復習）　$V(x) = \frac{1}{2} m\omega^2 x^2$

$$\phi(x) = f(x)\, e^{-\frac{x^2}{2b^2}} \quad ; \quad b = \sqrt{\frac{\hbar}{m\omega}}$$

$$\downarrow \quad \frac{d^2 f}{dy^2} - 2y\frac{df}{dy} + \left(\frac{2E}{\hbar\omega} - 1\right) f(y) = 0 \qquad \left(y = \frac{x}{b}\right)$$

$$f(y) = \sum_{n=0}^{\infty} C_n y^n$$

$$\downarrow$$

$$\underbrace{\sum_{n=0}^{\infty} n(n-1) C_n y^{n-2}} - 2\sum_{n=0}^{\infty} n C_n y^n + \left(\frac{2E}{\hbar\omega} - 1\right)\sum_{n=0}^{\infty} C_n y^n = 0$$

$$\parallel$$

$$2C_2 + 6C_3 y + 12 C_4 y^2 + \cdots$$

$$\parallel$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) C_{n+2} y^n$$

$$\rightarrow \sum_{n=0}^{\infty} \left[(n+2)(n+1) C_{n+2} - \left(2n - \left(\frac{2E}{\hbar\omega} - 1\right)\right) C_n \right] y^n = 0$$

$$\downarrow \quad \frac{C_{n+2}}{C_n} = \frac{2n - \left(\frac{2E}{\hbar\omega} - 1\right)}{(n+2)(n+1)} \sim \frac{2}{n} \quad (n\text{が大きいとき})$$

すなわち C_n と C_{n+2} は同符号

(note)　$C_{2n+2} \sim \frac{1}{n} C_{2n} \sim \frac{1}{n(n-1)} C_{2n-2} \sim \frac{1}{n!}$

$$\downarrow \quad f(y) = \sum_{n'=0}^{\infty} C_{2n'} y^{2n'} \sim \sum_{n'} \frac{y^{2n'}}{(n'-1)!} = \sum_{n'} n' \frac{y^{2n'}}{n'!} > e^{y^2}$$

$$\phi(x) = e^{-y^2/2} f(y) > e^{-\frac{y^2}{2}} e^{y^2} = e^{\frac{y^2}{2}} \quad (\text{発散})$$

(note) $f(y) = \sum_{n=0}^{\infty} C_n y^n$

$$0 = f'' - 2y f' + \left(\frac{2E}{\hbar\omega} - 1\right) f$$

$$= \sum_n n(n-1) C_n y^{n-2} - 2 \sum_n n C_n y^n + \left(\frac{2E}{\hbar\omega} - 1\right) \sum_n C_n y^n$$

$$= \sum_n \left[(n+2)(n+1) C_{n+2} - 2n C_n + \left(\frac{2E}{\hbar\omega} - 1\right) C_n \right] y^n$$

$\Downarrow$

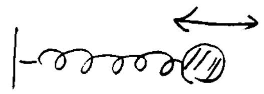
$$(n+2)(n+1) C_{n+2} = \left(2n - \left(\frac{2E}{\hbar\omega} - 1\right)\right) C_n$$

$\Downarrow$

$2n = \frac{2E}{\hbar\omega} - 1$ となれば $C_{n+2} = 0$, 及び
$C_k = 0 \ (k > n+2)$

となり y^n の和は途中で止まる。
→ 波動関数が発散しない。

(note) 「振動」とは？



$$\Psi(x, t=0) = \phi_0(x) \longrightarrow \Psi(x,t) = e^{-i\hbar\omega t/2\hbar} \phi_0(x)$$

このとき $\rho(x,t) = |\Psi(x,t)|^2 = |\phi_0(x)|^2$

t に依らない → 「振動」していない

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}$$

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$$\int_{-\infty}^{\infty} x^2 e^{-\alpha x^2} dx = \frac{1}{2\alpha}\sqrt{\frac{\pi}{\alpha}}$$

$$\Psi(x, 0) = \phi_0(x) + \alpha\phi_1(x) \qquad (|\alpha| \ll 1)$$

$$\rightarrow \quad \Psi(x,t) = e^{-i\omega t/2}\phi_0(x) + \alpha e^{-3i\omega t/2}\phi_1(x)$$

このとき $|\Psi(x,t)|^2 = |\phi_0(x)|^2 + \alpha^2|\phi_1(x)|^2$

$$+ \alpha(e^{i\omega t} + e^{-i\omega t})\phi_0(x)\phi_1(x)$$

$$\sim |\phi_0(x)|^2 + 2\alpha \underbrace{\cos\omega t}_{\text{振動}} \phi_0(x)\phi_1(x) + O(\alpha^2)$$

cf. ビリアル定理

$$\phi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$$

$$\phi_0'(x) = -\frac{m\omega}{\hbar}x\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$$

$$\Rightarrow \quad \left\langle \frac{1}{2}m\omega^2x^2 \right\rangle = \frac{m\omega^2}{2}\cdot\sqrt{\frac{m\omega}{\pi\hbar}} \int_{-\infty}^{\infty} x^2 e^{-\frac{m\omega}{\hbar}x^2} dx$$

$$= \frac{m\omega^2}{2}\cancel{\sqrt{\frac{m\omega}{\pi\hbar}}}\cdot\frac{\hbar}{2m\omega}\cancel{\sqrt{\frac{\pi\hbar}{m\omega}}} = \frac{1}{4}\hbar\omega$$

$$\left\langle \frac{p^2}{2m} \right\rangle = \frac{-\hbar^2}{2m}\int_{-\infty}^{\infty} dx\, \phi_0(x)\phi_0''(x) = \frac{1}{2m}\int_{-\infty}^{\infty} dx\,[\phi_0'(x)]^2$$

$$= \frac{\hbar^2}{2m}\cdot\left(\frac{m\omega}{\hbar}\right)^2\sqrt{\frac{m\omega}{\pi\hbar}}\int_{-\infty}^{\infty} x^2 e^{-\frac{m\omega}{\hbar}x^2} dx = \frac{1}{4}\hbar\omega$$

$$\Rightarrow$$

$$\langle T \rangle = \langle V \rangle = \frac{1}{4}\hbar\omega$$

一般には $\langle T \rangle = \frac{1}{2}\left\langle x\frac{dV}{dx} \right\rangle$

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3.9. 調和振動子ポテンシャル：代数学的解法

$$a^{+} = \sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} - i\frac{\hat{p}}{\sqrt{2m\omega\hbar}}$$

$$\rightarrow\ a = (a^{+})^{+} = \sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} + i\frac{\hat{p}}{\sqrt{2m\omega\hbar}}$$

(note) $$[a, a^{+}] = \sqrt{\frac{m\omega}{2\hbar}}\cdot\left(\frac{-i}{\sqrt{2m\omega\hbar}}\right)[x, p] + \frac{i}{\sqrt{2m\omega\hbar}}\cdot\sqrt{\frac{m\omega}{2\hbar}}\,[p, x]$$

$$= -\frac{i}{\hbar}[x, p] = 1.$$

$$\downarrow$$

$$[a, (a^{+})^{n}] = n(a^{+})^{n-1}$$

- $[a, a^{+}] = 1$
- $[a, (a^{+})^{n}] = n(a^{+})^{n-1}$ とすると

$$[a, (a^{+})^{n+1}] = (a^{+})^{n}[a, a^{+}] + [a, (a^{+})^{n}]a^{+}$$

$$= (a^{+})^{n} + n(a^{+})^{n} = (n+1)(a^{+})^{n}$$

任意の関数 $f(a^{+})$ に対して

$$f(a^{+}) = \sum_{n=0}^{\infty}\frac{1}{n!}f^{(n)}(0)\,(a^{+})^{n}$$

$$\rightarrow\ [a, f(a^{+})] = \sum_{n=0}^{\infty}\frac{1}{n!}f^{(n)}(0)\cdot n(a^{+})^{n-1}$$

$$= f'(a^{+})$$

すなわち $a = \dfrac{d}{da^{+}}$ としてよい。

同様に $a^{+} = -\dfrac{d}{da}$; $[a^{+}, g(a)] = -g'(a)$

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$$a + a^{\dagger} = 2 \cdot \sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} \quad \rightarrow \quad \hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\,(a + a^{\dagger})$$

$$a^{\dagger} - a = -2 \cdot \frac{i}{\sqrt{2m\omega\hbar}}\,\hat{p} \quad \rightarrow \quad \hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}\,(a^{\dagger} - a)$$

$$H = \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega^2 \hat{x}^2$$

$$= -\frac{1}{2\cancel{m}} \cdot \frac{m\omega\hbar}{2}(a^{\dagger} - a)^2 + \frac{1}{2}\cancel{m}\omega^2 \cdot \frac{\hbar}{2\cancel{m\omega}}(a + a^{\dagger})^2$$

$$= \frac{\hbar\omega}{4}\left(\cancel{-a^{\dagger}a^{\dagger}} + a^{\dagger}a + aa^{\dagger} \cancel{- aa} + \cancel{a^{\dagger}a^{\dagger}} + a^{\dagger}a + aa^{\dagger} \cancel{+ aa}\right) = \frac{\hbar\omega}{2}(aa^{\dagger} + a^{\dagger}a)$$

$$= \hbar\omega\left(a^{\dagger}a + \frac{1}{2}\right)$$

(note) $[H, a] = [\hbar\omega a^{\dagger}, a]\,a = -\hbar\omega a$

$[H, a^{\dagger}] = a^{\dagger}[\hbar\omega a, a^{\dagger}] = \hbar\omega a^{\dagger}$

$H|\varphi\rangle = E|\varphi\rangle$ とすると

$$H(a|\varphi\rangle) = (-\hbar\omega a + aH)|\varphi\rangle = (E - \hbar\omega)(a|\varphi\rangle)$$

$$H(a^{\dagger}|\varphi\rangle) = (\hbar\omega a^{\dagger} + a^{\dagger}H)|\varphi\rangle = (E + \hbar\omega)(a^{\dagger}|\varphi\rangle)$$

固有状態

$E + \hbar\omega$
$\uparrow a^{\dagger}$
E
$\downarrow a$
$E - \hbar\omega$

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基底状態（最低エネルギー状態）： $a|0\rangle = 0$

$$\rightarrow \quad H|0\rangle = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)|0\rangle = \underbrace{\frac{1}{2}\hbar\omega}_{\text{固有値}}|0\rangle$$

$a|0\rangle = 0$

$$\rightarrow \quad 0 = \langle x|a|0\rangle = \langle x|\sqrt{\frac{m\omega}{2\hbar}}\hat{x} + i\frac{1}{\sqrt{2m\omega\hbar}}\hat{p}|0\rangle$$

$$= \left(\sqrt{\frac{m\omega}{2\hbar}}x + i\frac{1}{\sqrt{2m\omega\hbar}}\cdot\frac{\hbar}{i}\frac{d}{dx}\right)\underbrace{\langle x|0\rangle}_{\parallel \atop \phi_0(x)}$$

$$\rightarrow \quad \phi_0'(x) = -\sqrt{\frac{2m\omega}{\hbar}}\cdot\sqrt{\frac{m\omega}{2\hbar}}x\phi_0(x)$$

$$= -\frac{m\omega}{\hbar}x\phi_0(x)$$

$$\rightarrow \quad \phi_0(x) \propto e^{-\frac{m\omega}{2\hbar}x^2}$$

励起状態： $|n\rangle = A\cdot(a^\dagger)^n|0\rangle$

規格化：

$$1 = \langle n|n\rangle = A^2\langle 0|a^n(a^\dagger)^n|0\rangle$$

$$a(a^\dagger)^n|0\rangle = \left((a^\dagger)^n a + n(a^\dagger)^{n-1}\right)|0\rangle = n(a^\dagger)^{n-1}|0\rangle$$

$$\downarrow$$

$$a^2(a^\dagger)^n|0\rangle = na(a^\dagger)^{n-1}|0\rangle = n(n-1)(a^\dagger)^{n-2}|0\rangle$$

$\rightarrow$ これをくり返すと $(a^\dagger)^n a^n|0\rangle = n!\,|0\rangle$

$$\downarrow$$

$$1 = A^2\cdot n! \quad \downarrow \quad A = \frac{1}{\sqrt{n!}}$$

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すなわち $|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle$

$$\rightarrow \quad a|n\rangle = \frac{1}{\sqrt{n!}}\left[\cancel{(a^\dagger)^n a} + n(a^\dagger)^{n-1}\right]|0\rangle$$

$$= \sqrt{n}\,\frac{1}{\sqrt{(n-1)!}}(a^\dagger)^{n-1}|0\rangle = \sqrt{n}\,|n-1\rangle$$

$$a^\dagger|n\rangle = \sqrt{n+1}\cdot\frac{(a^\dagger)^{n+1}}{\sqrt{(n+1)!}}|0\rangle = \sqrt{n+1}\,|n+1\rangle$$

(note) $n=0$ のとき $a|0\rangle = 0$, $a^\dagger|0\rangle = |1\rangle$

(note) $\hat{n} = a^\dagger a$

$$\hat{n}|n\rangle = a^\dagger a|n\rangle = \sqrt{n}\,a^\dagger|n-1\rangle = n|n\rangle.$$

(note) $\frac{1}{2}m\omega^2 x^2 = \frac{m\omega^2}{2}\cdot\frac{\hbar}{2m\omega}(a^{\dagger 2} + a^2 + \underbrace{aa^\dagger}_{=a^\dagger a+1} + a^\dagger a)$

$$\langle n|(a^\dagger)^2|n\rangle \propto \langle n|n+2\rangle = 0$$
$$\langle n|a^2|n\rangle \propto \langle n|n-2\rangle = 0$$

$$\downarrow$$

$$\langle n|\tfrac{1}{2}m\omega^2 x^2|n\rangle = \frac{\hbar\omega}{4}\cdot(2n+1) = \frac{1}{2}\cdot\underbrace{\hbar\omega\left(n+\tfrac{1}{2}\right)}_{=E_n}$$

$$\downarrow$$

$$\langle n|\tfrac{1}{2}m\omega^2 x^2|n\rangle = \langle n|\frac{p^2}{2m}|n\rangle = \frac{1}{2}E_n$$

(ビリアル定理)

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(note) 超対称性量子力学

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V(x)\right)\varphi_0(x)=0$$

(エネルギーのスケールは適当にとる)

$$\rightarrow \quad V\varphi_0 = \frac{\hbar^2}{2m}\varphi_0''$$

$$\downarrow \quad H = \frac{\hbar^2}{2m}\left(-\frac{d^2}{dx^2}+\frac{\varphi_0''}{\varphi_0}\right)$$

$$A^\dagger = \frac{\hbar}{\sqrt{2m}}\left(-\frac{d}{dx}-\frac{\varphi_0'}{\varphi_0}\right),\quad A = \frac{\hbar}{\sqrt{2m}}\left(\frac{d}{dx}-\frac{\varphi_0'}{\varphi_0}\right)$$

とすると

$$A\varphi_0 = \frac{\hbar}{\sqrt{2m}}\left(\varphi_0' - \frac{\varphi_0'}{\varphi_0}\varphi_0\right) = 0$$

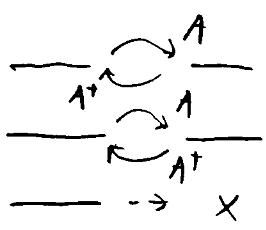
$$A^\dagger A = \frac{\hbar^2}{2m}\left(-\frac{d^2}{dx^2} + \underbrace{\left(\frac{\varphi_0'}{\varphi_0}\right)'}_{=\frac{\varphi_0\varphi_0''-\varphi_0'^2}{\varphi_0^2}} + \cancel{\frac{\varphi_0'}{\varphi_0}\frac{d}{dx}} - \cancel{\frac{\varphi_0'}{\varphi_0}\frac{d}{dx}} \cancel{+\left(\frac{\varphi_0'}{\varphi_0}\right)^2}\right)$$

$$= H.$$

$$\tilde{H} = AA^\dagger$$

もし $H\varphi_n = A^\dagger A\varphi_n = E_n\varphi_n$ なら

$$\tilde{H}(A\varphi_n) = A\underline{A^\dagger A}\varphi_n = E_n(A\varphi_n)$$



H $\quad$ $\tilde{H}$