○ 時間に依存するSchrödinger方程式が厳密に解ける例.

例1：調和振動子＋線形結合　（コヒーレント状態）

$$H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right) + f(t)(a + a^\dagger)$$

$$\begin{aligned} i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle &= H|\psi(t)\rangle \\ &= \left[\hbar\omega\left(a^\dagger a + \frac{1}{2}\right) + f(t)(a+a^\dagger)\right]|\psi(t)\rangle \end{aligned}$$

$\downarrow$

解：

$$\begin{aligned} |\psi(t)\rangle &= \exp[\alpha(t)a^\dagger - \alpha^*(t)a]|0\rangle e^{i\eta(t)} \\ &= |\alpha(t)\rangle e^{i\eta(t)} \end{aligned}$$

$$\begin{cases} \alpha(t) = -\frac{i}{\hbar} e^{-i\omega t}\int_0^t f(\tau) e^{i\omega\tau} d\tau \\ \eta(t) = -\frac{1}{2}\omega t - \frac{1}{2\hbar}\int_0^t f(\tau)(\alpha(\tau) + \alpha^*(\tau)) d\tau \end{cases}$$

ここで $|\alpha\rangle$ はコヒーレント状態で，$a|\alpha\rangle = \alpha|\alpha\rangle$ を満たす。

(note) コヒーレント状態：

$$\begin{aligned} |\alpha\rangle &= e^{\alpha a^\dagger - \alpha^* a}|0\rangle = e^{\alpha a^\dagger} e^{-\alpha^* a} e^{-\frac{1}{2}|\alpha|^2}|0\rangle \\ &= e^{-\frac{1}{2}|\alpha|^2} e^{\alpha a^\dagger}|0\rangle \end{aligned}$$

$\downarrow$

$$a|\alpha\rangle = \frac{\partial}{\partial a^\dagger}|\alpha\rangle = \alpha|\alpha\rangle.$$

$[a, a^\dagger] = 1$

(証明)

(note) $e^{\alpha a^\dagger - \alpha^* a} = e^{\alpha a^\dagger} e^{-\alpha^* a} e^{-\frac{1}{2}|\alpha|^2}$

$$右辺 = [\hbar\omega(a^\dagger a + \tfrac{1}{2}) + f(a+a^\dagger)]\,|\alpha\rangle e^{i\eta}$$

$$= \{[\hbar\omega\alpha + f]a^\dagger + (\tfrac{1}{2}\hbar\omega + f\alpha)\}\,|\alpha\rangle e^{i\eta}$$

$$左辺 = i\hbar\frac{\partial}{\partial t}\,[e^{\alpha a^\dagger} e^{-\alpha^* a} e^{-\frac{1}{2}|\alpha|^2}]\,e^{i\eta}\,|0\rangle$$

$$= i\hbar\{[\dot\alpha a^\dagger - \tfrac{1}{2}(\dot\alpha\alpha^* + \alpha\dot\alpha^*) + i\dot\eta]\,|\alpha(t)\rangle e^{i\eta}$$

$$\cancel{-\dot\alpha^* e^{\alpha a^\dagger} a\, e^{-\alpha^* a} e^{-\frac{1}{2}|\alpha|^2} e^{i\eta}|0\rangle}\}$$

0

(note) $\dot\alpha = -i\omega\alpha - \frac{i}{\hbar}\cancel{e^{-i\omega t}} f(t) \cancel{e^{i\omega t}}$

$$= -i\omega\alpha - \frac{i}{\hbar}f$$

$$\dot\alpha^* = i\omega\alpha^* + \frac{i}{\hbar}f$$

$$\dot\eta = -\frac{1}{2}\omega - \frac{f}{2\hbar}(\alpha+\alpha^*)$$

↓

$$左辺 = i\hbar\,[(-i\omega\alpha - \tfrac{i}{\hbar}f)a^\dagger$$

$$-\tfrac{1}{2}(\cancel{-i\omega\alpha\alpha^*} \cancel{- \tfrac{i}{\hbar}f\alpha^*} \cancel{+ i\omega\alpha\alpha^*} + \tfrac{i}{\hbar}f\alpha)$$

$$+ i(-\tfrac{1}{2}\omega - \tfrac{f}{2\hbar}\cancel{(\alpha+\alpha^*)})]\,|\alpha(t)\rangle e^{i\eta}$$

$$= [(\hbar\omega\alpha + f)a^\dagger + (\tfrac{1}{2}\hbar\omega + f\alpha)]\,|\alpha(t)\rangle e^{i\eta}$$

$= 右辺$

例2: 2準位問題 (Rabi の公式)

$$H = \begin{pmatrix} -\frac{\varepsilon}{2} & Ve^{i\omega t} \\ Ve^{-i\omega t} & \frac{\varepsilon}{2} \end{pmatrix}$$

$$i\hbar \frac{\partial}{\partial t} \begin{pmatrix} \phi_0(t) \\ \phi_1(t) \end{pmatrix} = H \begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix} = \begin{pmatrix} -\frac{\varepsilon}{2} & Ve^{i\omega t} \\ Ve^{-i\omega t} & \frac{\varepsilon}{2} \end{pmatrix} \begin{pmatrix} \phi_0(t) \\ \phi_1(t) \end{pmatrix}.$$

$\downarrow$

解:

$$\phi_0(t) = e^{-i(-\frac{\varepsilon}{2})t/\hbar}\, c_0(t)$$

$$c_0(t) = \frac{i\hbar}{V} e^{i\omega t} e^{-i\varepsilon t/\hbar} \times 2iA\left(i\beta\, e^{i\beta t}\sin\gamma t + \gamma\, e^{i\beta t}\cos\gamma t\right)$$

$$\phi_1(t) = e^{-i(\frac{\varepsilon}{2})t/\hbar}\, c_1(t)$$

$$c_1(t) = 2iA\, e^{i\beta t}\sin\gamma t$$

$$\begin{cases} A = -\dfrac{V}{\sqrt{(\varepsilon - \hbar\omega)^2 + 4V^2}} \\ \beta = \dfrac{1}{2\hbar}(\varepsilon - \hbar\omega) \\ \gamma = \dfrac{1}{2\hbar}\sqrt{(\varepsilon - \hbar\omega)^2 + 4V^2} \end{cases}$$

[証明]

(note)

$$\dot{c}_1 = i\beta c_1 + 2iA\, e^{i\beta t}\cdot \gamma\cos\gamma t$$

$$\ddot{c}_1 = i\beta \dot{c}_1 - 2A\beta e^{i\beta t}\,\gamma\cos\gamma t - \gamma^2 c_1$$

$$= i\beta \dot{c}_1 - \gamma^2 c_1 + i\beta(\dot{c}_1 - i\beta c_1)$$

$$= 2i\beta \dot{c}_1 + (-\gamma^2+\beta^2)c_1$$

$$= \frac{i}{\hbar}(\varepsilon - \hbar\omega)\dot{c}_1 - \frac{V^2}{\hbar^2}c_1$$

$$c_0 = \frac{i\hbar}{V} e^{i\omega t} e^{-i\varepsilon t/\hbar}\dot{c}_1$$

$$\dot{c}_0 = i\omega c_0 - \frac{i\varepsilon}{\hbar}c_0 + \frac{i\hbar}{V}e^{i\omega t}e^{-i\varepsilon t/\hbar}\ddot{c}_1$$

$$= i\omega c_0 - \frac{i\varepsilon}{\hbar}c_0 + \frac{i\hbar}{V}e^{i\omega t}e^{-i\varepsilon t/\hbar}\left[\left(\frac{i\varepsilon}{\hbar} - i\omega\right)\dot{c}_1 - \frac{V^2}{\hbar^2}c_1\right]$$

c_0

$$= \frac{i\hbar}{V}e^{i\omega t}e^{-i\varepsilon t/\hbar}\cdot\left(-\frac{V^2}{\hbar^2}\right)c_1$$

$\Downarrow$

$$\begin{cases} i\hbar\,\dot{c}_0 = V e^{i\omega t} e^{-i\varepsilon t/\hbar} c_1 \\ i\hbar\,\dot{c}_1 = V e^{-i\omega t} e^{i\varepsilon t/\hbar} c_0 \end{cases}$$

$\Downarrow$

$$\begin{cases} i\hbar\,\dot{\phi}_0 = -\frac{\varepsilon}{2}\phi_0 + e^{i\varepsilon t/2\hbar}\cdot i\hbar\,\dot{c}_0 \\ \qquad = -\frac{\varepsilon}{2}\phi_0 + V e^{i\omega t}\boxed{e^{-i\varepsilon t/2\hbar}c_1} \\ \qquad = -\frac{\varepsilon}{2}\phi_0 + V e^{i\omega t}\phi_1 \\ i\hbar\,\dot{\phi}_1 = \cdots = \frac{\varepsilon}{2}\phi_1 + V e^{-i\omega t}\phi_0 \end{cases}$$

遷移確率

$$P_1(t) = |C_1|^2 = 4A^2 \sin^2 \gamma t$$

$$= \frac{4V^2}{(\varepsilon - \hbar\omega)^2 + 4V^2} \sin^2 \left\{ \frac{1}{2} \sqrt{\left(\frac{\varepsilon - \hbar\omega}{\hbar}\right)^2 + \frac{4V^2}{\hbar^2}} \; t \right\}$$

(Rabiの公式)