Martingales for Determinantal Log-Gases

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Let $B(t), t \geq 0$ be the one-dimensional standard Brownian motion (BM). For each time $t \in [0, \infty)$, a collection of all measurable events generated by $\{B(s) : 0 \leq s \leq t\}$ is denoted by $\mathcal{F}(t)$. The family $\{\mathcal{F}(t) : t \geq 0\}$ is nondecreasing in the sense $\mathcal{F}(s) \subset \mathcal{F}(t) \subset {}^{\exists}\mathcal{F}$ for $0 \leq s < t < \infty$, and is called a filtration. For $0 \leq s \leq t < \infty$, the expectation of a measurable bounded function $f(t, \cdot)$ of BM conditioned by $\mathcal{F}(s)$ is denoted by $\mathrm{E}[f(t, B(t))|\mathcal{F}(s)]$. It is given by

$$\mathrm{E}[f(t, B(t))|\mathcal{F}(s)] = \int_{\mathbb{R}} dy\, f(t, y) p(t - s, y|B(s)), \quad \text{a.s.} \tag{1}$$

with the transition probability density $p(t, y|x) = \mathbf{1}_{(t>0)} e^{-(x-y)^2/2t}/\sqrt{2\pi t} + \mathbf{1}_{(t=0)}\delta(x - y)$. If a process $\mathcal{M}(t, B(t))$, defined as a functional of BM, satisfies

$$\mathrm{E}[\mathcal{M}(t, B(t))|\mathcal{F}(s)] = \mathcal{M}(s, B(s)) \quad \text{for all } 0 \leq s \leq t, \tag{2}$$

it is called a martingale. It represents a fluctuation preserving its mean value in time. It is easy to confirm the following; $B(t)$ is a martingale; $B(t)^2$ is not a martingale, but $B(t)^2 - t$ is so; for each $n \in \mathbb{N}$, $m_n(t, B(t)) \equiv (t/2)^{n/2} H_n(B(t)/\sqrt{2t})$ is a martingale, where H_n is the Hermite polynomial of degree n. We consider a transformation $\mathcal{I}[f(W)|(t, x)] : f(W) \to \widehat{f}(t, x)$ such that $\mathcal{I}[W^n|(t, x)] = m_n(t, x), \forall n \in \mathbb{N}$, which is realized as an integral transformation.

Consider a finite or an infinite sequence of points on $\mathbb{R}$, $\{x_j\}_{j \in \mathbb{I}}$, such that the point-mass measure $\xi(\cdot) = \sum_{j \in \mathbb{I}} \delta_{x_j}(\cdot)$ has no accumulation points and the entire functions of $z \in \mathbb{C}$, $\Phi_\xi^\zeta(z) = \prod_{j \in \mathbb{I}} (z - x_j)/(\zeta - x_j)$ parameterized by $\zeta \in \mathbb{C} \setminus \operatorname{supp} \xi$, are well-defined. Then, for $v \in \operatorname{supp} \xi$, $(s, x) \in [0, \infty) \times \mathbb{R}$,

$$\mathcal{M}_\xi^v((s, x)|(t, B(t))) = \mathcal{I}\left[\frac{1}{2\pi i} \oint_{C(\delta_v)} \frac{p(s, x|\zeta)}{p(s, x|v)} \frac{1}{W - \zeta} \Phi_\xi^\zeta(W) d\zeta \,\middle|\, (t, B(t))\right], \quad t \in [0, \infty), \tag{3}$$

is a martingale, where $i = \sqrt{-1}$ and $C(\delta_v)$ is a closed contour on $\mathbb{C}$ encircling a point v on $\mathbb{R}$ once in the positive direction. We proved that equilibrium and nonequilibrium dynamics of Dyson's Brownian motion model with $\beta = 2$ (the noncolliding Brownian motion) started at ξ are described by $\{\mathcal{M}_\xi^v((s, x)|(t, y)) : v \in \operatorname{supp} \xi, (s, x), (t, y) \in [0, \infty) \times \mathbb{R}\}$ [1, 2].

In the present talk, exact-solvability of 'determinantal log-gases' is discussed from the view point of martingales [2]. I would like to include the discussions on the Airy process with the Tracy-Widom distribution, O'Connell's process related with the KPZ phenomena, trigonometric, hyperbolic, and elliptic extensions of Dyson's Brownian motion model, and discretized models.

References

[1] M. Katori, H. Tanemura, Complex Brownian motion representation of the Dyson model, Electron. Commun. Probab. **18** (4), 1-16 (2013).

[2] M. Katori, Determinantal martingales and noncolliding diffusion processes, Stochastic Process. Appl. (2014), http://dx.doi.org/10.1016/j.spa.2014.06.002.