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## 4.3. 磁場中の"水素"原子

(note) 古典的運動方程式(ローレンツ力)

$$m\frac{d^2\mathbf{r}}{dt^2} = -e\left[\mathbf{E}(\mathbf{r},t) + \frac{1}{c}\mathbf{v}\times\mathbf{B}(\mathbf{r},t)\right]$$

$$\mathbf{E} = -\frac{1}{c}\frac{\partial A(\mathbf{r},t)}{\partial t} - \nabla\phi(\mathbf{r},t)$$

$$\mathbf{B} = \nabla\times A$$

(note) $A \to A - \nabla f(\mathbf{r},t)$

$\phi \to \phi + \frac{1}{c}\frac{\partial}{\partial t}f(\mathbf{r},t)$

としてもマクスウェル方程式は不変
(ゲージ不変性)

$\rightsquigarrow$ $\rho(\mathbf{r})$ が時間に依存しない時

$$\nabla\cdot A(\mathbf{r},t) = 0$$

となるようなゲージをとると便利
(クーロン・ゲージ)

$$(\boldsymbol{v}\times(-\nabla\times\boldsymbol{A}))_i = \varepsilon_{kji}\, v_k\, (-\nabla\times\boldsymbol{A})_j$$

$$= \underbrace{\varepsilon_{ikj}\,\varepsilon_{i'k'j}}_{=\delta_{ii'}\delta_{kk'}-\delta_{ik'}\delta_{ki'}}\, v_k\,(-\partial_{i'}A_{k'})$$

$$= v_k(-\partial_i A_k + \partial_k A_i)$$

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(note)

$$H = \frac{1}{2m}\left(\boldsymbol{p} + \frac{e}{c}\boldsymbol{A}\right)^2 - e\phi(\boldsymbol{r})$$

とすると古典的な運動方程式が得られる。

〈証明〉

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial p_i} = \frac{1}{m}\left(p_i + \frac{e}{c}A_i\right)$$

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial x_i} = -\frac{1}{m}\left(p_k + \frac{e}{c}A_k\right)\frac{e}{c}\frac{\partial A_k}{\partial x_i} + e\frac{\partial \phi}{\partial x_i}$$

$$\downarrow$$

$$m\frac{d^2x_i}{dt^2} = \frac{d}{dt}\left(p_i + \frac{e}{c}A_i\right) = \frac{dp_i}{dt} + \frac{e}{c}\left(\frac{\partial A_i}{\partial t} + \frac{\partial A_i}{\partial x_k}\frac{\partial x_k}{\partial t}\right)$$

$$= -\frac{e}{c}\underbrace{\frac{1}{m}\left(p_k + \frac{e}{c}A_k\right)}_{\frac{dx_k}{dt}}\frac{\partial A_k}{\partial x_i} + e\frac{\partial \phi}{\partial x_i} + \frac{e}{c}\frac{\partial A_i}{\partial t} + \frac{e}{c}\frac{\partial A_i}{\partial x_k}\frac{dx_k}{dt}$$

$$= e\underbrace{\left(\frac{\partial \phi}{\partial x_i} + \frac{1}{c}\frac{\partial A_i}{\partial t}\right)}_{=\,-E_i} + \frac{e}{c}\underbrace{\left(\frac{\partial A_i}{\partial x_k} - \frac{\partial A_k}{\partial x_i}\right)\frac{dx_k}{dt}}_{=\,-(\boldsymbol{v}\times(\nabla\times\boldsymbol{A}))_i\,=\,-(\boldsymbol{v}\times\boldsymbol{B})_i}$$

$$= -e\left(E_i + (\boldsymbol{v}\times\boldsymbol{B})_i\right)$$

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磁場中の水素原子は

$$\boxed{H = \frac{1}{2\mu}\left(\mathbb{P} + \frac{e}{c}\mathbb{A}\right)^2 - \underbrace{\frac{Ze^2}{r}}_{-e\phi}}$$

で記述される。

(note) $\frac{1}{2\mu}\left(\mathbb{P} + \frac{e}{c}\mathbb{A}\right)^2 = \frac{1}{2\mu}\left(\mathbb{P}^2 + \frac{e}{c}\mathbb{P}\cdot\mathbb{A} + \frac{e}{c}\mathbb{A}\cdot\mathbb{P} + \underbrace{\frac{e^2}{c^2}\mathbb{A}^2}_{e\text{の2次}(\to\text{小})}\right)$

$$\underbrace{\mathbb{P}}_{=\frac{\hbar}{i}\nabla}\cdot\mathbb{A}\,\psi = \underbrace{(\mathbb{P}\cdot\mathbb{A})}_{=0}\psi + \mathbb{A}\cdot(\mathbb{P}\psi)$$

(クーロン・ゲージ $\nabla\cdot\mathbb{A} = 0$)

$$\boxed{H \sim \frac{\mathbb{P}^2}{2\mu} - \frac{Ze^2}{r} + \frac{e}{\mu c}\mathbb{A}\cdot\mathbb{P}}$$

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• 一様磁場の場合

$\mathbb{A} = -\frac{1}{2}\mathbb{r}\times\mathbb{B}$ ととれる。

(note) $\mathbb{A} = -\frac{1}{2}(yB_z - zB_y,\ zB_x - xB_z,\ xB_y - yB_x)$

↓

$$\nabla\times\mathbb{A} = (\partial_y A_z - \partial_z A_y,\ \partial_z A_x - \partial_x A_z,\ \partial_x A_y - \partial_y A_x)$$
$$= (+\tfrac{1}{2}B_x + \tfrac{1}{2}B_x,\ B_y,\ B_z)$$
$$= (B_x,\ B_y,\ B_z)$$

$$\frac{e}{\mu c}\mathbb{A}\cdot\mathbb{P} = -\frac{e}{2\mu c}(\mathbb{r}\times\mathbb{B})\cdot\mathbb{P} = \frac{e}{2\mu c}(\mathbb{B}\times\mathbb{r})\cdot\mathbb{P}$$

(note) $(\mathbb{A}\times\mathbb{B})\cdot\mathbb{C} = \varepsilon_{ijk}A_iB_jC_k$
$= \varepsilon_{jki}A_iB_jC_k = \mathbb{A}\cdot(\mathbb{B}\times\mathbb{C})$

$$= \frac{e}{2\mu c}\mathbb{B}\cdot(\mathbb{r}\times\mathbb{P}) = \frac{e}{2\mu c}\mathbb{B}\cdot\mathbb{L}$$

$$\boxed{H = \frac{\mathbb{P}^2}{2\mu} - \frac{Ze^2}{r} + \frac{e}{2\mu c}\mathbb{B}\cdot\mathbb{L}}$$

$\mathbb{B}$の向きを$z$軸にとると

$$\boxed{H = \frac{\mathbb{P}^2}{2\mu} - \frac{Ze^2}{r^2} + \frac{e}{2\mu c}BL_z}$$

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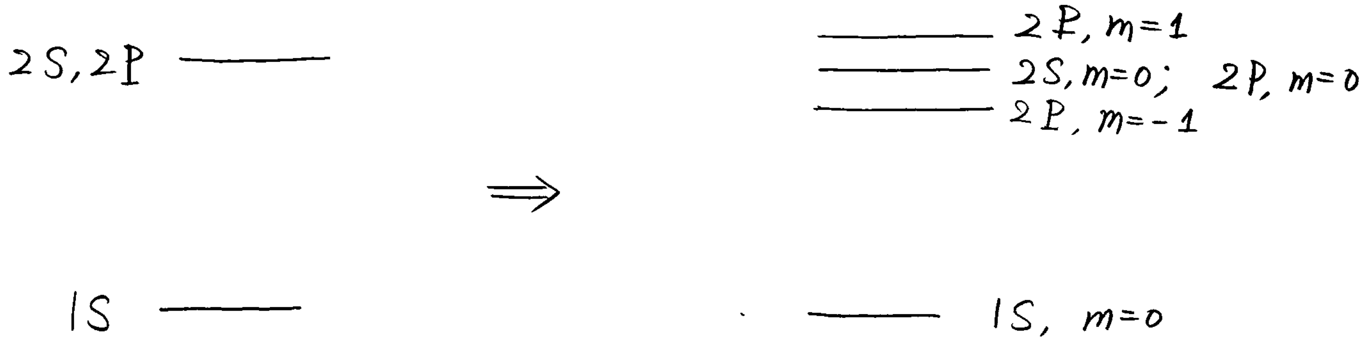
(note)

$$H_0 = \frac{\mathbf{p}^2}{2\mu} - \frac{Ze^2}{r}$$

の固有関数 $\psi_{n\ell m}(\mathbf{r}) = R_{n\ell}(r)\,Y_{\ell m}(\hat{\mathbf{r}})$
は $L_z$ の固有関数

$\Downarrow$

$$H\, R_{n\ell}(r)\,Y_{\ell m}(\hat{\mathbf{r}}) = \Big(\underbrace{-\frac{(Z\alpha)^2}{2n^2}\mu c^2}_{H_0} + \overbrace{\frac{e}{2\mu c}B}^{\omega_L\ \equiv}\cdot m\hbar\Big) \times R_{n\ell}(r)\,Y_{\ell m}(\hat{\mathbf{r}})$$




ゼーマン効果

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[note] ランダウ準位 (ポテンシャルがない場合の荷電粒子)

$\mathbb{B} = B\,\mathbb{e}_z$ は $\mathbb{A} = (0, Bx, 0)$ としても可。

$$\nabla \times \mathbb{A} = (\partial_y A_z - \partial_z A_y,\ \partial_z A_x - \partial_x A_z,\ \partial_x A_y - \partial_y A_x)$$
$$= (0, 0, B)$$

↓

$$H = \frac{1}{2m}\left(\mathbb{P} + \frac{e}{c}\mathbb{A}\right)^2 = \frac{1}{2m}\left(p_x^2 + \left(p_y + \frac{e}{c}Bx\right)^2 + p_z^2\right)$$
$$= \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2 + \frac{2e}{c}Bxp_y + \frac{e^2}{c^2}B^2x^2\right)$$

(note) $[H, p_y] = [H, p_z] = 0$

↓ $p_y, p_z, H$ の同時固有状態

○ 簡単のため, $p_z$ の固有値が 0, $p_y$ の固有値が $k\hbar$ の状態を考える

↓

$$\psi(x, y) = e^{iky}\phi(x)$$

↓

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{k^2\hbar^2}{2m} + \underbrace{\frac{eB}{mc}k\hbar\, x + \frac{e^2B^2}{2mc^2}x^2}\right)\phi(x) = E\phi(x)$$

$$= \frac{1}{2m}\left(\frac{eB}{c}\right)^2\left(x^2 + 2\cdot\frac{k\hbar c}{eB}\cdot x\right)$$

$$= \frac{1}{2m}\left(\frac{eB}{c}\right)^2\left(x + \frac{k\hbar c}{eB}\right)^2 - \frac{k^2\hbar^2}{2m}$$

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$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+\underbrace{\frac{e^2B^2}{2mc^2}\left(x+\frac{k\hbar c}{eB}\right)^2}_{\displaystyle \| \atop \displaystyle \frac{1}{2}m\omega^2(x+x_0)^2}\right)\phi(x)=E\,\phi(x)$$

$$\omega=\frac{eB}{mc}\,,\quad x_0=\frac{k\hbar c}{eB}$$

$$E_n=\hbar\omega\left(n+\frac{1}{2}\right)\qquad n=0,1,2,\cdots$$

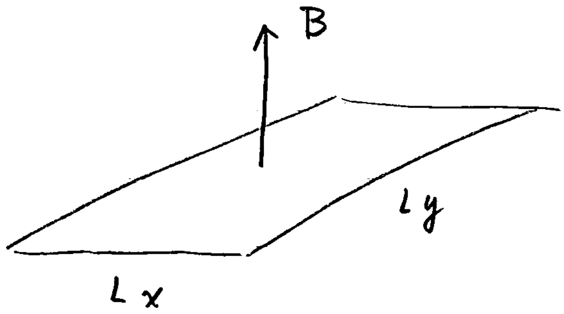
ランダウ準位.
(kには依らない)

波動関数：$\psi(x,y)=e^{iky}\,\phi_{HO}(x+x_0)$
$=e^{i\cdot\frac{eB}{\hbar c}x_0 y}\,\phi_{HO}(x+x_0)$

cf. 量子ホール効果 $\qquad k=\frac{eB}{\hbar c}x_0$

。もし電子が $L_x \times L_y$ の2次元 $xy$ 面内に閉じこめ
られているとすると

長さ




$$\psi(x, y) = \psi(x, y+L_y) \quad \text{(周期境界条件)}$$

$$\rightarrow \quad e^{iky} = e^{ik(y+L_y)}$$

$$\Downarrow \quad kL_y = 2\pi n_y \quad (n_y = 0, 1, 2, \cdots)$$

$$\Downarrow \quad \frac{eB}{\hbar c} x_0 L_y = 2\pi n_y$$

(note) $\quad 0 \leq x_0 \leq L_x$

$$\Downarrow \quad \boxed{0 \leq n_y \leq \frac{eB}{2\pi\hbar c} \underbrace{L_x L_y}_{= S}}$$

$\Downarrow$ 各 $n$ に対し、$n_y = 0, 1, \cdots, \frac{eB}{2\pi\hbar c} S$ 個のレベルが縮退。

$\longleftrightarrow$ 単位面積当たりの縮退度：$\frac{eB}{2\pi\hbar c}$