Department of Physics
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2.3. リップマン・シュウィンガー方程式

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi = E\psi$$

$$\rightarrow \left(\underbrace{-\frac{\hbar^2}{2m}\nabla^2}_{=\hat{H}_0} - E\right)\psi = -V\psi$$

形式解

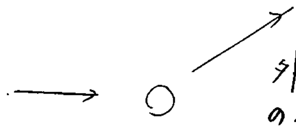
$$\boxed{\psi = \phi - \frac{1}{\hat{H}_0 - E - i\eta} V\psi}$$

リップマン・シュウィンガー方程式

$(\hat{H}_0 - E)\phi = 0$

η：正の微小量

(note) グリーン関数



$$\hat{G}^{(+)} = +\frac{1}{\hat{H}_0 - E - i\eta}$$

外向きの境界条件

座標表示では

$$G^{(+)}(\mathbf{r}, \mathbf{r}') = +\langle \mathbf{r} | \frac{1}{\hat{H}_0 - E - i\eta} | \mathbf{r}' \rangle$$

$$= \frac{2m}{\hbar^2} \cdot \frac{1}{4\pi} \frac{e^{+ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \qquad \left(k = \sqrt{\frac{2m}{\hbar^2}E}\right)$$

(http://www.nucl.phys.tohoku.ac.jp/~hagino/lecture2/konan08/green.pdf を参照)

$$\int \frac{d\mathbf{k}}{(2\pi)^3}\, e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} = \delta(\mathbf{r}-\mathbf{r}')$$

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（参考） グリーン関数

$$G^{(+)}(\mathbf{r},\mathbf{r}') = \langle \mathbf{r}| \frac{1}{\hat{H}_0 - E - i\eta} |\mathbf{r}'\rangle$$

$$\langle \mathbf{r}|\hat{G}^{(+)}|\mathbf{r}'\rangle = \int d\mathbf{k}' \langle \mathbf{r}|\mathbf{k}'\rangle \frac{1}{\frac{k'^2\hbar^2}{2m} - \frac{k^2\hbar^2}{2m} - i\eta} \langle \mathbf{k}'|\mathbf{r}'\rangle$$

$$= \int \frac{d\mathbf{k}'}{(2\pi)^3} e^{i\mathbf{k}'\cdot\mathbf{r}} \cdot \frac{2m}{\hbar^2} \frac{1}{k'^2 - k^2 - i\eta} e^{-i\mathbf{k}'\cdot\mathbf{r}'}$$

$$= \frac{2m}{\hbar^2}\cdot\frac{1}{(2\pi)^3} \int k'^2 dk' d\hat{k}'\, e^{ik's\cos\theta} \frac{1}{k'^2 - k^2 - i\eta}$$

$$(\mathbf{s} \equiv \mathbf{r} - \mathbf{r}')$$

$$= \frac{1}{(2\pi)^3}\cdot\frac{2m}{\hbar^2} \int_0^\infty k'^2 dk' \cdot \underbrace{2\pi\int_{-1}^{1} d(\cos\theta) \frac{e^{ik's\cos\theta}}{k'^2 - k^2 - i\eta}}_{\frac{2\pi}{k'^2-k^2-i\eta}\cdot\frac{1}{ik's}\left(e^{ik's} - e^{-ik's}\right)}$$

$$= \frac{1}{i}\cdot\frac{1}{(2\pi)^2}\cdot\frac{2m}{\hbar^2} \underbrace{\int_0^\infty dk' \frac{k'dk'}{k'^2-k^2-i\eta}\left(\frac{e^{ik's}}{s} - \frac{e^{-ik's}}{s}\right)}_{\int_{-\infty}^{\infty} dk' \frac{k'dk'}{k'^2-k^2-i\eta}\cdot\frac{e^{ik's}}{s}}$$

$$= \int_{-\infty}^{\infty} dk' \frac{k'dk'}{(k'+k+i\eta)(k'-k-i\eta)}\cdot\frac{e^{ik's}}{s}$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} dk' \left(\frac{1}{k'+k+i\eta} + \frac{1}{k'-k-i\eta}\right)\cdot\frac{e^{ik's}}{s}$$

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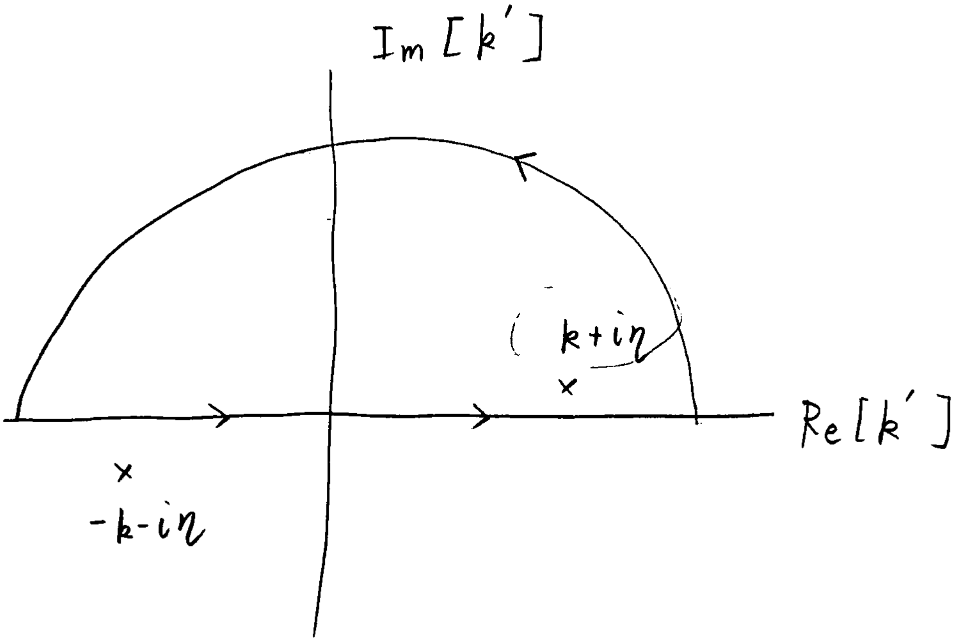

2 $$G^{(+)}(\mathbf{r}, \mathbf{r}') = \frac{1}{\cancel{i}} \cdot \frac{1}{(2\pi)^2} \cdot \frac{2m}{\hbar^2} \cdot \frac{1}{2} \cdot 2\pi \cancel{i} \cdot \frac{e^{iks}}{s}$$

$$= \frac{m}{2\pi \hbar^2} \cdot \frac{e^{iks}}{s}$$

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(note)

$$\psi(\mathbf{r}) = \phi(\mathbf{r}) - \int d\mathbf{r}'\, G^{(+)}(\mathbf{r},\mathbf{r}')\, V(\mathbf{r}')\, \psi(\mathbf{r}')$$

$$= e^{i\mathbf{k}\cdot\mathbf{r}} - \frac{2m}{\hbar^2}\cdot\frac{1}{4\pi}\int d\mathbf{r}'\, \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\, V(\mathbf{r}')\, \psi(\mathbf{r}')$$

$r \to \infty$ では

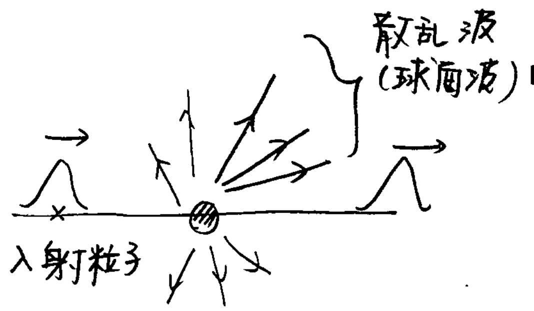
$$k|\mathbf{r}-\mathbf{r}'| = k\sqrt{r^2 - 2\mathbf{r}\cdot\mathbf{r}' + r'^2} \sim kr - \underbrace{\left(k\frac{\mathbf{r}}{r}\right)}_{=\ \mathbf{k}'}\cdot\mathbf{r}'$$

$$\downarrow$$

$$\psi(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} \underbrace{- \frac{m}{2\pi\hbar^2}\int d\mathbf{r}'\, e^{-i\mathbf{k}'\cdot\mathbf{r}'}\, V(\mathbf{r}')\, \psi(\mathbf{r}')}_{=\ f(\theta)\ \text{散乱振幅}} \cdot \frac{e^{ikr}}{r}$$

$|\mathbf{r}-\mathbf{r}'| \sim r$

$$= \underbrace{e^{i\mathbf{k}\cdot\mathbf{r}}}_{\text{入射波}} + \underbrace{f(\theta)\frac{e^{ikr}}{r}}_{\text{散乱波}}$$



ボルン近似

$$f(\theta) = -\frac{m}{2\pi\hbar^2}\int d\mathbf{r}'\, e^{-i\mathbf{k}'\cdot\mathbf{r}'}\, V(\mathbf{r}')\, \psi(\mathbf{r}')$$

$$\psi(\mathbf{r}') \simeq \phi(\mathbf{r}') = e^{i\mathbf{k}\cdot\mathbf{r}'}$$

$$= -\frac{m}{2\pi\hbar^2}\int d\mathbf{r}'\, e^{-i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}'}\, V(\mathbf{r}')$$

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2$$ ← フェルミの黄金則による導出と一致.

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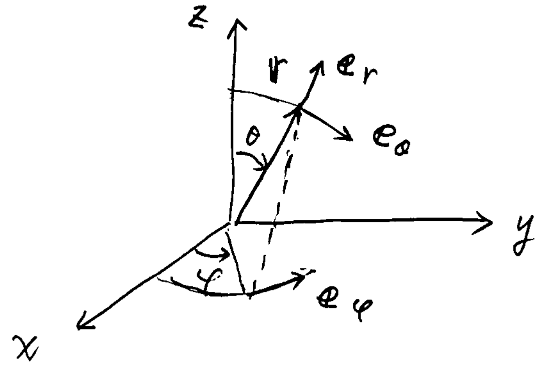
∘ $\dfrac{d\sigma}{d\Omega} = |f(\theta)|^2$ の導出

散乱波 $\psi_{sc}(\mathbf{r}) \sim f(\theta)\dfrac{e^{ikr}}{r}$ に伴うフラックス

$$\mathbf{j} = \frac{\hbar}{2im}\left[\psi_{sc}^* \nabla \psi_{sc} - c.c.\right]$$

(note) $\boxed{\nabla = \mathbf{e}_r \dfrac{\partial}{\partial r} + \dfrac{1}{r}\mathbf{e}_\theta \dfrac{\partial}{\partial \theta} + \dfrac{1}{r\sin\theta}\mathbf{e}_\varphi \dfrac{\partial}{\partial \varphi}}$



$$\therefore\ \mathbf{j} = \frac{\hbar}{2im}\left[f^*(\theta)\frac{e^{-ikr}}{r}\left(\mathbf{e}_r\partial_r + \frac{1}{r}\mathbf{e}_\theta\partial_\theta\right)f(\theta)\frac{e^{ikr}}{r} - c.c.\right]$$

$$= \frac{\hbar}{2im}\left[f^*(\theta)\frac{e^{-ikr}}{r}\left\{f(\theta)\left(\frac{e^{ikr}}{r}\cdot ik - \frac{1}{r^2}e^{ikr}\right)\mathbf{e}_r + \frac{e^{ikr}}{r^2}f'(\theta)\mathbf{e}_\theta\right\} - c.c.\right]$$

$$\sim \frac{\hbar}{2im}\cdot ik\frac{|f(\theta)|^2}{r^2}\cdot 2\mathbf{e}_r \quad (r\to\infty)$$

$$= \frac{k\hbar}{m}\frac{|f(\theta)|^2}{r^2}\mathbf{e}_r$$

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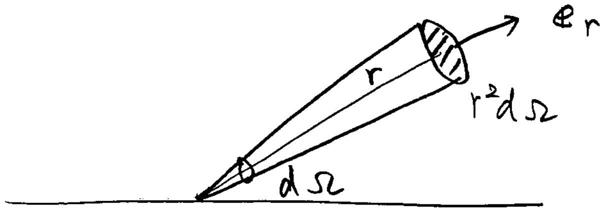

↓

単位時間に立体角 $d\Omega$ に散乱される
粒子の数

$$\frac{k\hbar}{m}\,\frac{|f(\theta)|^2}{\cancel{r^2}}\cdot\cancel{r^2}\,d\Omega$$

↓

散乱断面積：

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{1}{j_{in}}\cdot\frac{k\hbar}{m}\,|f(\theta)|^2 = |f(\theta)|^2}$$