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$[L_x, L_y] = i\hbar L_z$ など
$[\mathbb{L}^2, L_z] = 0$ など

$$\hat{\ell}^2 |Y_{\ell m}\rangle = \ell(\ell+1) |Y_{\ell m}\rangle \qquad [\hat{\mathbb{L}}^2 |Y_{\ell m}\rangle = \ell(\ell+1)\hbar^2 |Y_{\ell m}\rangle$$
$$\hat{\ell}_z |Y_{\ell m}\rangle = m |Y_{\ell m}\rangle \qquad \hat{L}_z |Y_{\ell m}\rangle = m\hbar |Y_{\ell m}\rangle]$$

$$\hat{\ell}^2 = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \frac{1}{\sin^2\theta}\frac{\partial}{\partial\varphi}$$
$$\hat{\ell}_z = \frac{1}{i}\frac{\partial}{\partial\varphi}$$

$$Y_{\ell m}(\theta,\varphi) = (-)^m \sqrt{\frac{2\ell+1}{4\pi}} \sqrt{\frac{(\ell-m)!}{(\ell+m)!}} P_\ell^m(\cos\theta) e^{im\varphi}$$

$$\begin{cases} \ell : \text{整数} \\ -\ell \le m \le \ell \quad (m \text{ は } 1 \text{ ずつかわる}) \end{cases}$$

規格化：

$$\underbrace{\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta\, d\theta}_{\equiv\, d\hat{\mathbb{r}}} \; Y_{\ell m}^*(\theta,\varphi)\, Y_{\ell' m'}(\theta,\varphi) = \delta_{\ell,\ell'}\delta_{m,m'}$$

cf. $d\mathbb{r} = r^2 dr \boxed{\sin\theta\, d\theta\, d\varphi}$

- 球面調和関数

以上より l^2 と l_z の同時固有関数は

$$\boxed{Y_{lm}(\theta,\varphi) = (-1)^m \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\, P_l^m(\cos\theta)\, e^{im\varphi}}$$

球面調和関数

$$\boxed{\begin{aligned} l^2 |Y_{lm}\rangle &= l(l+1)\,|Y_{lm}\rangle \\ l_z |Y_{lm}\rangle &= m\,|Y_{lm}\rangle \end{aligned}}$$

l：整数（ゼロ又は正）
$-l \leq m \leq l$

$l=0$: $Y_{00} = \frac{1}{\sqrt{4\pi}}$

$l=1$: $Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{i\varphi}$

$Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta$

$Y_{1-1} = \sqrt{\frac{3}{8\pi}} \sin\theta\, e^{-i\varphi}$

$l=2$: $Y_{20} = \sqrt{\frac{5}{4\pi}} \cdot \frac{1}{2}(3\cos^2\theta - 1)$

※ 位相の $(-1)^m$ は慣習 (convention)

$(-)^m$ がある場合

$$e^{i\varphi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right) Y_{lm} = \pm\sqrt{(l-m)(l+m+1)}\; Y_{l\,m+1}$$

$(-)^m$ がない場合

(note)

$$r Y_{1\pm1}(\theta,\varphi) \propto r\sin\theta\, e^{\pm i\varphi}$$
$$= r\sin\theta(\cos\varphi \pm i\sin\varphi)$$
$$= x \pm iy$$

$$r Y_{10}(\theta,\varphi) \propto r\cos\theta = z$$

$$r^2 Y_{20}(\theta,\varphi) \propto 3r^2\cos^2\theta - r^2 = 3z^2 - r^2$$
$$(= 2z^2 - x^2 - y^2)$$

cf. 3次元調和振動子

$$H = \frac{\mathbb{P}^2}{2m} + \frac{1}{2}m\omega^2 \underbrace{(x^2+y^2+z^2)}_{= r^2}$$

基底状態：$(n_x, n_y, n_z) = (0,0,0)$

$$\psi_{000}(\mathbf{r}) \propto e^{-\alpha x^2} e^{-\alpha y^2} e^{-\alpha z^2} = e^{-\alpha(x^2+y^2+z^2)}$$
$$= e^{-\alpha r^2}$$
$$\left(\alpha = \frac{m\omega}{2\hbar}\right)$$

→ 角度に依存しない ↔ $l=0$ の状態

第一励起状態：$(n_x, n_y, n_z) = (1,0,0), (0,1,0), (0,0,1)$

$$\psi_{100}(\mathbf{r}) \propto x e^{-\alpha r^2},\quad \psi_{010} \propto y e^{-\alpha r^2},\quad \psi_{001} \propto z e^{-\alpha r^2}$$

$$\rightarrow \left.\begin{aligned} &\psi_{100} \pm i\psi_{010} \propto r e^{-\alpha r^2} Y_{1\pm1}(\theta,\varphi) \\ &\psi_{001} \propto r e^{-\alpha r^2} Y_{10}(\theta,\varphi) \end{aligned}\right\}$$ 第一励起状態は $l=1$ の状態

cf, 実際に固有関数になっているかチェック

(note)

$$\left[-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)-\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]\left(\sin\theta\, e^{\pm i\varphi}\right)$$

$$=\left[-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\cos\theta\right)-\frac{1}{\sin^2\theta}\cdot(-1)\sin\theta\right]e^{\pm i\varphi}$$

$$=\left[-\frac{1}{\sin\theta}\left(\underbrace{\cos^2\theta-\sin^2\theta}_{1-2\sin^2\theta}\right)+\frac{1}{\sin\theta}\right]e^{\pm i\varphi}$$

$$=\left(-\cancel{\frac{1}{\sin\theta}}+2\sin\theta+\cancel{\frac{1}{\sin\theta}}\right)e^{\pm i\varphi}$$

$$=2\sin\theta\, e^{\pm i\varphi}=1\cdot(1+1)\sin\theta\, e^{\pm i\varphi}$$

(note)

$$\left[-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)-\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]\left(3\cos^2\theta-1\right)$$

$$=-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\cdot 6\cos\theta\cdot(-\sin\theta)\right)$$

$$=\frac{6}{\sin\theta}\frac{\partial}{\partial\theta}\left(\cos\theta\underbrace{\sin^2\theta}_{1-\cos^2\theta}\right)$$

$$=\frac{6}{\sin\theta}\left(-\sin\theta-3\cos^2\theta\cdot(-\sin\theta)\right)$$

$$=6\cdot\left(3\cos^2\theta-1\right)$$

$$=2\times(2+1)\cdot\left(3\cos^2\theta-1\right)$$

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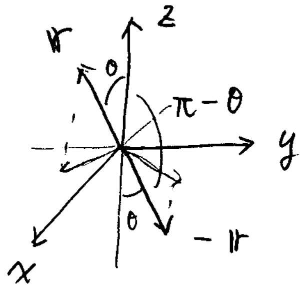
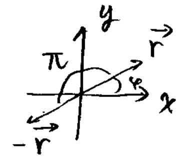
球面調和関数の性質



$$Y_{l-m}(\underbrace{\hat{r}}_{=(\theta,\varphi)}) = (-1)^m Y_{lm}(\hat{r})^*$$

重要 $\Longrightarrow$

$$Y_{lm}(-\hat{r}) = Y_{lm}(\pi-\theta, \varphi+\pi) = \underbrace{(-1)^l}_{\text{パリティ}} Y_{lm}(\hat{r})$$



{ 偶数の l は正パリティ
{ 奇数 〃 負 〃

規格化：

(note) $d\mathbf{r} = r^2 dr \boxed{\sin\theta d\theta d\varphi}$

$$\underbrace{\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta d\theta}_{\equiv \int d\hat{r}} Y^*_{lm}(\theta,\varphi) Y_{l'm'}(\theta,\varphi) = \delta_{l,l'}\delta_{m,m'}$$

$$\langle Y_{lm} | Y_{l'm'} \rangle = \delta_{ll'}\delta_{m,m'}$$

加法定理

$$\frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_{lm}(\hat{r}_1) Y_{lm}(\hat{r}_2)^* = P_l(\cos\underbrace{\gamma}_{})$$

ルジャンドル多項式

$\hat{r}_1$ と $\hat{r}_2$ のなす角

cf.
$rY_{1\pm1} \propto (x \pm iy)$

2.4 昇降演算子

(猪木・川合の7.4章)

$L_\pm = L_x \pm i L_y$ を定義する(昇降演算子)。

(note) $(L_+)^\dagger = L_-$, $(L_-)^\dagger = L_+$

$$L_+ L_- = (L_x + iL_y)(L_x - iL_y)$$
$$= L_x^2 + L_y^2 - i(L_x L_y - L_y L_x)$$
$$= L_x^2 + L_y^2 - i \underbrace{[L_x, L_y]}_{= i\hbar L_z}$$
$$= L_x^2 + L_y^2 + \hbar L_z$$

$$\Downarrow \quad \mathbb{L}^2 = L_x^2 + L_y^2 + L_z^2 = L_+ L_- + L_z^2 - \hbar L_z$$

同様に $\mathbb{L}^2 = L_- L_+ + L_z^2 + \hbar L_z$

(note) $[L_+, L_z] = [L_x + iL_y, L_z]$
$$= -i\hbar L_y + i \cdot i\hbar L_x = -\hbar L_+$$

同様に $[L_-, L_z] = \hbar L_-$

(note) $[\mathbb{L}^2, L_\pm] = 0$

$\uparrow$

$[\mathbb{L}^2, L_x] = [\mathbb{L}^2, L_y] = 0$

$[L_\pm, L_z] = \mp \hbar L_\pm$

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$$\begin{cases} \mathbb{L}^2 \,|Y_{\ell m}\rangle = \ell(\ell+1)\hbar^2 \,|Y_{\ell m}\rangle \\ L_z \,|Y_{\ell m}\rangle = m\hbar \,|Y_{\ell m}\rangle \end{cases}$$

$L_\pm |Y_{\ell m}\rangle$ という状態を考える。

$$\mathbb{L}^2 L_\pm |Y_{\ell m}\rangle \underset{\substack{\uparrow \\ [\mathbb{L}^2, L_\pm] = 0}}{=} L_\pm \mathbb{L}^2 |Y_{\ell m}\rangle = \ell(\ell+1)\hbar^2 L_\pm |Y_{\ell m}\rangle$$

∪

⤵ $L_\pm |Y_{\ell m}\rangle$ は $\mathbb{L}^2$ の固有状態で
固有値 $\ell(\ell+1)\hbar^2$

$$L_z L_\pm |Y_{\ell m}\rangle = (L_\pm L_z \pm \hbar L_\pm)|Y_{\ell m}\rangle$$
$$= (m\pm 1)\hbar \, L_\pm |Y_{\ell m}\rangle$$

⤵ $L_\pm |Y_{\ell m}\rangle$ は L_z の固有状態で
固有値は $(m\pm1)\hbar$

∪

⤵

$$L_\pm |Y_{\ell m}\rangle = \alpha_\pm |Y_{\ell\, m\pm1}\rangle$$

m が ± 1 変わる
(昇降演算子)

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• $\alpha_\pm$ の決定

$$L_\pm |Y_{\ell m}\rangle = \alpha_\pm |Y_{\ell m\pm 1}\rangle$$

$\downarrow$

$$\langle Y_{\ell m}| \underbrace{(L_\pm)^\dagger}_{\substack{\parallel \\ L_\mp L_\pm}} L_\pm |Y_{\ell m}\rangle = |\alpha_\pm|^2 \underbrace{\langle Y_{\ell m\pm 1}|Y_{\ell m\pm 1}\rangle}_{\substack{\parallel \\ 1}}$$
$$= |\alpha_\pm|^2$$

(note) $L_- L_+ = \mathbb{L}^2 - L_z^2 - \hbar L_z$
$L_+ L_- = \mathbb{L}^2 - L_z^2 + \hbar L_z$

$\downarrow$

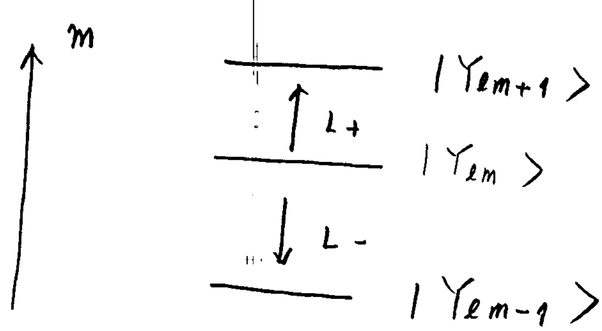
$$\text{左辺} = \ell(\ell+1)\hbar^2 - m^2\hbar^2 \mp m\hbar^2$$
$$= \hbar^2[\ell(\ell+1) - m(m\pm 1)]$$

$\downarrow$

$$\boxed{\alpha_\pm = \hbar\sqrt{\ell(\ell+1) - m(m\pm 1)}}$$

$\downarrow$

$$L_\pm |Y_{\ell m}\rangle = \hbar\sqrt{\ell(\ell+1) - m(m\pm 1)}\ |Y_{\ell m\pm 1}\rangle$$



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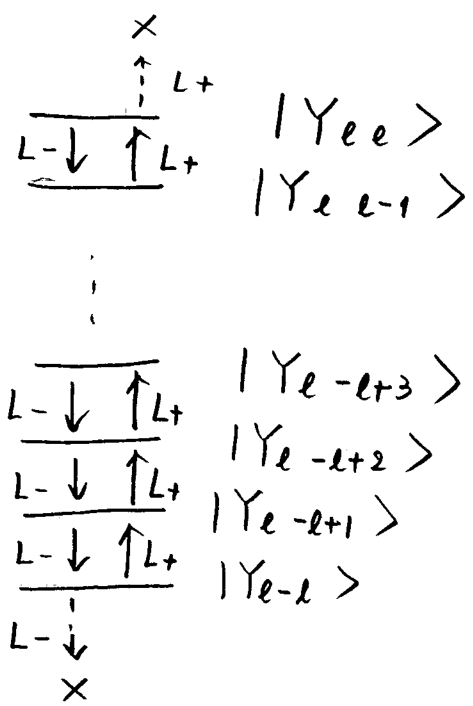
(note)

$$L_+ |Y_{\ell\ell}\rangle = \hbar\sqrt{\ell(\ell+1)-\ell(\ell+1)}\,|Y_{\ell\,\ell+1}\rangle$$
$$= 0$$
$$L_- |Y_{\ell\,-\ell}\rangle = \hbar\sqrt{\ell(\ell+1)+\ell(-\ell-1)}\,|Y_{\ell\,-\ell-1}\rangle$$
$$= 0$$



※角運動量の合成のときに重要となる

※半整数スピンの場合にも拡張可.

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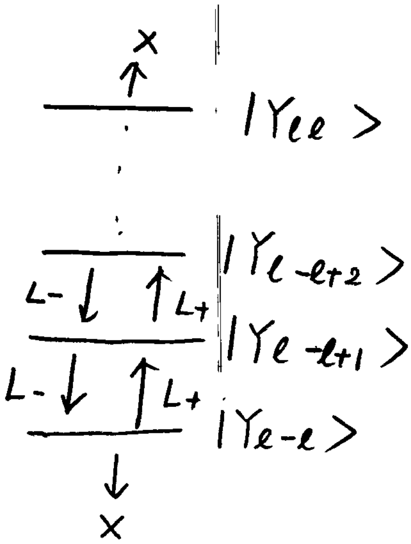
(note) 調和振動子との比較

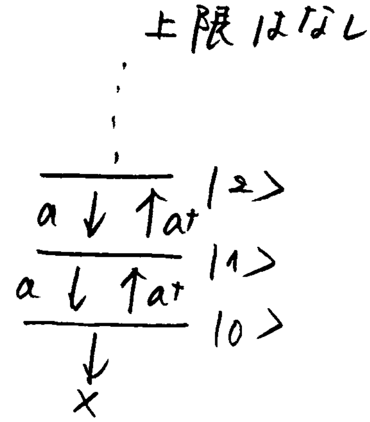
$$\mathbb{L}^2 = L_+ L_- + L_z^2 - \hbar L_z \qquad H = \left(a^\dagger a + \frac{1}{2}\right)\hbar\omega$$

$$L_- |Y_{l\,-l}\rangle = 0 \qquad a|0\rangle = 0$$

$$L_+ |Y_{lm}\rangle \propto |Y_{l\,m+1}\rangle \qquad a^\dagger |n\rangle \propto |n+1\rangle$$

$$L_- |Y_{lm}\rangle \propto |Y_{l\,m-1}\rangle \qquad a|n\rangle \propto |n-1\rangle$$





$$|Y_{l\,-l+n}\rangle \propto (L_+)^n |Y_{l\,-l}\rangle \qquad |n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle$$

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(note) m について:

$$L_\pm |Y_{lm}\rangle = \underbrace{\hbar\sqrt{l(l+1)-m(m\pm1)}}_{=\alpha_\pm} |Y_{l\,m\pm1}\rangle$$

⇓ $l(l+1) \geq m(m\pm1)$ （$\sqrt{\ }$ の中身）

→ $\boxed{-l \leq m \leq l}$

最小の m を m_{min} とする。

→ $L_-|Y_{l\,m_{min}}\rangle = 0$

⇓ $m_{min} = -l$

同様に最大の m は $L_+|Y_{l\,m_{max}}\rangle = 0$ より

$m_{max} = +l$

$L_\pm$ により m の値は 1 ずつ変わる

⇓ $m = -l, -l+1, \cdots, l-1, l$

この条件のみからは l として $\begin{cases} \text{整数} \\ \text{半整数} \end{cases}$ の両方の場合

が許される。 cf. スピン

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(note) $\hat{\ell}^2$ の固有値が $\ell(\ell+1)$ となること：

$\hat{\ell}^2$ の固有値を λ とする。

$$\hat{L}_+ |Y_{\ell\ell}\rangle = 0 \rightarrow \underbrace{\hat{L}_- \hat{L}_+}_{\shortparallel \atop \hat{\mathbb{L}}^2 - \hat{L}_z^2 - \hbar L_z} |Y_{\ell\ell}\rangle = 0$$

$$= (\lambda - m^2 - m)\hbar^2 |Y_{\ell\ell}\rangle = 0$$

$$(m = \ell)$$

$\curlyvee$

$$\lambda = \ell^2 + \ell = \ell(\ell+1)$$