・<u>球面調和関数</u>

以上より ℓ^2 と ℓ_z の同時固有関数は

$$\boxed{Y_{\ell m}(\theta,\varphi) = (-1)^m \sqrt{\frac{2\ell+1}{4\pi}\,\frac{(\ell-m)!}{(\ell+m)!}}\; P_\ell^m(\cos\theta)\, e^{im\varphi}}$$

球面調和関数

$$\boxed{\begin{aligned} \ell^2\,|Y_{\ell m}\rangle &= \ell(\ell+1)\,|Y_{\ell m}\rangle \\ \ell_z\,|Y_{\ell m}\rangle &= m\,|Y_{\ell m}\rangle \end{aligned}}$$

ℓ：整数（ゼロ 又は 正）

$-\ell \le m \le \ell$

$\ell = 0$ ： $Y_{00} = \dfrac{1}{\sqrt{4\pi}}$

$\ell = 1$ ： $Y_{11} = -\sqrt{\dfrac{3}{8\pi}}\,\sin\theta\, e^{i\varphi}$

$Y_{10} = \sqrt{\dfrac{3}{4\pi}}\,\cos\theta$

$Y_{1-1} = \sqrt{\dfrac{3}{8\pi}}\,\sin\theta\, e^{-i\varphi}$

$\ell = 2$ ： $Y_{20} = \sqrt{\dfrac{5}{4\pi}}\cdot\dfrac{1}{2}\,(3\cos^2\theta - 1)$

(note)

$$r Y_{1\pm1}(\theta,\varphi) \propto r\sin\theta\, e^{\pm i\varphi}$$
$$= r\sin\theta\,(\cos\varphi \pm i\sin\varphi)$$
$$= x \pm i y$$

$$r Y_{10}(\theta,\varphi) \propto r\cos\theta = z$$

$$r^2 Y_{20}(\theta,\varphi) \propto 3r^2\cos^2\theta - r^2 = 3z^2 - r^2$$
$$(= 2z^2 - x^2 - y^2)$$

(note)

$$\left[-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)-\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]\left(\sin\theta\, e^{\pm i\varphi}\right)$$

$$=\left[-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\cos\theta\right)-\frac{1}{\sin^2\theta}\cdot(-1)\sin\theta\right]e^{\pm i\varphi}$$

$$=\left[-\frac{1}{\sin\theta}\left(\underbrace{\cos^2\theta-\sin^2\theta}_{=1-2\sin^2\theta}\right)+\frac{1}{\sin\theta}\right]e^{\pm i\varphi}$$

$$=\left(-\cancel{\frac{1}{\sin\theta}}+2\sin\theta+\cancel{\frac{1}{\sin\theta}}\right)e^{\pm i\varphi}$$

$$=2\sin\theta\, e^{\pm i\varphi}\quad=1\cdot(1+1)\sin\theta\, e^{\pm i\varphi}$$

(note)

$$\left[-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)-\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]\left(3\cos^2\theta-1\right)$$

$$=-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\cdot 6\cos\theta\cdot(-\sin\theta)\right)$$

$$=\frac{6}{\sin\theta}\frac{\partial}{\partial\theta}\left(\cos\theta\,\underbrace{\sin^2\theta}_{=1-\cos^2\theta}\right)$$

$$=\frac{6}{\sin\theta}\left(-\sin\theta-3\cos^2\theta\cdot(-\sin\theta)\right)$$

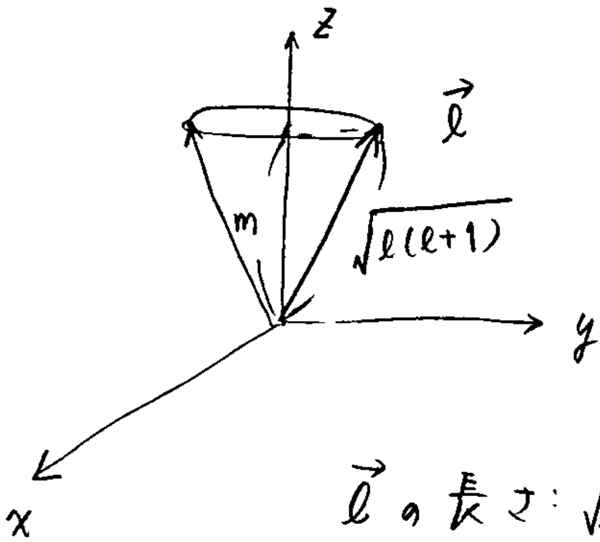
$$=6\cdot\left(3\cos^2\theta-1\right)$$

$$=2\times(2+1)\cdot\left(3\cos^2\theta-1\right)$$

(note) $\hat{l}^2 |Y_{lm}\rangle = l(l+1) |Y_{lm}\rangle$

$\hat{l}_z |Y_{lm}\rangle = m |Y_{lm}\rangle$

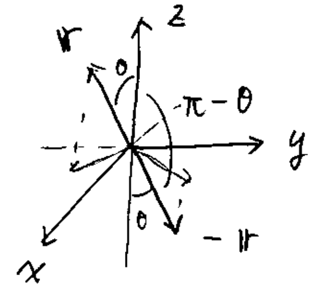
$-l \le m \le l$



$\vec{l}$ の長さ: $\sqrt{l(l+1)} > l$

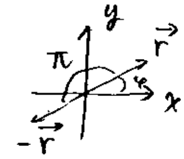
$\Updownarrow$ l_z を決めると l_x, l_y が決まらない。

性質



$$Y_{l\,-m}(\underbrace{\hat{r}}_{=(\theta,\varphi)}) = (-1)^m Y_{lm}(\hat{r})^*$$

$$Y_{lm}(-\hat{r}) = Y_{lm}(\pi-\theta, \varphi+\pi) = \underbrace{(-1)^l}_{\text{パリティ}} Y_{lm}(\hat{r})$$



規格化：

(note) $d\mathbf{r} = r^2 dr \boxed{\sin\theta d\theta d\varphi}$

$$\underbrace{\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta d\theta}_{=\int d\hat{r}} \; Y_{lm}^*(\theta,\varphi) Y_{l'm'}(\theta,\varphi) = \delta_{l,l'}\delta_{m,m'}$$

$$\langle Y_{lm} | Y_{l'm'} \rangle = \delta_{ll'}\delta_{m,m'}$$

加法定理

$$\frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y_{lm}(\hat{r}_1) Y_{lm}(\hat{r}_2)^* = P_l(\cos\gamma)$$

P_l：ルジャンドル多項式

γ：$\hat{r}_1$ と $\hat{r}_2$ のなす角

2.4 昇降演算子

$L_\pm \equiv L_x \pm i L_y$ を定義する（昇降演算子）。

(note) $(L_+)^\dagger = L_-$, $(L_-)^\dagger = L_+$

$$
\begin{aligned}
L_+ L_- &= (L_x + i L_y)(L_x - i L_y) \\
&= L_x^2 + L_y^2 - i (L_x L_y - L_y L_x) \\
&= L_x^2 + L_y^2 - i \underbrace{[L_x, L_y]}_{= i\hbar L_z} \\
&= L_x^2 + L_y^2 + \hbar L_z
\end{aligned}
$$

$\therefore$ $\mathbb{L}^2 = L_+ L_- + L_z^2 - \hbar L_z$

同様に $\mathbb{L}^2 = L_- L_+ + L_z^2 + \hbar L_z$

(note) $[L_+, L_z] = [L_x + i L_y, L_z]$

$= -i\hbar L_y + i \cdot i\hbar L_x = -\hbar L_+$

同様に $[L_-, L_z] = \hbar L_-$

(note) $[\mathbb{L}^2, L_\pm] = 0$

$[\mathbb{L}^2, L_x] = [\mathbb{L}^2, L_y] = 0$

$$[L_\pm, L_z] = \mp \hbar L_\pm$$

$$\begin{cases} \mathbb{L}^2 \, |Y_{\ell m}\rangle = \ell(\ell+1)\hbar^2 \, |Y_{\ell m}\rangle \\ L_z \, |Y_{\ell m}\rangle = m\hbar \, |Y_{\ell m}\rangle \end{cases}$$

$L_\pm |Y_{\ell m}\rangle$ という状態を考える。

$$\mathbb{L}^2 \, L_\pm |Y_{\ell m}\rangle = L_\pm \, \mathbb{L}^2 |Y_{\ell m}\rangle = \ell(\ell+1)\hbar^2 \, L_\pm |Y_{\ell m}\rangle$$

$[\mathbb{L}^2, L_\pm] = 0$

↓ $L_\pm |Y_{\ell m}\rangle$ は $\mathbb{L}^2$ の固有状態で
固有値 $\ell(\ell+1)\hbar^2$

$$\begin{aligned} L_z \, L_\pm |Y_{\ell m}\rangle &= (L_\pm L_z \pm \hbar L_\pm)|Y_{\ell m}\rangle \\ &= (m\pm 1)\hbar \, L_\pm |Y_{\ell m}\rangle \end{aligned}$$

↓ $L_\pm |Y_{\ell m}\rangle$ は L_z の固有状態で
固有値は $(m\pm 1)\hbar$

↓

$$L_\pm |Y_{\ell m}\rangle = \alpha_\pm |Y_{\ell\, m\pm 1}\rangle$$

m が ± 1 変わる
(昇降演算子)

- $\alpha_\pm$ の決定

$$L_\pm |Y_{\ell m}\rangle = \alpha_\pm |Y_{\ell m\pm1}\rangle$$

$\downarrow$

$$\langle Y_{\ell m}| \underbrace{(L_\pm)^\dagger L_\pm}_{\substack{\parallel \\ L_\mp L_\pm}} |Y_{\ell m}\rangle = |\alpha_\pm|^2 \langle Y_{\ell m\pm1}|Y_{\ell m\pm1}\rangle$$
$$= |\alpha_\pm|^2$$

(note) $L_- L_+ = \mathbb{L}^2 - L_z^2 - \hbar L_z$

$L_+ L_- = \mathbb{L}^2 - L_z^2 + \hbar L_z$

$\downarrow$

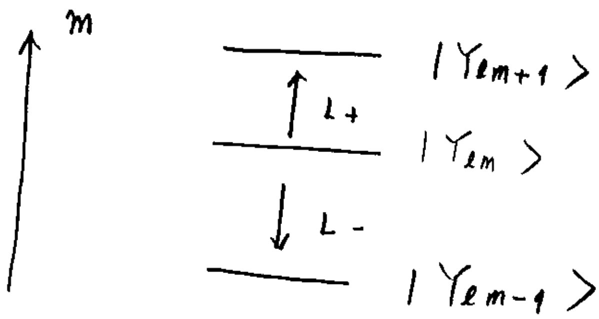
$$左辺 = \ell(\ell+1)\hbar^2 - m^2\hbar^2 \mp m\hbar^2$$
$$= \hbar^2 [\ell(\ell+1) - m(m\pm1)]$$

$\downarrow$

$$\boxed{\alpha_\pm = \hbar\sqrt{\ell(\ell+1) - m(m\pm1)}}$$

$\downarrow$

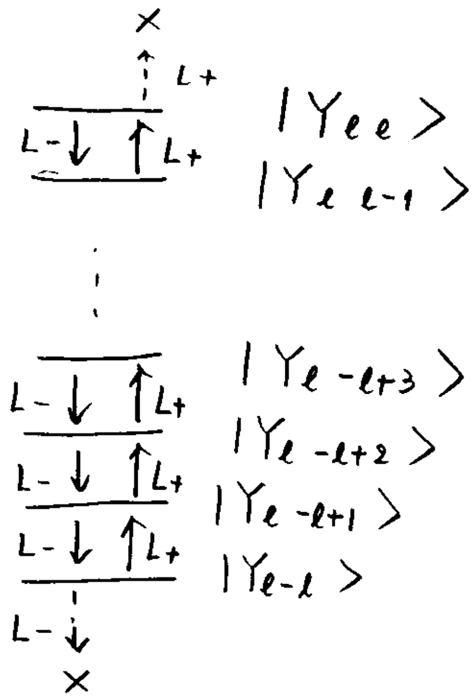
$$L_\pm |Y_{\ell m}\rangle = \hbar\sqrt{\ell(\ell+1) - m(m\pm1)}\ |Y_{\ell m\pm1}\rangle$$



(note)

$$L_+ |Y_{\ell\ell}\rangle = \hbar \underbrace{\sqrt{\ell(\ell+1)-\ell(\ell+1)}}_{=0} \, |Y_{\ell\,\ell+1}\rangle$$

$$L_- |Y_{\ell\,-\ell}\rangle = \hbar \underbrace{\sqrt{\ell(\ell+1)+\ell(-\ell-1)}}_{=0} \, |Y_{\ell\,-\ell-1}\rangle$$



※角運動量の合成のときに重要となる