

波動関数の漸近形

$$\psi(r) \xrightarrow{r \rightarrow \infty} e^{ik \cdot r} + \underbrace{f(\theta)}_{\text{散乱振幅}} \frac{e^{ikr}}{r}$$

微分散乱断面積 $\frac{d\sigma}{d\Omega} = |f(\theta)|^2$

(note)

$$\psi(r) \rightarrow \frac{i}{2k} \sum_l (2l+1) i^l \frac{1}{r} \left[\underbrace{e^{-i(kr - \frac{l\pi}{2})}}_{\text{内向波}} - S_l \underbrace{e^{i(kr - \frac{l\pi}{2})}}_{\text{外向波}} \right] \times P_l(\cos\theta)$$

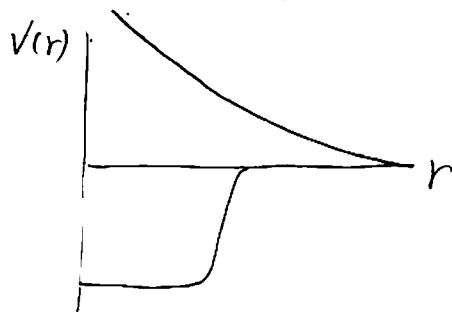
$$f(\theta) = \sum_l \frac{(2l+1)}{2} \frac{S_l - 1}{2ik} P_l(\cos\theta)$$

$$\begin{aligned} \sigma &= \int d\Omega \frac{d\sigma}{d\Omega} = \frac{\pi}{k^2} \sum_l (2l+1) |e^{2i\delta_l} - 1|^2 \\ &= \frac{\pi}{k^2} \sum_l (2l+1) \sin^2 \delta_l \end{aligned}$$

5.4. 低エネルギー - 散乱

○ 一般的な考察

$$\left[-\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} + \frac{l(l+1)\hbar^2}{2\mu r^2} + V(r) - E \right] u_l(r) = 0$$



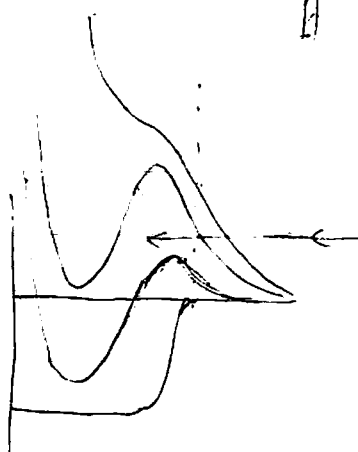
as $l \rightarrow \infty$
 $\frac{l(l+1)\hbar^2}{2\mu r^2} \gg V(r)$
 $\delta_l \rightarrow 0$
 less important
 cf. 低エネルギー近似

(note) black disk



$$L_{max} = kR$$

(note)



反応が起るためには
 interaction range まで
 到達する必要がある

(note) $E \rightarrow 0$

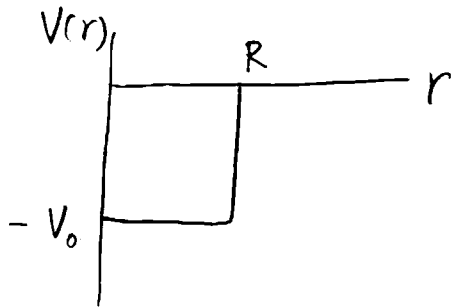
only $l=0$ のみ寄与



高エネルギー近似 $\delta_l = 0$

• $E \rightarrow 0$ の振る舞い:

(15') square well potential



$$\psi(r) = \frac{u(r)}{r} Y_{00}(\hat{r})$$

$$r < R : u(r) = A \sin \tilde{k} r$$

$$r \geq R : u(r) = B \sin(kr + \delta)$$

$$k = \sqrt{\frac{2m}{\hbar^2} E}, \quad \tilde{k} = \sqrt{\frac{2m}{\hbar^2} (E + V_0)}$$

$r = R$ の matching:

$$\begin{cases} A \sin \tilde{k} R = B \sin(kR + \delta) = B (\sin kR \cos \delta + \cos kR \sin \delta) \\ \tilde{k} A \cos \tilde{k} R = k B \cos(kR + \delta) \\ \qquad \qquad \qquad \sim k B (\cos \delta - kR \sin \delta) \qquad \qquad B (kR \cos \delta + \sin \delta) \end{cases}$$

\rightarrow

$$\frac{1}{\tilde{k}} \tan \tilde{k} R = \frac{1}{k} \cdot \frac{\sin \delta + kR \cos \delta}{\cos \delta - kR \sin \delta}$$

$$\sim \frac{1}{k} \tan \delta \sim \frac{1}{k} \delta$$

$$\rightarrow \delta \propto k$$

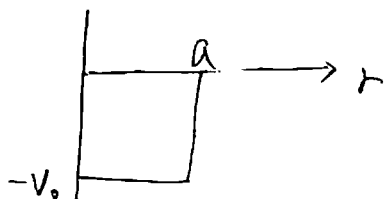
一般に $\delta_l \propto k^{2l+1}$

$$\uparrow \quad j_l(kr) \sim \frac{(kr)^l}{(2l+1)!!} \quad (k \rightarrow 0)$$

$$\rightarrow \delta_{l=0} \gg \delta_{l=1} \gg \delta_{l=2} \gg \dots \quad (k \sim 0)$$

- $E \rightarrow 0$ の振るまい (Lの値: threshold の振るまい)
(一般のLの場合)

(例) square well potential



$$\psi(r) = R_l(r) Y_{lm}(\hat{r})$$

$$r < a$$

$$R_l(r) = A j_l(kr)$$

$$r > a$$

$$R_l(r) = B j_l(kr) + C n_l(kr)$$

$$k = \sqrt{\frac{2m}{\hbar^2} (E + V_0)}, \quad k = \sqrt{\frac{2m}{\hbar^2} E}$$

(note)

$$R_l(r) \rightarrow \left[B \sin\left(kr - \frac{l\pi}{2}\right) - C \cos\left(kr - \frac{l\pi}{2}\right) \right] \cdot \frac{1}{kr}$$

$$\Leftrightarrow \frac{1}{kr} \sin\left(kr - \frac{l\pi}{2} + \delta_l\right) = \frac{1}{kr} \left[\sin\left(kr - \frac{l\pi}{2}\right) \cos \delta_l + \cos\left(kr - \frac{l\pi}{2}\right) \sin \delta_l \right]$$

$$\Rightarrow \boxed{\tan \delta_l = -\frac{C}{B}}$$

~~~~~

matching at  $r=a$ :

$$\frac{R_l'}{R_l} = \frac{k j_l'(ka)}{j_l(ka)} = k \frac{j_l'(ka) + \frac{C}{B} n_l'(ka)}{j_l(ka) + \frac{C}{B} n_l(ka)}$$

$$\Rightarrow \tan \delta_l = -\frac{C}{B} = \frac{k j_l'(ka) j_l(ka) - k j_l(ka) j_l'(ka)}{k n_l(ka) j_l(ka) - k n_l(ka) j_l'(ka)}$$

$ka \ll l$  の極限:

$$j_l(ka) \sim \frac{(ka)^l}{(2l+1)!!}, \quad n_l(ka) \sim -\frac{(2l-1)!!}{(ka)^{l+1}}$$

↓

$$\begin{aligned} \tan \delta_l &\sim \frac{k \cdot \frac{l(ka)^{l-1}}{(2l+1)!!} j_l(ka) - k \cdot \frac{(ka)^l}{(2l+1)!!} j_l'(ka)}{k \cdot (l+1) \frac{(2l-1)!!}{(ka)^{l+2}} j_l(ka) + k \cdot \frac{(2l-1)!!}{(ka)^{l+1}} j_l'(ka)} \\ &= \frac{2l+1}{[(2l+1)!!]^2} \cdot (ka)^{2l+1} \frac{k l j_l(ka) - k \cdot ka j_l'(ka)}{k(l+1) j_l(ka) + k ka j_l'(ka)} \\ &\equiv -C_l k^{2l+1} \end{aligned}$$

(square well potential 以外 7.6 式) )

↓

$$\boxed{\tan \delta_l \sim \delta_l \sim -C_l k^{2l+1}} \quad (k \rightarrow 0)$$

(note)

$$\text{as } k \rightarrow 0$$

$$\delta_{l=0} \gg \delta_{l=1} \gg \delta_{l=2} \dots$$

(note)

$$\begin{aligned} \sigma_{\text{tot}} &= \frac{4\pi}{k^2} \sum_l (2l+1) \sin^2 \delta_l \sim \frac{4\pi}{k^2} \sum_l (2l+1) \delta_l^2 \\ &\sim \frac{4\pi}{k^2} \sum_l (2l+1) C_l^2 k^{4l+2} \\ &\rightarrow 4\pi C_{l=0}^2 \quad (k \rightarrow 0) \end{aligned}$$

• 散乱長

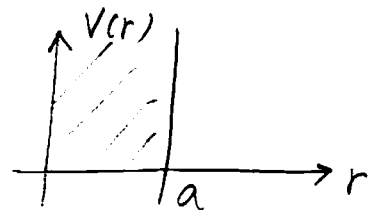
$$\frac{1}{k} \tan \delta \sim \frac{1}{\tilde{k}} \tan \tilde{k} R$$

↪

$$k \cot \delta = \tilde{k} \cot \tilde{k} R \equiv -\frac{1}{a} \quad \text{散乱長}$$

(note) 剛性球'に及ぶ散乱 ( $l=0$ )

$$V(r) = \begin{cases} \infty & (r < a) \\ 0 & (r \geq a) \end{cases}$$



$$U(r) = \sin(kr + \delta)$$

$$U(r=a) = 0 = \sin(ka + \delta)$$

$$\Rightarrow \boxed{\delta = -ka}$$

$$\left( k \cot \delta \sim \frac{k}{\delta} = -\frac{1}{a} \right)$$

◦ effective range (有効距離)

一般に  $k \cot \delta \sim -\frac{1}{a} + \frac{1}{2} r_0 k^2 + \dots$

↓ 低エネルギー-散乱は  $a, r_0$  の2個のパラメーターで記述でき、ポテンシャルの詳細によらない。

↓

$V(r)$  の座標依存性をゼロにするためには高エネルギー-散乱が必要。

• effective interaction (pseudo potential)

$$k \cot \delta \sim -\frac{1}{a} + \frac{1}{2} k^2 r_0 + \dots$$

低エネルギーではポテンシャルの詳細は重要ではない。

$$\leadsto V_{\text{eff}}(r) = \frac{2\pi \hbar^2}{m_r} a \delta(r)$$

reduced mass

係数は散乱長さ  $a$  に依存するように決定。

cf. BEC (アルカリ原子の希薄気体)

cf. H. Esbensen et al. PRC 56(1997) 3054

Fetter - Walecka

Eqs. (11.14), (11.53)

|     |                                      |       |
|-----|--------------------------------------|-------|
| 原子  | $1 \text{ \AA} = 10^{-8} \text{ cm}$ | eV    |
| 原子核 | $1 \text{ fm} = 10^{-13} \text{ cm}$ | MeV   |
| 素粒子 | $\leq 10^{-16} \text{ cm}$           | > GeV |



## 5.5 共鳴散乱

$$\sigma_{\text{tot}} = \frac{4\pi}{k^2} \sum_l (2l+1) \sin^2 \delta_l$$

$$\sigma_l \equiv \frac{4\pi}{k^2} (2l+1) \sin^2 \delta_l \leq \frac{4\pi}{k^2} (2l+1)$$

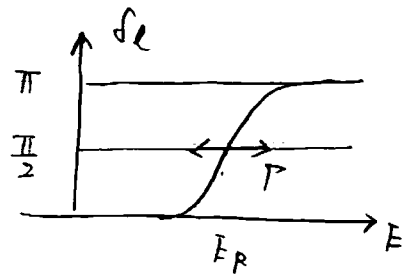
↑ 等号は  $\delta_l = (n + \frac{1}{2})\pi$  の時

assume  $\delta_l(E = E_R) = \frac{\pi}{2}$

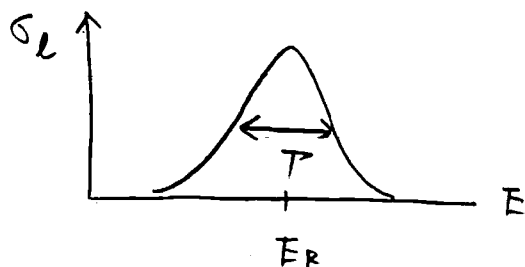
$$\cot \delta_l(E) \sim \cot \underbrace{\delta_l(E_R)}_{\frac{\pi}{2}} - \frac{2}{\Gamma} (E - E_R) + \dots$$

$$= -\frac{2}{\Gamma} (E - E_R)$$

$$-\frac{2}{\Gamma} = \frac{d}{dE} \cot \delta_l \Big|_{E=E_R}$$



$$\begin{aligned} \sigma_l(E) &= \frac{4\pi}{k^2} (2l+1) \cdot \frac{1}{1 + \cot^2 \delta_l} = \frac{4\pi}{k^2} (2l+1) \cdot \frac{1}{1 + \frac{4}{\Gamma^2} (E - E_R)^2} \\ &= \frac{4\pi}{k^2} \cdot (2l+1) \cdot \frac{\frac{\Gamma^2}{4}}{\frac{\Gamma^2}{4} + (E - E_R)^2} \end{aligned}$$



(Breit-Wigner  
の公式)

$\Gamma$ : 共鳴幅

$V(r)=0$  の場合の 2 つの独立解:

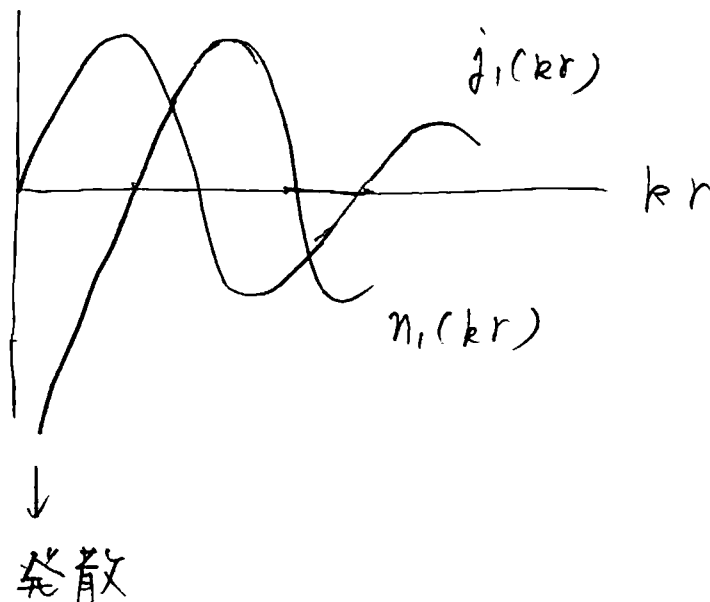
$j_l(kr)$ ,  $n_l(kr)$

$$j_l(kr) \rightarrow \frac{1}{kr} \sin\left(kr - \frac{l\pi}{2}\right) \quad (r \rightarrow \infty)$$

$$\sim \frac{(kr)^l}{(2l+1)!!} \quad (r \rightarrow 0)$$

$$n_l(kr) \rightarrow \frac{1}{kr} \cos\left(kr - \frac{l\pi}{2}\right) \quad (r \rightarrow \infty)$$

$$\sim -\frac{(2l-1)!!}{(kr)^{l+1}} \quad (r \rightarrow 0)$$

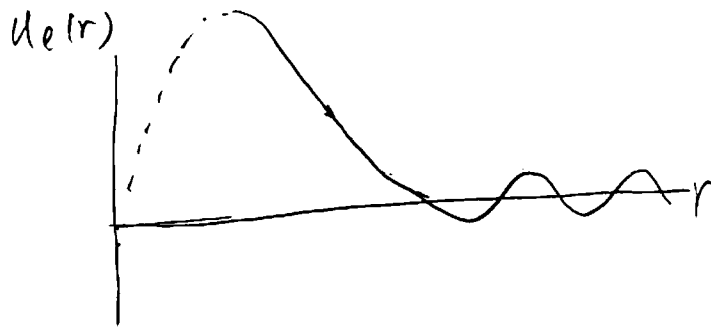


位相のずれ:

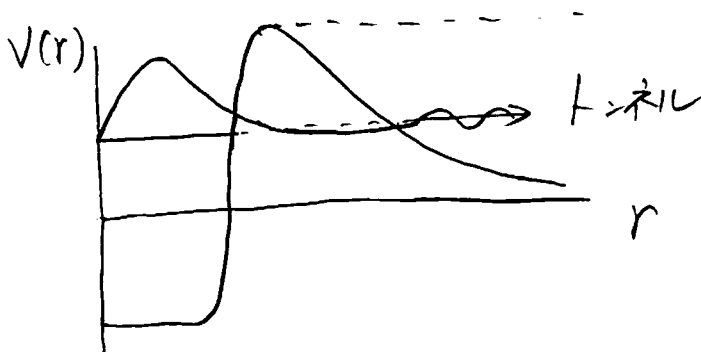
$$u_e(r) \rightarrow \sin(kr - \frac{\ell\pi}{2} + \delta_\ell)$$

$$\delta_\ell = \frac{\pi}{2} \ell \text{ と}$$

$$u_e(r) \rightarrow \sin(kr - \frac{\ell\pi}{2} + \frac{\pi}{2}) = \cos(kr - \frac{\ell\pi}{2}) \\ = n_e(kr)$$



準束縛状態



↔ 共鳴散乱

共鳴からずれたエネルギーでは

