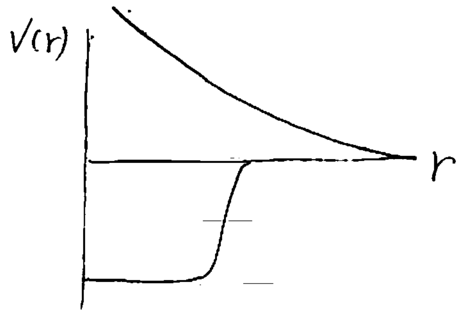
5.4. 低エネルギー散乱

○一般的な考察

$$\left[-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2}+\frac{l(l+1)\hbar^2}{2\mu r^2}+V(r)-E\right]u_l(r)=0$$



as $l \to \infty$

$$\frac{l(l+1)\hbar^2}{2\mu r^2} \gg \underset{\text{less important}}{V(r)}$$

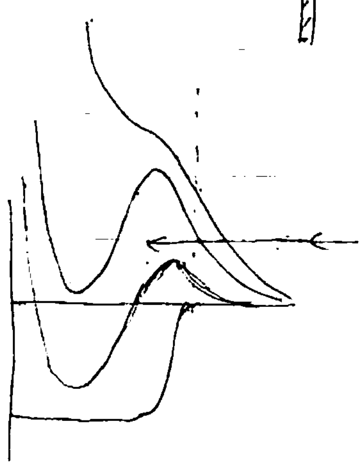
$\Updownarrow$

$\delta_l \to 0$　　cf. ボルン近似

(note) black disk

$L_{max} = kR$

(note)

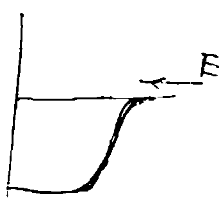


反応が起きるためには interaction range までトンネルする必要あり

(note) $E \to 0$ only $l=0$ のみ寄与

$\downarrow$

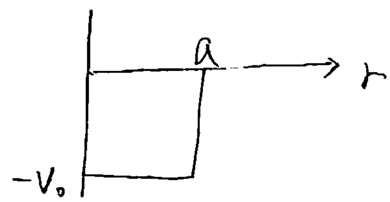
遠心力バリア $=0$



∘ $E \to 0$ での振るまい（しきい値：threshold の振るまい）

（例）square well potential

$$\psi(\mathbf{r}) = R_\ell(r)\, Y_{\ell m}(\hat{\mathbf{r}})$$



$r < a$

$$R_\ell(r) = A\, j_\ell(Kr)$$

$r \geq a$

$$R_\ell(r) = B\, j_\ell(kr) + C\, n_\ell(kr)$$

$$K = \sqrt{\frac{2m}{\hbar^2}(E+V_0)}, \quad k = \sqrt{\frac{2m}{\hbar^2}E}$$

(note)

$$R_\ell(r) \to \left[B \sin\left(kr - \frac{\ell\pi}{2}\right) - C\cos\left(kr - \frac{\ell\pi}{2}\right)\right]\cdot\frac{1}{kr}$$

$$\Leftrightarrow \frac{1}{kr}\sin\left(kr - \frac{\ell\pi}{2} + \delta_\ell\right) = \frac{1}{kr}\left[\sin\left(kr - \frac{\ell\pi}{2}\right)\cos\delta_\ell + \cos\left(kr - \frac{\ell\pi}{2}\right)\sin\delta_\ell\right]$$

$$\rightarrow \boxed{\tan\delta_\ell = -\frac{C}{B}}$$

matching at $r = a$:

$$\frac{R_\ell'}{R_\ell} = \frac{K\, j_\ell'(Ka)}{j_\ell(Ka)} = k\,\frac{j_\ell'(ka) + \frac{C}{B} n_\ell'(ka)}{j_\ell(ka) + \frac{C}{B} n_\ell(ka)}$$

$$\rightarrow \tan\delta_\ell = -\frac{C}{B} = \frac{k\, j_\ell'(ka)\, j_\ell(Ka) - K\, j_\ell'(Ka)\, j_\ell(ka)}{k\, n_\ell'(ka)\, j_\ell(Ka) - K\, n_\ell(ka)\, j_\ell'(Ka)}$$

$ka \ll l$ の極限:

$$j_l(ka) \sim \frac{(ka)^l}{(2l+1)!!}, \quad n_l(ka) \sim -\frac{(2l-1)!!}{(ka)^{l+1}}$$

$\downarrow$

$$\tan\delta_l \sim \frac{k \cdot \frac{l(ka)^{l-1}}{(2l+1)!!} j_l(Ka) - K \cdot \frac{(ka)^l}{(2l+1)!!} j_l'(Ka)}{k \cdot (l+1) \frac{(2l-1)!!}{(ka)^{l+2}} \cdot j_l(Ka) + K \cdot \frac{(2l-1)!!}{(ka)^{l+1}} j_l'(Ka)}$$

$$= \frac{2l+1}{[(2l+1)!!]^2} \cdot (ka)^{2l+1} \frac{kl \; j_l(Ka) - K \cdot ka \, j_l'(Ka)}{k(l+1) j_l(Ka) + kKa \, j_l'(Ka)}$$

$$\equiv -C_l \, k^{2l+1}$$

(square well potential 以外でも成り立つ)

$\downarrow$

$$\boxed{\tan\delta_l \sim \delta_l \sim -C_l \, k^{2l+1}} \qquad (k \to 0)$$

(note)

as $k \to 0$

$$\delta_{l=0} \gg \delta_{l=1} \gg \delta_{l=2} \cdots$$

(note)

$$\sigma_{tot} = \frac{4\pi}{k^2} \sum_l (2l+1) \sin^2\delta_l \sim \frac{4\pi}{k^2} \sum_l (2l+1) \delta_l^2$$

$$\sim \frac{4\pi}{k^2} \sum_l (2l+1) C_l^2 \, k^{4l+2}$$

$$\longrightarrow 4\pi C_{l=0}^2 \qquad (k \to 0)$$

◦ 散乱長

$$\tan \delta_l = - c_l \, k^{2l+1}$$

$l = 0$:

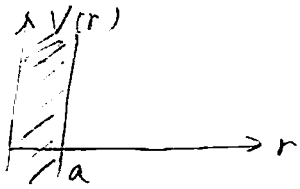
$$k \cot \delta_l = - \frac{1}{c_l} \equiv - \frac{1}{a}$$

散乱長

$$\sigma_{tot} = 4\pi a^2$$

(note) 剛体球による散乱 (s-wave)

$$V(r) = \begin{cases} \infty & (r < a) \\ 0 & (r > a) \end{cases}$$



$$u(r) = \sin(kr + \delta_0)$$

$$u(r=a) = 0 \quad \Rightarrow \quad \boxed{\delta_0 = -ka}$$

$$j_0(x) = \frac{\sin x}{x}$$
$$j_0'(x) = \frac{\cos x}{x} - \frac{\sin x}{x^2}$$

・散乱長の意味

square well ($l=0$):

$$\tan\delta_0 = kR \cdot \frac{-kR\, j_0'(kR)}{j_0(kR) + kR\, j_0'(kR)}$$

$$= kR \cdot \frac{-kR \cdot \left(\frac{\cos kR}{kR} - \frac{\sin kR}{k^2R^2}\right)}{\cancel{\frac{\sin kR}{kR}} + \cancel{kR}\left(\frac{\cos kR}{\cancel{kR}} - \cancel{\frac{\sin kR}{k^2R^2}}\right)}$$

$$= kR\left(\frac{\tan kR}{kR} - 1\right)$$

$\downarrow$

$$\boxed{a = R\left(1 - \frac{\tan kR}{kR}\right)}$$

(note)

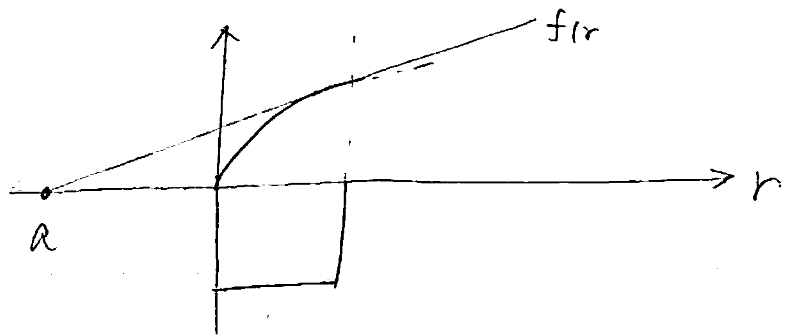
$$u_0(r) = A\sin kr \qquad (r<R)$$

$$f(r) \equiv u_0(R) + u_0'(R)(r-R)$$

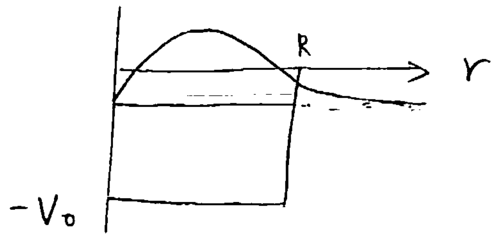
$$= A\left(\sin kR + k\cos kR\cdot(r-R)\right)$$

$\downarrow$

$$f(r=a) = A\left(\sin kR + k\cos kR\cdot\left(\cancel{R} - \frac{\tan kR}{k} - \cancel{R}\right)\right) = 0$$



(note) 束縛状態からのアプローチ　　(E < 0)



$r < R$

$$u(r) = A\sin k'r$$

$r > R$

$$u(r) = Be^{-\tilde{k}r}$$

$$k' = \sqrt{\frac{2m}{\hbar^2}(E+V_0)}\ ,\quad \tilde{k} = \sqrt{\frac{2m}{\hbar^2}|E|}$$

matching at $r = R$:

$$\frac{k'\cos k'R}{\sin k'R} = \frac{-\tilde{k}e^{-\tilde{k}R}}{e^{-\tilde{k}R}}$$

$\Downarrow$

$$\boxed{k'\cot k'R = -\tilde{k}} \quad \leftrightarrow \ -\frac{1}{a}$$

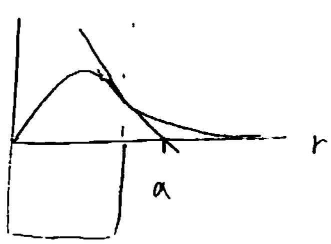
$\Downarrow$

$$a = R\left(1 - \frac{\tan kR}{kR}\right) \approx R\left(1 - \frac{\tan k'R}{k'R}\right)$$

$$= R + \frac{1}{\tilde{k}} \approx \frac{1}{\tilde{k}}$$

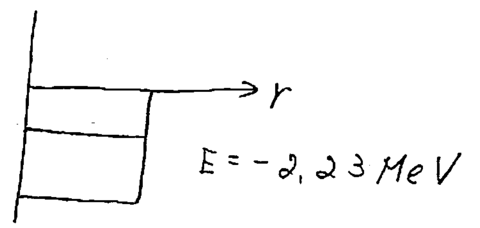
(zero range limit : $R \to 0$)

$\Downarrow$ 弱束縛状態がある場合は, $a > 0$
ない場合は $a < 0$



・陽子-中性子 散乱

重陽子



$S = 1$ (spin triplet)

⇓

$$a = \frac{1}{k} = \sqrt{\frac{\hbar^2}{2|m|E}} = \sqrt{\frac{\hbar^2 c^2}{\cancel{2}\,\frac{m_N c^2}{\cancel{2}}|E|}}$$

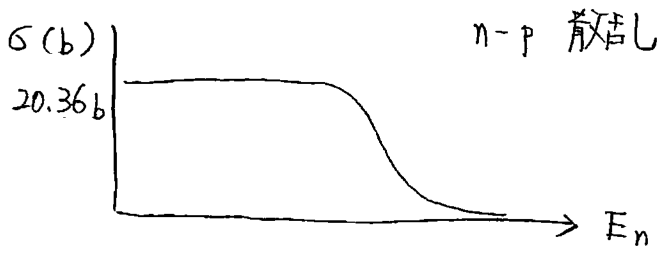
$$= \sqrt{\frac{197^2}{940 \times 2.23}} = 4.31 \text{ fm}$$

$1\,b = 10^{-24}\,cm^2$
$1\,fm^2 = 10^{-26}\,cm^2$
$= 10^{-2}\,b$

⇓

$$\sigma = 4\pi a^2 = 233\ fm^2 = 2.33\ b.$$

実験:



n-p 散乱 ← スピン偏極なし

$$\sigma = \frac{3}{4}\sigma_t + \frac{1}{4}\sigma_s = \frac{3}{4} \times 2.33 + \frac{1}{4}\sigma_s$$

⇓

$\sigma_s = 74.45\ b$ ⇒ $a_s = \pm 23$ fm.

高精度
実験データの解析

$$\begin{cases} a_t = 5.415 \pm 0.012\ fm & r_t = 1.704 \pm 0.028\ fm \\ a_s = -23.806 \pm 0.028\ fm & r_s = 2.49 \pm 0.24\ fm \end{cases}$$

○有効距離の理論

$$\begin{cases} -\frac{\hbar^2}{2m}\psi''(k,r)+V(r)\psi(k,r)-\frac{\hbar^2k^2}{2m}\psi(k,r)=0 \\ -\frac{\hbar^2}{2m}\psi''(0,r)+V(r)\psi(0,r) \qquad\qquad\qquad =0 \end{cases}$$

$\downarrow$

$$-\cancel{\frac{\hbar^2}{2m}}\left(\psi(0)\psi''(k,r)-\psi(k,r)\psi''(0,r)\right)-\frac{k^2\cancel{\hbar^2}}{\cancel{2m}}\psi(0,r)\psi(k,r)=0$$

$\Downarrow$

$$\frac{d}{dr}\{\underbrace{\psi(k,r)\psi'(0,r)-\psi'(k,r)\psi(0,r)}_{\substack{\parallel \\ W[\psi(k,r)\psi(0,r)]}}\}=k^2\psi(0,r)\psi(k,r)$$

(Wronskian)

(note)

$$\varphi(k,r)\equiv\frac{1}{\sin\delta}\sin(kr+\delta)$$

$$\downarrow\left(-\frac{\hbar^2}{2m}\frac{d^2}{dr^2}-\frac{k^2\hbar^2}{2m}\right)\varphi(k,r)=0$$

$\Downarrow$

$$\frac{d}{dr}W[\varphi(k,r)\varphi(0,r)]=k^2\varphi(0,r)\varphi(k,r)$$

$$\delta(k) = -\delta(-k)$$

(note)
$$\varphi(0,r) \sim \frac{\sin\delta}{\sin\delta} + \frac{k\cos\delta}{\sin\delta}\cdot r = 1 - \frac{r}{a}$$

(note)
$$W[\varphi(k,r)\,\varphi(0,r)]_{r=0} = \underbrace{\varphi(k,0)}_{=1}\,\underbrace{\varphi'(0,0)}_{=-\frac{1}{a}} - \underbrace{\varphi'(k,0)}_{=k\cot\delta}\,\underbrace{\varphi(0,0)}_{=1}$$

$$= -\frac{1}{a} - k\cot\delta.$$

$$W[\psi(k,r)\,\psi(0,r)]_{r=0} = 0$$

(note)
$$\psi(k,r) \to \varphi(k,r) \qquad (r\to\infty)$$

$$\downarrow$$

$$\int_0^\infty dr \frac{d}{dr}\left\{ W[\varphi(k,r)\,\varphi(0,r)] - W[\psi(k,r)\,\psi(0,r)] \right\}$$

$$= -W[\varphi(k,r)\varphi(0,r)]_{r=0} = +\frac{1}{a} + k\cot\delta$$

$$= k^2 \int_0^\infty dr\,\left(\varphi(k,r)\varphi(0,r) - \psi(k,r)\psi(0,r)\right)$$

$$\downarrow$$

$$k\cot\delta = -\frac{1}{a} + k^2\int_0^\infty dr\,\left(\varphi(k,r)\varphi(0,r) - \psi(k,r)\psi(0,r)\right)$$

$$\sim -\frac{1}{a} + k^2 \int_0^\infty dr\,\left(\varphi(0,r)^2 - \psi(0,r)^2\right)$$

$$\equiv -\frac{1}{a} + \frac{1}{2}k^2 \underset{\sim}{r_0}$$ effective range

↓

低エネルギー散乱は a, r_0 の2個のパラメーターで記述でき，ポテンシャルの詳細によらない．

↓

$V(r)$ の座標依存性をギロンするためには高エネルギー散乱が必要

∘ effective interaction :

$$k \cot \delta \sim -\frac{1}{a} + \frac{1}{2} k^2 r_0 + \cdots$$

低エネルギーではポテンシャルの詳細は重要ではない。

$$\rightsquigarrow \quad V_{eff}(\mathbf{r}) = \frac{2\pi \hbar^2}{m_r} a \, \delta(\mathbf{r})$$

reduced mass

係数は散乱長が a になるように決定。

cf. BEC（アルカリ原子の希薄気体）

cf、H. Esbensen et al. PRC 56 ('97) 3054

Fetter - Walecka
Eqs. (11.14), (11.53)

原子	$1\,\mathring{A} = 10^{-8}$ cm	eV
原子核	$1\,\mathrm{fm} = 10^{-13}$ cm	MeV
素粒子	$\lesssim 10^{-16}$ cm	$>$ GeV

5.5 共鳴散乱

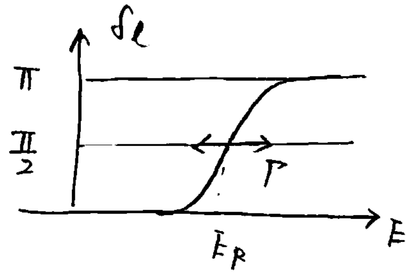
$$\sigma_{tot} = \frac{4\pi}{k^2} \sum_\ell (2\ell+1) \sin^2 \delta_\ell$$

$$\sigma_\ell \equiv \frac{4\pi}{k^2} (2\ell+1) \sin^2 \delta_\ell \leq \frac{4\pi}{k^2} (2\ell+1)$$

↑ 等号は $\delta_\ell = (n+\frac{1}{2})\pi$ の時

assume $\quad \delta_\ell(E=E_R) = \frac{\pi}{2}$

$$\cot \delta_\ell(E) \sim \cot \underbrace{\delta_\ell(E_R)}_{= \frac{\pi}{2}} - \frac{2}{\Gamma}(E-E_R) + \cdots$$



$$= -\frac{2}{\Gamma}(E-E_R)$$

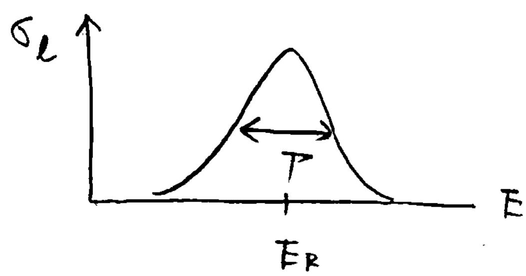
$$-\frac{2}{\Gamma} = \frac{d}{dE} \cot \delta_\ell \Big|_{E=E_R}$$

↓

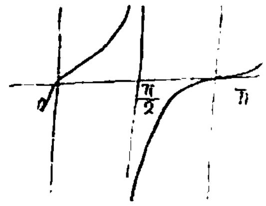
$$\sigma_\ell(E) = \frac{4\pi}{k^2}(2\ell+1) \cdot \frac{1}{1+\cot^2 \delta_\ell} = \frac{4\pi}{k^2}(2\ell+1) \cdot \frac{1}{1+\frac{4}{\Gamma^2}(E-E_R)^2}$$

$$= \frac{4\pi}{k^2} \cdot (2\ell+1) \cdot \frac{\frac{\Gamma^2}{4}}{\frac{\Gamma^2}{4} + (E-E_R)^2}$$

(Breit-Wigner の公式)



Γ: 共鳴幅



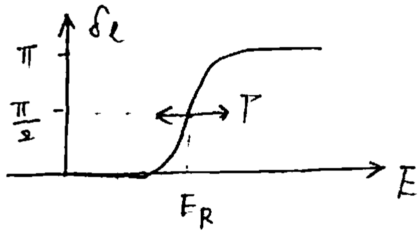
5.5. 共鳴散乱

$$\tan\delta_l = \frac{k j_l'(ka) j_l(\kappa a) - \kappa j_l'(\kappa a) j_l(ka)}{k n_l'(ka) j_l(\kappa a) - \kappa n_l(ka) j_l'(\kappa a)}$$

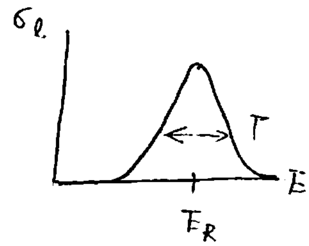
分母 = 0 のとき $\tan\delta_l = \pm\infty$ $\rightarrow \delta_l = \frac{\pi}{2}$

$\rightarrow$ $\sigma_l(k) = \frac{4\pi}{k^2}(2l+1)\sin^2\delta_l(k)$: maximum

共鳴散乱



のように振るまうとき

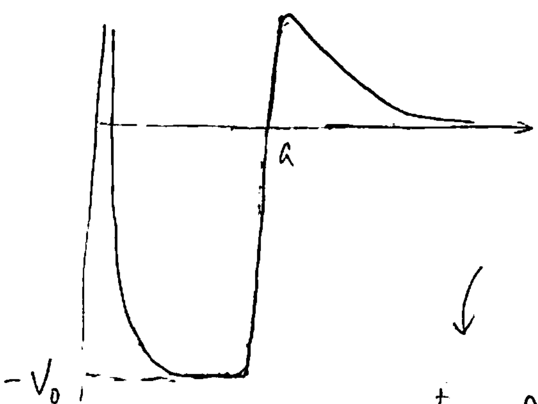


↓

共鳴散乱

・共鳴の条件

$V_0 \gg 1$ のポテンシャル, 大きい l の場合



$\boxed{ka \gg l \gg \kappa a}$

$j_l(ka) \sim \frac{(ka)^l}{(2l+1)!!}$, $n_l(ka) \sim -\frac{(2l-1)!!}{(ka)^{l+1}}$

↓

$$\tan\delta_l = \frac{2l+1}{[(2l+1)!!]^2}(ka)^{2l+1}\frac{l j_l(\kappa a) - \kappa a j_l'(\kappa a)}{(l+1) j_l(\kappa a) + \kappa a j_l'(\kappa a)}$$

$\Downarrow$ 共鳴：

$$(l+1)\underbrace{j_l(ka)}_{\sim\ \frac{1}{ka}\sin\left(ka-\frac{l\pi}{2}\right)} + ka\,\underbrace{j_l'(ka)}_{\sim\ -\frac{1}{(ka)^2}\sin\left(ka-\frac{l\pi}{2}\right)+\frac{1}{ka}\cos\left(ka-\frac{l\pi}{2}\right)} = 0$$

$\Downarrow$

$$\frac{l+1}{ka}\underbrace{\sin\left(ka-\frac{l\pi}{2}\right)}_{=\ \cos\left(ka-\frac{l\pi}{2}-\frac{\pi}{2}\right)} + \underbrace{\cos\left(ka-\frac{l\pi}{2}\right)}_{=\ -\sin\left(ka-\frac{l\pi}{2}-\frac{\pi}{2}\right)} \sim 0.$$

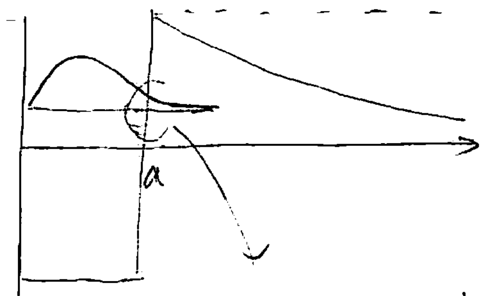
$\Downarrow$

$$\tan\left(ka-\frac{l+1}{2}\pi\right) \sim \frac{l+1}{ka}$$

$\Downarrow$

$$\boxed{ka-\frac{l+1}{2}\pi = n\pi + \frac{l+1}{ka}}$$

(note) 束縛状態のための条件:



かたむきが 0

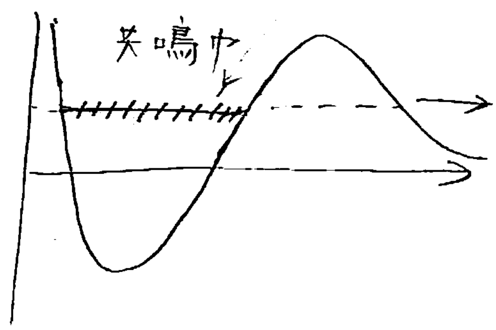
$\Leftrightarrow \quad j_\ell'(ka) \approx \frac{1}{ka} \cos\left(ka - \frac{\ell\pi}{2}\right) = 0$

$\Leftrightarrow \quad \boxed{ka - \frac{\ell\pi}{2} = \left(n + \frac{1}{2}\right)\pi}$

$\Updownarrow$

共鳴条件と等しい

トンネル : 準安定状態



ポテンシャルが高ければ
トンネルの確率 小

→ 共鳴巾 : 小

$$\frac{1+i\tan\theta}{1-i\tan\theta}=\frac{1-\tan^2\theta+2i\tan\theta}{1+\tan^2\theta}=\cos^2\theta-\sin^2\theta+2i\cos\theta\sin\theta$$
$$=\cos 2\theta+i\sin 2\theta$$
$$=e^{i2\theta}$$

・ Breit-Wigner の公式：

$$\cot\delta_l \sim \underbrace{\cot\delta_l(E_r)}_{\frac{\pi}{2}}-\frac{2}{\Gamma}(E-E_r)+\cdots$$

$$=-\frac{2}{\Gamma}(E-E_r)+\cdots$$

$$-\frac{2}{\Gamma}=\frac{d}{dE}\cot\delta_l\Big|_{E=E_r}$$

$\downarrow$

$$\sigma_l(E)=\frac{4\pi}{k^2}(2l+1)\sin^2\delta_l=\frac{4\pi}{k^2}(2l+1)\cdot\frac{1}{1+\cot^2\delta_l}$$

$$=\frac{4\pi}{k^2}(2l+1)\cdot\frac{1}{1+\frac{4}{\Gamma^2}(E-E_r)^2}$$

$$\boxed{=\frac{4\pi}{k^2}(2l+1)\cdot\frac{\frac{\Gamma^2}{4}}{\frac{\Gamma^2}{4}+(E-E_r)^2}}$$

Γ：共鳴幅

(note)

$$f_l(k)=\frac{e^{2i\delta_l}-1}{2ik}=\frac{1}{2ik}\left(\frac{1+i\tan\delta_l}{1-i\tan\delta_l}-1\right)$$

$$=\frac{1}{2ik}\left(\frac{\cot\delta_l+i}{\cot\delta_l-i}-1\right)=\frac{1}{\cancel{2ik}}\cdot\frac{\cancel{2i}}{\cot\delta_l-i}$$

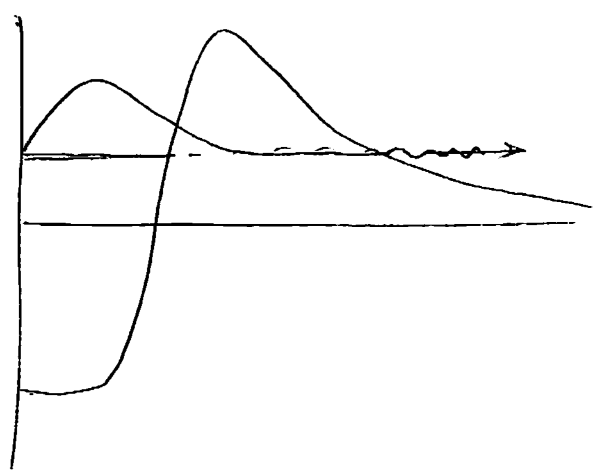
$$=\frac{1}{k}\cdot\frac{1}{-\frac{2}{\Gamma}(E-E_r)-i}=\frac{1}{k}\cdot\frac{-\frac{\Gamma}{2}}{(E-E_r)+i\frac{\Gamma}{2}}$$

(note) ガモフ 状態

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + \frac{l(l+1)\hbar^2}{2mr^2} + V(r) - E \right] \psi(r) = 0$$

$$\psi(r) \sim \begin{cases} r^{l+1} & (r \to 0) \\ e^{i\left(kr - \frac{l\pi}{2}\right)} & (r \to \infty) \end{cases}$$

$$\to \quad E = \underbrace{E_R}_{\substack{\uparrow \\ \text{共鳴} \\ \text{エネルギー}}} - i\frac{\overset{\text{共鳴巾}}{\Gamma}}{2}$$



$$P(t) = |\langle \psi(0) | \psi(t) \rangle|^2 = |\langle \psi(0) | e^{-iHt/\hbar} | \psi(0) \rangle|^2$$

$$= \left| e^{-i\left(E_R - i\frac{\Gamma}{2}\right)t/\hbar} \right|^2 = e^{-\Gamma t/\hbar}$$

$$j_0(x) = \frac{\sin x}{x} \qquad n_0(x) = -\frac{\cos x}{x}$$

$$j_0'(x) = \frac{\cos x}{x} - \frac{\sin x}{x^2} \qquad n_0'(x) = \frac{\sin x}{x} + \frac{\cos x}{x^2}$$

∘ Levinson の定理

$$\tan\delta_0 = \frac{k\, j_0'(kR)\, j_0(\kappa R) - \kappa\, j_0'(\kappa R)\, j_0(kR)}{k\, n_0'(kR)\, j_0(\kappa R) - \kappa\, n_0(kR)\, j_0'(\kappa R)}$$

$$= \frac{k\left(\frac{\cos kR}{kR} - \cancel{\frac{\sin kR}{k^2R^2}}\right)\frac{\sin \kappa R}{\kappa R} - \kappa\left(\frac{\cos \kappa R}{\kappa R} - \cancel{\frac{\sin \kappa R}{\kappa^2 R^2}}\right)\frac{\sin kR}{kR}}{k\left(\frac{\sin kR}{kR} + \cancel{\frac{\cos kR}{k^2R^2}}\right)\frac{\sin \kappa R}{\kappa R} + \kappa\frac{\cos kR}{kR}\left(\frac{\cos \kappa R}{\kappa R} - \cancel{\frac{\sin \kappa R}{\kappa^2R^2}}\right)}$$

$$= \frac{k \tan \kappa R - \kappa \tan kR}{\kappa + k \tan kR \tan \kappa R}$$

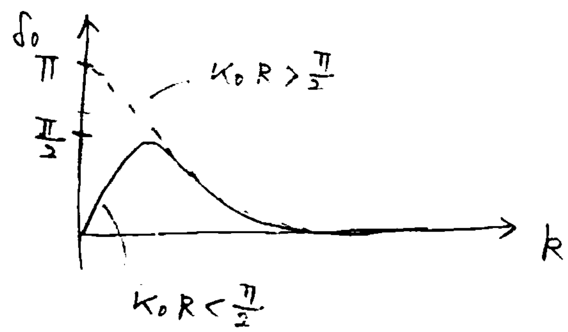
$$k = \sqrt{\frac{2m}{\hbar^2}E}\ , \quad \kappa = \sqrt{\frac{2m}{\hbar^2}(E+V_0)}\ , \quad \kappa_0 \equiv \sqrt{\frac{2m}{\hbar^2}V_0}$$

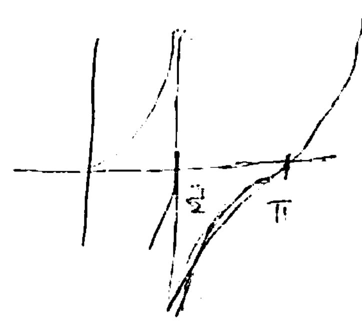
· $\kappa/k \sim 1 \quad (E\to\infty)$: $\quad \tan\delta_0 = 0$

$\Downarrow \quad \delta_0(\infty) = 0.$

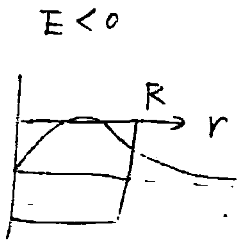
· $E \to 0\ (k\to 0) \quad \Downarrow \quad \tan\delta_0 \to \frac{k}{\kappa_0}\tan\kappa_0 R - kR$ $\quad \to 0$

if	$\kappa_0 R \ll 1$	$\tan\delta_0 \to 0+$	$\Downarrow\ \delta_0 \to 0$
	$\kappa_0 R = \frac{\pi}{2}$	$\tan\delta_0 \to \infty$	$\Downarrow\ \delta_0 \to \frac{\pi}{2}$
	$\kappa_0 R = \frac{\pi}{2}+$	$\tan\delta_0 \to 0-$	$\Downarrow\ \delta_0 \to \pi$





(note)



$r < R$

$u(r) = A \sin \kappa r$

$r > R$

$u(r) = B e^{-\tilde{\kappa} r}$

$\tilde{\kappa} = \sqrt{\frac{2m}{\hbar^2}|E|}$

matching: $\kappa \underbrace{\cot \kappa R}_{\frac{\cos \kappa R}{\sin \kappa R}} = -\tilde{\kappa}$

⇓ bound state が存在するための条件: $\kappa R > \frac{\pi}{2}$

⇓

bound state が存在する時: $\delta_0(0) = \pi$

しない時: $\delta_0(0) = 0$.

一般に $\boxed{\delta_\ell(0) - \delta_\ell(\infty) = N_B \pi}$ N_B: # of bound state

Levinson's theorem