量子力学 II

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1. 3次元球対称ポテンシャル中の運動

1次元ポテンシャル中の運動 ← 量子力学 I

$$\underbrace{\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V(x)\right)}_{\text{ハミルトニアン}\ H=\frac{p^2}{2m}+V(x)}\psi(x)=E\,\psi(x)$$

$$\int_{-\infty}^{\infty}dx\,|\psi(x)|^2=1 \quad \text{(規格化条件)}$$

⤳ 3次元ポテンシャルへの拡張

$$H=\frac{1}{2m}\left(p_x^2+p_y^2+p_z^2\right)+V(x,y,z)$$
$$=\frac{\mathbf{p}^2}{2m}+V(\mathbf{r})$$

座標表示 : $\mathbf{p}=\frac{\hbar}{i}\nabla \qquad \left(p_k=\frac{\hbar}{i}\frac{\partial}{\partial x_k}\right)$

$$H=-\frac{\hbar^2}{2m}\nabla^2+V(\mathbf{r})$$

シュレーディンガー方程式 :

$$\left(-\frac{\hbar^2}{2m}\nabla^2+V(\mathbf{r})\right)\psi(\mathbf{r})=E\,\psi(\mathbf{r})$$

波動関数の規格化 :

$$\underbrace{\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dy\int_{-\infty}^{\infty}dz}_{=\int d\mathbf{r}}\,|\psi(\mathbf{r})|^2=1$$

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ポテンシャルが変数分離型のとき.

例) 3次元調和振動子

$$V(x,y,z) = \frac{1}{2}m\omega_x^2 x^2 + \frac{1}{2}m\omega_y^2 y^2 + \frac{1}{2}m\omega_z^2 z^2$$

$\downarrow$

$$H = \underbrace{\frac{p_x^2}{2m} + \frac{1}{2}m\omega_x x^2}_{= h_x} + \underbrace{\frac{p_y^2}{2m} + \frac{1}{2}m\omega_y y^2}_{= h_y} + \underbrace{\frac{p_z^2}{2m} + \frac{1}{2}m\omega_z z^2}_{= h_z}$$

シュレーディンガー方程式の解

$$\Psi(x,y,z) = \phi^{(1)}_{n_x}(x)\,\phi^{(2)}_{n_y}(y)\,\phi^{(3)}_{n_z}(z)$$

$$\left(\frac{p_x^2}{2m} + \frac{1}{2}m\omega_x x^2\right)\phi^{(1)}_{n_x}(x) = \left(n_x + \frac{1}{2}\right)\hbar\omega_x \times \phi^{(1)}_{n_x}(x)$$

など.

$$E = \left(n_x + \frac{1}{2}\right)\hbar\omega_x + \left(n_y + \frac{1}{2}\right)\hbar\omega_y + \left(n_z + \frac{1}{2}\right)\hbar\omega_z$$

(note) $\omega_x = \omega_y = \omega_z \equiv \omega$ のとき

$$E = \left(n_x + n_y + n_z + \frac{3}{2}\right)\hbar\omega$$

$$V(x,y,z) = \frac{1}{2}m\omega^2 \underbrace{(x^2 + y^2 + z^2)}_{= r^2}$$

→ 極座標を使っても解ける.

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（複習）変数分離

$$H = \underbrace{\frac{p_x^2}{2m} + V(x)}_{=h_x} + \underbrace{\frac{p_y^2}{2m} + V(y)}_{=h_y}$$

$$H\Psi(x,y) = E\Psi(x,y)$$

$\Psi(x,y) = X(x)Y(y)$ とおく。

$$H\Psi = (h_x X)Y + X(h_y Y)$$

$$h_x X = E_x X$$
$$h_y Y = E_y Y$$
のとき

$$H\Psi = (E_x + E_y)XY = \underbrace{(E_x + E_y)}_{=E}\Psi$$

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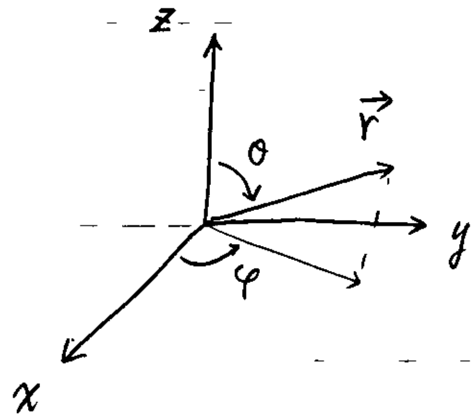
ポテンシャルが球対称の時（$\mathbf{r}$の大きさ $r=|\mathbf{r}|$ にしか依存しない場合）

$$H = \frac{\mathbf{P}^2}{2m} + V(r)$$

→ 極座標表示をすることにより
変数分離が可能



$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$$

微分オペレーター ∇ を極座標を使って書く：

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial \varphi}{\partial x}\frac{\partial}{\partial \varphi}$$

(note) $r = \sqrt{x^2+y^2+z^2} \rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$

(note) $z = r\cos\theta$ の両辺を x で微分すると

$$0 = \underbrace{\frac{\partial r}{\partial x}}_{= \frac{x}{r}}\cos\theta - r\sin\theta\frac{\partial \theta}{\partial x}$$

$$\Rightarrow \frac{\partial \theta}{\partial x} = \frac{x\cos\theta}{r^2\sin\theta} = \frac{1}{r}\cos\theta\cos\varphi$$

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(note) $y = r\sin\theta\sin\varphi$ の両辺を x で微分

↓

$$0 = \frac{\partial r}{\partial x}\sin\theta\sin\varphi + r\cos\theta\frac{\partial\theta}{\partial x}\sin\varphi + r\sin\theta\cos\varphi\frac{\partial\varphi}{\partial x}$$

$$= \sin^2\theta\,\underline{\sin\varphi\cos\varphi} + \cos^2\theta\,\underline{\cos\varphi\sin\varphi} + \underline{r\sin\theta}\cos\varphi\frac{\partial\varphi}{\partial x}$$

$$= \sin\varphi\cancel{\cos\varphi} + r\sin\theta\cancel{\cos\varphi}\frac{\partial\varphi}{\partial x}$$

$$\therefore\quad \frac{\partial\varphi}{\partial x} = -\frac{\sin\varphi}{r\sin\theta}$$

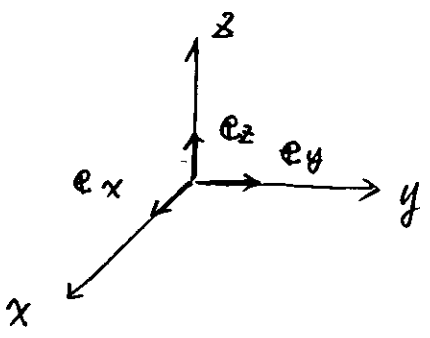
$$\therefore\quad \frac{\partial}{\partial x} = \sin\theta\cos\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\varphi\frac{\partial}{\partial\theta} - \frac{\sin\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}$$

$\frac{\partial}{\partial y}$, $\frac{\partial}{\partial z}$ も同様に求められる。

↓

$$\nabla = \left[\sin\theta\cos\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\varphi\frac{\partial}{\partial\theta} - \frac{\sin\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right]\mathbf{e}_x + \left[\sin\theta\sin\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\sin\varphi\frac{\partial}{\partial\theta} + \frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right]\mathbf{e}_y + \left[\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right]\mathbf{e}_z$$

$\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ は x, y, z 方向の単位ベクトル



$$|\mathbf{e}_k| = 1$$

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さらにもう一度微分演算を行なえば
∇^2の極座標表示が得られる（うしろのページ参照）

$$\boxed{\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{1}{r^2}\ell^2$$

$$-\ell^2 = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

シュレーディンガー方程式：

$$\boxed{\left[-\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{\ell^2\hbar^2}{2mr^2} + V(r) - E\right]\psi(\mathbf{r}) = 0}$$

角度に関係する演算子はℓ^2のみ。

もしℓ^2の固有状態　$\ell^2 Y(\theta,\varphi) = \lambda Y(\theta,\varphi)$
がわかったとして

$$\psi(\mathbf{r}) = R(r)Y(\theta,\varphi)$$

と変数分離すれば

$$\left[-\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{\lambda\hbar^2}{2mr^2} + V(r) - E\right]R(r) = 0$$

となり、rのみの2階の微分方程式となる。

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[参考] ∇^2 の極座標表示の導出

$$\frac{\partial^2}{\partial x^2} = \left(\sin\theta\cos\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\varphi\frac{\partial}{\partial\theta} - \frac{\sin\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi} \right)^2$$

$$= \sin^2\theta\cos^2\varphi\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\cos^2\theta\cos^2\varphi\frac{\partial^2}{\partial\theta^2} + \frac{\sin^2\varphi}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

$$+ \frac{2}{r}\sin\theta\cos\theta\cos^2\varphi\frac{\partial^2}{\partial r\partial\theta} \cancel{- \frac{2}{r^2}\frac{\cos\theta}{\sin\theta}\sin\varphi\cos\varphi\frac{\partial^2}{\partial\theta\partial\varphi}}$$

$$\cancel{-\frac{2}{r}\sin\varphi\cos\varphi\frac{\partial^2}{\partial r\partial\varphi}}$$

$$-\frac{1}{r^2}\sin\theta\cos\theta\cos^2\varphi\frac{\partial}{\partial\theta} \cancel{+ \frac{1}{r^2}\sin\varphi\cos\varphi\frac{\partial}{\partial\varphi}}$$

$$+\frac{1}{r}\cos^2\theta\cos^2\varphi\frac{\partial}{\partial r} - \frac{1}{r^2}\sin\theta\cos\theta\cos^2\varphi\frac{\partial}{\partial\theta} \cancel{+ \frac{\cos\theta\cos\varphi}{r}\frac{\sin\varphi}{r} \times \frac{\cos\theta}{\sin^2\theta}\frac{\partial}{\partial\varphi}}$$

$$+\frac{\sin\varphi}{r\sin\theta}\sin\theta\sin\varphi\frac{\partial}{\partial r} + \frac{\sin\varphi}{r\sin\theta}\cdot\frac{1}{r}\cos\theta\sin\varphi\frac{\partial}{\partial\theta}$$

$$\cancel{+\frac{\sin\varphi}{r\sin\theta}\cdot\frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}}$$

$$\frac{\partial^2}{\partial y^2} = \left(\sin\theta\sin\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\sin\varphi\frac{\partial}{\partial\theta} + \frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi} \right)^2$$

$$= \sin^2\theta\sin^2\varphi\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\cos^2\theta\sin^2\varphi\frac{\partial^2}{\partial\theta^2} + \frac{\cos^2\varphi}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

$$+\frac{2}{r}\sin\theta\cos\theta\sin^2\varphi\frac{\partial^2}{\partial r\partial\theta} \cancel{+ \frac{2}{r^2}\frac{\cos\theta}{\sin\theta}\sin\varphi\cos\varphi\frac{\partial^2}{\partial\theta\partial\varphi}}$$

$$\cancel{+\frac{2}{r}\sin\varphi\cos\varphi\frac{\partial^2}{\partial r\partial\varphi}}$$

$$-\frac{1}{r^2}\sin\theta\cos\theta\sin^2\varphi\frac{\partial}{\partial\theta} \cancel{- \frac{1}{r^2}\sin\varphi\cos\varphi\frac{\partial}{\partial\varphi}}$$

$$+\frac{1}{r}\cos^2\theta\sin^2\varphi\frac{\partial}{\partial r} - \frac{1}{r^2}\sin\theta\cos\theta\sin^2\varphi\frac{\partial}{\partial\theta} \cancel{+ \frac{\cos\theta\sin\varphi}{r}\cdot\frac{\cos\varphi}{r}\left(-\frac{\cos\theta}{\sin^2\theta}\right)\frac{\partial}{\partial\varphi}}$$

$$+\frac{\cos\varphi}{r\sin\theta}\sin\theta\cos\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\varphi\cdot\frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\theta}$$

$$\cancel{-\frac{\cos\varphi\sin\varphi}{r^2\sin^2\theta}\frac{\partial}{\partial\varphi}}$$

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$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} = \sin^2\theta\frac{\partial^2}{\partial r^2}+\frac{1}{r^2}\cos^2\theta\frac{\partial^2}{\partial\theta^2}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

$$+\cancel{\frac{2}{r}\sin\theta\cos\theta\frac{\partial^2}{\partial r\partial\theta}}-\cancel{\frac{1}{r^2}\sin\theta\cos\theta\frac{\partial}{\partial\theta}}$$

$$+\frac{1}{r}\cos^2\theta\frac{\partial}{\partial r}-\cancel{\frac{1}{r^2}\sin\theta\cos\theta\frac{\partial}{\partial\theta}}+\frac{1}{r}\frac{\partial}{\partial r}$$

$$+\frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta}$$

$$\frac{\partial^2}{\partial z^2} = \left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)^2$$

$$=\cos^2\theta\frac{\partial^2}{\partial r^2}+\frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2}-\cancel{\frac{2\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial r\partial\theta}}$$

$$+\cancel{\frac{1}{r^2}\sin\theta\cos\theta\frac{\partial}{\partial\theta}}+\frac{\sin^2\theta}{r}\frac{\partial}{\partial r}+\cancel{\frac{1}{r^2}\sin\theta\cos\theta\frac{\partial}{\partial\theta}}$$

$$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{-1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}+\frac{2}{r}\frac{\partial}{\partial r}$$

$$+\frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta}$$

$$=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

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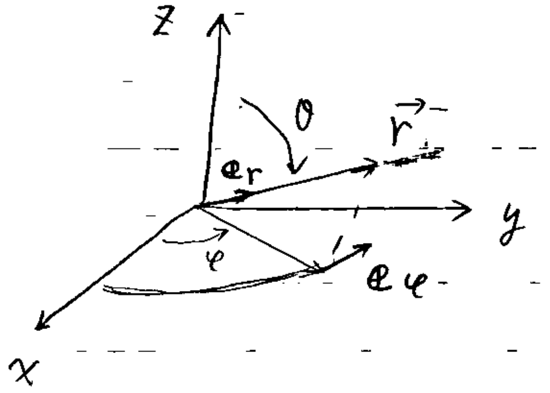
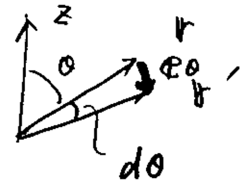
(note)

・e_x, e_y, e_z をさらに極座標を用いて表すと:



e_r : r の向きの単位ベクトル

e_θ :



$$e_r = \sin\theta\cos\varphi\, e_x + \sin\theta\sin\varphi\, e_y + \cos\theta\, e_z$$

$$\frac{\partial}{\partial\theta} e_r \longrightarrow e_\theta = \cos\theta\cos\varphi\, e_x + \cos\theta\sin\varphi\, e_y - \sin\theta\, e_z$$

$$\longrightarrow e_\varphi = -\sin\varphi\, e_x + \cos\varphi\, e_y$$

(note) $e_r \cdot e_\theta = \sin\theta\cos\theta\cos^2\varphi + \sin\theta\cos\theta\sin^2\varphi - \sin\theta\cos\theta = 0$

同様に $e_r \cdot e_\varphi = e_\theta \cdot e_\varphi = 0$

$$\nabla \cdot e_r = \sin^2\theta\cos^2\varphi\frac{\partial}{\partial r} + \cancel{\frac{1}{r}\sin\theta\cos\theta\cos^2\varphi\frac{\partial}{\partial\theta}} - \cancel{\frac{1}{r}\sin\varphi\cos\varphi\frac{\partial}{\partial\varphi}}$$
$$+ \sin^2\theta\sin^2\varphi\frac{\partial}{\partial r} + \cancel{\frac{1}{r}\sin\theta\cos\theta\sin^2\varphi\frac{\partial}{\partial\theta}} + \cancel{\frac{1}{r}\sin\varphi\cos\varphi\frac{\partial}{\partial\varphi}}$$
$$+ \cos^2\theta\frac{\partial}{\partial r} - \cancel{\frac{1}{r}\sin\theta\cos\theta\frac{\partial}{\partial\theta}}$$

$$= \frac{\partial}{\partial r}$$

$$\nabla \cdot e_\theta = \frac{1}{r}\frac{\partial}{\partial\theta}$$
$$\nabla \cdot e_\varphi = \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}$$

$$\therefore \boxed{\nabla = e_r\frac{\partial}{\partial r} + \frac{1}{r} e_\theta\frac{\partial}{\partial\theta} + \frac{1}{r\sin\theta} e_\varphi\frac{\partial}{\partial\varphi}}$$

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2. 角運動量

2.1. 角運動量演算子の極座標表示

$$\ell^2 = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

は角運動量の2乗を$\hbar^2$で割ったもの:

$$\mathbb{L}^2 = (\mathbb{r}\times\mathbb{p})^2 = \ell^2\hbar^2$$

(note) $L_z = x p_y - y p_x = \frac{\hbar}{i}\left(x\,\partial_y - y\,\partial_x\right)$

$$= \frac{\hbar}{i}\Big\{ r\sin\theta\cos\varphi\left(\cancel{\sin\theta\sin\varphi\,\partial_r + \frac{1}{r}\cos\theta\sin\varphi\,\partial_\theta} + \frac{\cos\varphi}{r\sin\theta}\,\partial_\varphi\right) - r\sin\theta\sin\varphi\left(\cancel{\sin\theta\cos\varphi\,\partial_r + \frac{1}{r}\cos\theta\cos\varphi\,\partial_\theta} - \frac{\sin\varphi}{r\sin\theta}\,\partial_\varphi\right)\Big\}$$

$$= \frac{\hbar}{i}\frac{\partial}{\partial\varphi}$$

同様に

$$L_x = \frac{\hbar}{i}\left(-\sin\varphi\,\partial_\theta - \frac{\cos\theta}{\sin\theta}\cos\varphi\,\partial_\varphi\right)$$

$$L_y = \frac{\hbar}{i}\left(\cos\varphi\,\partial_\theta - \frac{\cos\theta}{\sin\theta}\sin\varphi\,\partial_\varphi\right)$$

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$$
\begin{aligned}
L_x^2 + L_y^2 + L_z^2 &= -\hbar^2\left(-\sin\varphi\,\partial_\theta - \frac{\cos\theta}{\sin\theta}\cos\varphi\,\partial_\varphi\right)^2 \\
&\quad - \hbar^2\left(\cos\varphi\,\partial_\theta - \frac{\cos\theta}{\sin\theta}\sin\varphi\,\partial_\varphi\right)^2 \\
&\quad - \hbar^2\,\partial_\varphi^2
\end{aligned}
$$

$$
\begin{aligned}
&= -\hbar^2\Big\{\partial_\theta^2 + \underline{\frac{\cos^2\theta}{\sin^2\theta}\,\partial_\varphi^2 + \partial_\varphi^2} \\
&\quad + \cancel{\sin\varphi\cos\varphi\left(\frac{-\sin\theta}{\sin\theta} - \frac{\cos^2\theta}{\sin^2\theta}\right)}\partial_\varphi \\
&\quad - \cancel{\sin\varphi\cos\varphi\left(\frac{-\sin\theta}{\sin\theta} - \frac{\cos^2\theta}{\sin^2\theta}\right)}\partial_\varphi \\
&\quad + \frac{\cos\theta}{\sin\theta}\cos^2\varphi\,\partial_\theta + \frac{\cos\theta}{\sin\theta}\sin^2\varphi\,\partial_\theta\Big\}
\end{aligned}
$$

$$
= -\hbar^2\left(\partial_\theta^2 + \frac{\cos\theta}{\sin\theta}\,\partial_\theta + \frac{1}{\sin^2\theta}\,\partial_\varphi^2\right)
$$

$$
= \ell^2\hbar^2
$$

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(別の導出法)

2. 角運動量

2.1. 角運動量演算子(の極座標表示)

$$\ell^2 = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

は 角運動量の2乗を $\hbar^2$ で割ったもの:

$$\mathbb{L}^2 = (\mathbb{r}\times\mathbb{p})^2 = \ell^2\hbar^2$$

(note) $\nabla = \mathbb{e}_r\,\partial_r + \frac{1}{r}\,\mathbb{e}_\theta\,\partial_\theta + \frac{1}{r\sin\theta}\,\mathbb{e}_\varphi\,\partial_\varphi$

(note) $\mathbb{r} = r\,\mathbb{e}_r$

↓

$$\mathbb{L} = \mathbb{r}\times\mathbb{p} = r\,\mathbb{e}_r \times \frac{\hbar}{i}\nabla$$

$$= \frac{\hbar}{i}\cancel{r}\left\{\cancel{\mathbb{e}_r\times\mathbb{e}_r\,\partial_r} + \frac{1}{\cancel{r}}(\mathbb{e}_r\times\mathbb{e}_\theta)\,\partial_\theta + \frac{1}{\cancel{r}\sin\theta}(\mathbb{e}_r\times\mathbb{e}_\varphi)\,\partial_\varphi\right\}$$

(note) $\mathbb{e}_r\times\mathbb{e}_\theta = \mathbb{e}_\varphi$
$\mathbb{e}_r\times\mathbb{e}_\varphi = -\mathbb{e}_\theta$

↓

$$\boxed{\mathbb{L} = \frac{\hbar}{i}\left(\mathbb{e}_\varphi\,\partial_\theta - \frac{1}{\sin\theta}\,\mathbb{e}_\theta\,\partial_\varphi\right)}$$

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(note) $\mathbf{e}_\theta = \cos\theta\cos\varphi\,\mathbf{e}_x + \cos\theta\sin\varphi\,\mathbf{e}_y - \sin\theta\,\mathbf{e}_z$
$\mathbf{e}_\varphi = -\sin\varphi\,\mathbf{e}_x + \cos\varphi\,\mathbf{e}_y$

$$\mathbb{L} = \underbrace{\frac{\hbar}{i}\left(-\sin\varphi\,\partial_\theta - \frac{\cos\theta}{\sin\theta}\cos\varphi\,\partial_\varphi\right)}_{L_x}\mathbf{e}_x$$
$$+\underbrace{\frac{\hbar}{i}\left(\cos\varphi\,\partial_\theta - \frac{\cos\theta}{\sin\theta}\sin\varphi\,\partial_\varphi\right)}_{L_y}\mathbf{e}_y$$
$$+\underbrace{\frac{\hbar}{i}\partial_\varphi}_{L_z}\mathbf{e}_z$$
$$= L_x\mathbf{e}_x + L_y\mathbf{e}_y + L_z\mathbf{e}_z$$

(note) $L_\pm \equiv L_x \pm i L_y$

$$= \frac{\hbar}{i}\left\{(-\sin\varphi \pm i\cos\varphi)\,\partial_\theta - \frac{\cos\theta}{\sin\theta}(\cos\varphi \pm i\sin\varphi)\,\partial_\varphi\right\}$$

$$= \hbar e^{\pm i\varphi}\left(\pm\partial_\theta + i\frac{\cos\theta}{\sin\theta}\partial_\varphi\right)$$

$$-\sin\varphi \pm i\cos\varphi = -\frac{e^{i\varphi}-e^{-i\varphi}}{2i} \pm i\,\frac{e^{i\varphi}+e^{-i\varphi}}{2}$$
$$= \pm i\cdot e^{\pm i\varphi}$$

$$\cos\varphi \pm i\sin\varphi = \mp i\cdot(-\sin\varphi \pm i\cos\varphi) = e^{\pm i\varphi}$$

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2.2. 角運動量演算子の性質

$$\hat{\mathbb{L}} = \hat{\mathbb{r}} \times \hat{\mathbb{p}}$$

(note) $[V(r), L_z] = [V(r), x p_y - y p_x]$

$$= x[V(r), p_y] - y[V(r), p_x]$$
$$= -\frac{\hbar}{i} x \frac{\partial}{\partial y} V(r) + \frac{\hbar}{i} y \frac{\partial}{\partial x} V(r)$$
$$= -\frac{\hbar}{i} \cdot x \cdot \frac{y}{r} V'(r) + \frac{\hbar}{i} y \cdot \frac{x}{r} V'(r) = 0$$

同様に $[\frac{\mathbb{P}^2}{2m}, L_z] = 0$

↑

スカラー量

⇃

$H = \frac{\mathbb{P}^2}{2m} + V(r)$ に対し $[H, L_z] = 0$

同様に $[H, L_x] = [H, L_y] = 0$

また、$[H, \mathbb{L}^2] = [H, L_x^2 + L_y^2 + L_z^2]$

$$= L_x[H, L_x] + [H, L_x]L_x + \cdots$$
$$= 0$$

⇃ $H, L_x, L_y, L_z, \mathbb{L}^2$ の同時固有状態を作れる? → no

↑

$[L_i, L_j] = i\hbar \varepsilon_{ijk} L_k$

角運動量演算子は非可換

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$$[L_x, L_y] = [yP_z - zP_y,\ zP_x - xP_z]$$
$$= y[P_z, z]P_x + x[z, P_z]P_y$$
$$= \frac{\hbar}{i}\ yP_x + i\hbar\ xP_y$$
$$= i\hbar\ (xP_y - yP_x) = i\hbar L_z$$

同様に　$[L_y, L_z] = i\hbar L_x$
　　　　$[L_z, L_x] = i\hbar L_y$

また，$[L_z, \mathbb{L}^2] = [L_z, L_x^2] + [L_z, L_y^2]$
$$= i\hbar\ (L_xL_y + L_yL_x)$$
$$- i\hbar\ (L_xL_y + L_yL_x) = 0$$ など．

$\Updownarrow$

同時固有状態；　H, $\mathbb{L}^2$, (L_x, L_y, L_z) のうち1つ
$\downarrow$
L_z