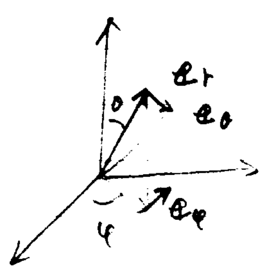
同様に

$$\partial_y = \sin\theta\sin\varphi\,\partial_r + \frac{1}{r}\cos\theta\sin\varphi\,\partial_\theta + \frac{\cos\varphi}{r\sin\theta}\partial_\varphi$$

$$\partial_z = \cos\theta\,\partial_r - \frac{\sin\theta}{r}\partial_\theta$$

$$\nabla = \left[\sin\theta\cos\varphi\,\partial_r + \frac{1}{r}\cos\theta\cos\varphi\,\partial_\theta - \frac{\sin\varphi}{r\sin\theta}\partial_\varphi\right]e_x$$
$$+\left[\sin\theta\sin\varphi\,\partial_r + \frac{1}{r}\cos\theta\sin\varphi\,\partial_\theta + \frac{\cos\varphi}{r\sin\theta}\partial_\varphi\right]e_y$$
$$+\left[\cos\theta\,\partial_r - \frac{1}{r}\sin\theta\,\partial_\theta\right]e_z$$

(note)



$$\begin{cases} e_r = \sin\theta\cos\varphi\, e_x + \sin\theta\sin\varphi\, e_y + \cos\theta\, e_z \\ e_\theta = \cos\theta\cos\varphi\, e_x + \cos\theta\sin\varphi\, e_y - \sin\theta\, e_z \\ e_\varphi = -\sin\varphi\, e_x + \cos\varphi\, e_y \end{cases}$$

(note) $\nabla\cdot e_r = \partial_r$, $\nabla\cdot e_\theta = \frac{1}{r}\partial_\theta$, $\nabla\cdot e_\varphi = \frac{1}{r\sin\theta}\partial_\varphi$

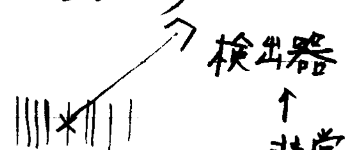
$$\boxed{\nabla = \partial_r\, e_r + \frac{1}{r}\partial_\theta\, e_\theta + \frac{1}{r\sin\theta}\partial_\varphi\, e_\varphi}$$

$$\therefore\quad \boldsymbol{j}_{sc} = \frac{\hbar}{2i\mu}\left[f^*(\theta)\frac{e^{-ikr}}{r}\left(\boldsymbol{e}_r\partial_r + \boldsymbol{e}_\theta\frac{1}{r}\partial_\theta\right)f(\theta)\frac{e^{ikr}}{r} - c.c.\right]$$

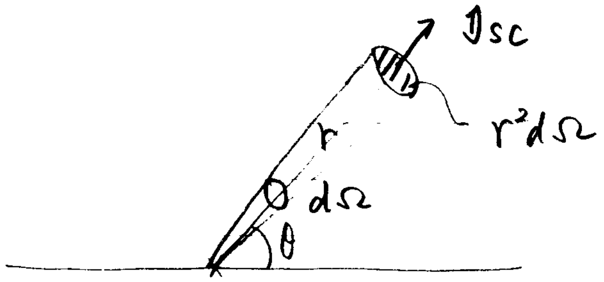
$$= \frac{\hbar}{2i\mu}\left[f^*(\theta)\cdot\frac{e^{-ikr}}{r}\left\{f(\theta)\left(\frac{ik}{r}e^{ikr} - \frac{1}{r^2}e^{ikr}\right)\boldsymbol{e}_r + \frac{e^{ikr}}{r^2}f'(\theta)\,\boldsymbol{e}_\theta\right\} - c.c.\right]$$

$$\sim \frac{\hbar}{2i\mu}\cdot 2ik\,\frac{|f(\theta)|^2}{r^2}\boldsymbol{e}_r \quad (r\to\infty)$$

$$= \frac{k\hbar}{\mu}\cdot\frac{|f(\theta)|^2}{r^2}\boldsymbol{e}_r$$



↑
非常に遠い
$\frac{1}{r} \gg \frac{1}{r^2}$



単位時間に立体角 $d\Omega$ に散乱される粒子数

$$= r^2 d\Omega\cdot\boldsymbol{j}_{sc}\cdot\boldsymbol{e}_r = \frac{k\hbar}{\mu}|f(\theta)|^2 d\Omega$$

$$\therefore\quad \boxed{\frac{d\sigma}{d\Omega} = \frac{1}{j_{in}}\cdot\frac{k\hbar}{\mu}|f(\theta)|^2 = |f(\theta)|^2}$$

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・ボルン近似

$$\psi = \phi - \frac{1}{H_0 - E - i\eta} V \psi$$

$$\sim \phi - \frac{1}{H_0 - E - i\eta} V \phi$$

$$f(\theta) = -\frac{\mu}{2\pi\hbar^2} \int d\mathbf{r}' \, e^{-i\mathbf{k}'\cdot\mathbf{r}'} V(\mathbf{r}') \phi(\mathbf{r}')$$

$$= -\frac{\mu}{2\pi\hbar^2} \int d\mathbf{r}' \, e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}'} V(\mathbf{r}')$$

$$= -\frac{\mu}{2\pi\hbar^2} \tilde{V}(\vec{q})$$

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \frac{\mu^2}{4\pi^2\hbar^4} |\tilde{V}(\vec{q})|^2$$

← フェルミの黄金則によるものと一致