Largest Eigenvalue Distributions & Their Applications

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Interface Fluctuations and KPZ Universality Class
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Outline

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- Limiting distribution of largest eigenvalue, $F_\beta(x)$, for $\beta = 1, 2, 4$.
  - Fredholm determinant & Painlevé representations
  - Tail behavior
  - Next-largest, etc. eigenvalue distributions

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- Beyond determinantal class
  - Wigner matrices & universality theorems
  - F_β for $\beta > 0$
  - ASEP

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- Role of F_β in data analysis

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This measure $\mathbb{P}_{\beta,N}$ induces a probability density on the eigenvalues which has the form

$$P_{\beta,N}(x_1,\ldots,x_N) = C_{N,\beta} \prod_{i<j} |x_i - x_j|^\beta \prod_j e^{-\beta x_j^2/2}, \quad \beta = 1, 2, 4.$$

Largest Eigenvalue Distributions

Let $x_{\max}$ denote the largest (random) eigenvalue. Then

$$\begin{aligned} F_{\beta,N}(x) : &= \mathbb{P}_{\beta,N}(x_{\max} < x) = \mathbb{P}_{\beta,N}(x_1 < x, \ldots, x_N < x) \\ &= \int_{-\infty}^{x} \cdots \int_{-\infty}^{x} P_{\beta,N}(x_1, \ldots, x_N)\, dx_1 \cdots dx_N \end{aligned}$$

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For $\beta = 2$ (unitary case) it was M. Gaudin who in 1961 showed that $F_{2,N}$ is a *Fredholm determinant*[1] whose kernel is

$$\begin{aligned} K_N(x,y) &= \left(\frac{N}{2}\right)^{\frac{1}{2}} \frac{\phi_N(x)\phi_{N-1}(y) - \phi_{N-1}(x)\phi_N(y)}{x-y}, \\ & \phi_n(x) = \text{harmonic oscillator wave fns.} \end{aligned}$$

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We are interested in a limit theorem as $N \to \infty$.

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Using the known asymptotics of Hermite polynomials in the transition region (*Plancherel-Rotach formulas*) it is an easy exercise to show that (at least formally) that the operator K_N converges to the *Airy kernel*[2]

$$K_{\mathrm{Airy}}(x, y) = \frac{\mathsf{Ai}(x)\,\mathsf{Ai}'(y) - \mathsf{Ai}'(x)\,\mathsf{Ai}(y)}{x - y}, \;\; \mathsf{Ai}(x) = \text{Airy function.}$$

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Thus in distribution

$$F_2(s) := \lim_{N\to\infty} \mathbb{P}_{2,N}\left[N^{2/3}\left(\frac{x_{\mathrm{max}}}{\sqrt{N}} - 2\right) \le s\right] = \det(I - K_{\mathrm{Airy}})$$

where K_{Airy} acts on $L^2((s,\infty))$.

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F. Dyson (1970) was the first to recognize that matrix kernels arise in the orthogonal and symplectic cases. This was further developed by M. L. Mehta, TW and others.

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Integrable Structure: Painlevé Representations

The *sine kernel* (bulk scaling limit) and the *Airy kernel* are both of the form

$$K(x,y) = \frac{\varphi(x)\psi(y) - \psi(x)\varphi(y)}{x-y} \tag{1}$$

where φ, ψ satisfy DEs of the form

$$\frac{d}{dx}\begin{pmatrix} \varphi(x) \\ \psi(x) \end{pmatrix} = \Omega(x)\begin{pmatrix} \varphi(x) \\ \psi(x) \end{pmatrix}$$

where $\Omega(x)$ is a 2×2 traceless matrix whose elements are rational functions of x.

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Let J be a union of open intervals: $J = \bigcup_{j=1}^{m}(a_{2j-1}, a_{2j})$, and consider the Fredholm determinant

$$\tau(a) := \det(I - K)$$

where K has kernel (1) and acts on $L^2(J)$.

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For the Airy kernel and $J = (s, \infty)$, the PDEs reduce to a single ODE—the *Painlevé II* DE:

$$\begin{aligned} F_2(s) &= \exp\left(-\int_s^\infty (x-s)q^2(x)\,dx\right) := (F(s))^2 \\ \frac{d^2q}{dx^2} &= x\,q + 2q^3 \\ q(x) &\sim \mathrm{Ai}(x),\ \ x\to\infty,\ \ \text{Hastings-McLeod solution} \end{aligned}$$

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Somewhat surprisingly, both F_1 and F_4 can be expressed in terms of the same Painlevé II function q, e.g.

$$F_1(s) = \exp\left(-\frac{1}{2}\int_s^\infty q(x)\, dx\right) (F_2(s))^{1/2} := E(x)F(x).$$

Historical remarks:

- The first connection between Fredholm deteminants/Toeplitz determinants and Painlevé functions was in the 2D Ising model where the 2-point massive scaling functions were shown to be expressible in terms of Painlevé III (Wu, McCoy, CT, Barouch, 1973–76).

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- Riemann-Hilbert methods by Its, Deift and others further developed this area particularly with regards to the difficult questions of asymptotics (connection formulas).

Some numerics

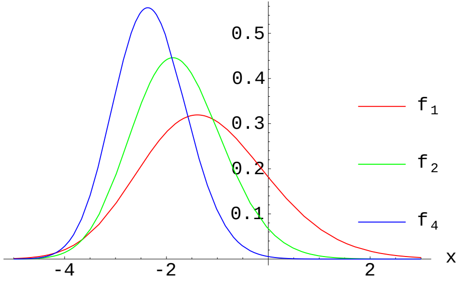

Figure : Largest eigenvalue densities $f_\beta(x) = dF_\beta/dx$, $\beta = 1, 2, 4$.

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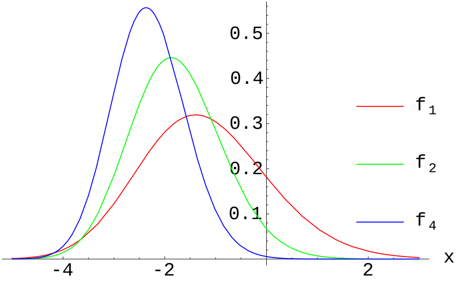
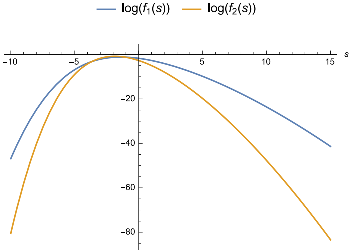


Figure : Largest eigenvalue densities $f_\beta(x) = dF_\beta/dx$, $\beta = 1, 2, 4$.

Table : The mean (μ_β), variance (σ_β^2), skewness (S_β) and kurtosis (K_β) of F_β.

β	μ_β	σ_β^2	S_β	K_β
1	-1.206 533 574	1.607 781 034	0.293 464 524	0.165 242 938
2	-1.771 086 807	0.813 194 792	0.224 084 203	0.093 448 087
4	-2.306 884 489	0.517 723 720	0.165 509 494	0.049 195 156

Tail behavior of F_β



As $x \to +\infty$

$$F(x) - 1 \sim -\frac{\exp(-\frac{4}{3}x^{3/2})}{32\pi x^{3/2}}, \quad E(x) - 1 \sim -\frac{\exp(-\frac{2}{3}x^{3/2})}{4\sqrt{\pi}x^{3/2}}$$

As $x \to -\infty$

$$F_1(x) \sim \tau_1 \frac{\exp(-\frac{1}{24}|x|^3 - \frac{1}{3\sqrt{2}}|x|^{3/2})}{|x|^{1/16}}, \quad F_2(x) \sim \tau_2 \frac{\exp(-\frac{1}{12}|x|^3)}{|x|^{1/8}}$$

$$\tau_1 = 2^{-11/48}\, e^{\zeta'(-1)}, \tau_2 = 2^{1/24}\, e^{\zeta'(-1)}$$

Next-largest, next-next-largest, etc. eigenvalue distributions

There are Painlevé II type representations. (TW, $\beta = 2$; M. DIENG, $\beta = 1, 4$.)

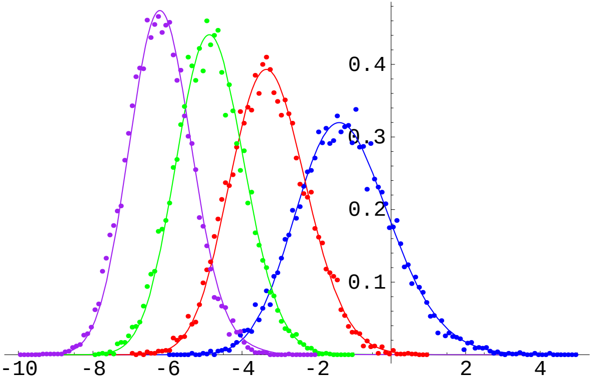
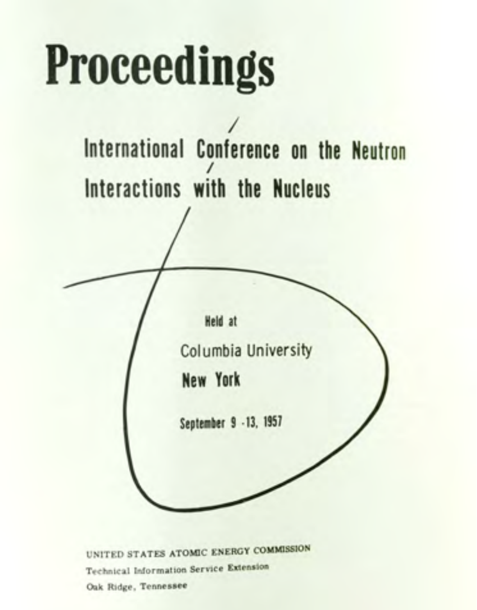


Figure : Histogram of the four largest (centered and normalized) eigenvalues for 10^4 realizations of 1000×1000 GOE matrices. Solid curves are F_1 and Dieng's limiting next-largest, etc. distributions. Figure courtesy of Dieng.

Beyond Determinantal Class

Some historical remarks concerning universality



SESSION IIB

INTERPRETATION OF LOW ENERGY NEUTRON SPECTROSCOPY

CHAIRMAN—W. W. Havens, Jr.

IIB1. DISTRIBUTION OF NEUTRON RESONANCE LEVEL SPACING.

E. P. WIGNER, *Princeton University*
Presented by E. P. Wigner

The problem of the spacing of levels is neither a terribly important one nor have I solved it. That is really the point which I want to make very definitely. As we go up in the energy scale it is evident that the detailed analyses which we have seen for low

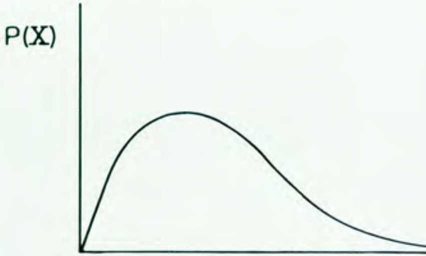


Fig. IIB1-1. Probability of a level spacing X.

Let me say only one more word. It is very likely that the curve in Figure I is a universal function. In other words, it doesn't depend on the details of the model with which you are working. There is one particular model in which the probability of the

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First proof for Wigner matrices: Tau & Vu (2009) and independently Erdös, Péché, Ramírez, Schlein, Yau (2009). Wigner matrices

$$A = \frac{1}{\sqrt{N}} (a_{ij}) , \text{ algebraically indep. elements } a_{ij} \text{ are i.i.d. random variables}$$

Universality theorem for largest eigenvalue of Wigner matrices

A. Soshnikov (1999): Properly centered and normalized largest eigenvalue converges in distribution to the corresponding Gaussian result; namely, F_β. Since Soshnikov's work the moment conditions in his theorem have been relaxed (Tao, Vu, Erdös, Yau and coworkers)

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> *The importance of universality for Wigner matrices is that these ensembles of matrices are representative of the "non-integrable" case; namely, "integrable techniques" such as Fredholm determinants, Riemann-Hilbert methods, Painlevé theory are not directly applicable to the Wigner case. Yet the limit theorems found by these integrable techniques continue to hold in a broader class of ensembles.*

The Beta Ensembles: F_β for $\beta > 0$

Recall

$$P_{\beta,N}(x_1, \ldots, x_N) = C_{N,\beta} \prod_{i<j} |x_i - x_j|^\beta \prod_j e^{-\beta x_j^2/2}, \qquad (2)$$

which makes sense for all real, positive β.

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B. Sutton and A. Edelman gave heuristic arguments that the rescaled operators

$$\tilde{H}_N^\beta = N^{1/6}\left(2\sqrt{N} - H_N^\beta\right)$$

converges to the differential operator

$$\mathcal{H}_\beta = -\frac{d^2}{dx^2} + x + \frac{2}{\sqrt{\beta}} b'_x$$

where b' is white noise.

Theorem (Ramírez, Rider, Virág): With probability one, for each $k \geq 0$ the set of eigenvalues of $\mathcal{H}_\beta$ has a well-defined $(k+1)$st lowest element Λ_k. Moreover, let $\lambda_1 \geq \lambda_2 \geq \cdots$ denote the eigenvalues of the Hermite β-ensemble H_N^β. Then the vector

$$\left(N^{1/6}(2\sqrt{N} - \lambda_{\beta,\ell})\right)_{\ell=1,\dots,k}$$

converges in distribution as $N \to \infty$ to $(\Lambda_0, \Lambda_1, \dots, \Lambda_{k-1})$. The distribution of $-\Lambda_0(\beta)$ defines F_β for all $\beta > 0$.

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As a corollary of their theorem

$$-\Lambda_0(\beta) = \sup_f \left\{ \frac{2}{\sqrt{\beta}} \int_0^\infty f^2(x)\, db(x) - \int_0^\infty \left[f'(x)^2 + xf^2(x)\right]\, dx \right\}$$

where the supremum is taken over the space of functions satisfying $f(0) = 0$ and $\int_0^\infty [(f')^2 + (1+x)f^2\, dx < \infty$ with norm one.

This last expression is a characterization of F_β independent of RMT.

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Theorem (TW) Let $m = [\sigma t]$, $\gamma = q - p > 0$ fixed, then

$$\lim_{t\to\infty} \mathbb{P}_{\mathbb{Z}^+}\left(x_m(t/\gamma) \le c_1(\sigma)t + c_2(\sigma)\, s\, t^{1/3}\right) = F_2(s)$$

uniformly for σ in compact subsets of $(0,1)$ where $c_1(\sigma) = -1 + 2\sqrt{\sigma}$, $c_2(\sigma) = \sigma^{-1/6}(1 - \sqrt{\sigma})^{2/3}$.

Here $x_m(t)$ is the position of the mth particle from the left.

Role of F_β in Data Analysis

The greatest root distribution is found everywhere in classical multivariate analysis. It describes the null hypothesis distribution for the union intersection test for any number of classical problems, including multiple response linear regression, MANOVA, canonical correlations, equality of covariance matrices and so on. However, the exact null distribution is difficult to calculate and work with, and so the use of extensive tables or special purpose software has always been necessary.

Role of F_β in Data Analysis

The greatest root distribution is found everywhere in classical multivariate analysis. It describes the null hypothesis distribution for the union intersection test for any number of classical problems, including multiple response linear regression, MANOVA, canonical correlations, equality of covariance matrices and so on. However, the exact null distribution is difficult to calculate and work with, and so the use of extensive tables or special purpose software has always been necessary.

Our main claim is that for many applied purposes, the Tracy–Widom approximation can often, if not quite always, substitute for the elaborate tables and computational procedures that have until now been needed.

Iain Johnstone, 2009

Review of multivariate statistical analysis

Let $x_1, x_2, \ldots, x_n$ be a random sample from $N_p(\mu, \Sigma)$. Form $n \times p$ data matrix X

$$X = \begin{pmatrix} \longleftarrow & x_1 & \longrightarrow \\ \longleftarrow & x_2 & \longrightarrow \\ & \vdots & \\ \longleftarrow & x_n & \longrightarrow \end{pmatrix}$$

then the $p \times p$ matrix

$$A = X^T X \sim W_p(n, \Sigma) = \text{Wishart distr.}$$

For $p = 1$, A is distributed as $\sigma^2 \chi^2_{(n)}$.

In the univariate case if X and Y are independent and χ^2-distributed with m and n degrees of freedom, respectively, then $X/(X+Y)$ has the beta distribution

$$\frac{1}{B(m/2, n/2)} t^{m/2-1}(1-t)^{-(m+n)/2}$$

Multivariate case: if $A \sim W_p(m, \Sigma)$ and $B \sim W_p(n, \Sigma)$, independent of A, then the matrix analogue of the beta distribution is

$$(A+B)^{-1}B$$

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The eigenvalues of $(A+B)^{-1}B$ have density (Jacobi ensemble)

$$C \prod_{i=1}^{\min(n,p)} \theta_i^{(|n-p|-1)/2}(1-\theta_i)^{(m-p-1)/2} \prod_{i \neq j} |\theta_i - \theta_j|.$$

Let

$$W(p,m,n) := \log\left(\frac{\theta(p,m,n)}{1-\theta(p,m,n)}\right)$$

Theorem (Johnstone): Assume

$$\lim_{p\to\infty}\frac{\min(p,n)}{m+n} > 0, \quad \lim_{p\to\infty}\frac{p}{m} < 1$$

then

$$\frac{W(p,m,n)-\mu(p,m,n)}{\sigma(p,m,n)} \overset{\mathcal{D}}{\Rightarrow} F_1$$

where

$$\begin{aligned}
\mu(p,m,n) &= 2\log\tan(\frac{\varphi+\gamma}{2}) \\
\sigma^3(p,m,n) &= \frac{16}{(m+n-1)^2}\frac{1}{\sin^2(\varphi+\gamma)\sin\varphi\sin\gamma} \\
\sin^2(\gamma/2) &= \frac{\min(p,n)-1/2}{m+n-1} \\
\sin^2(\varphi/2) &= \frac{\max(p,n)-1/2}{m+n-1}
\end{aligned}$$

Note: "-1/2" and "-1" make convergence $\mathrm{O}(p^{-2/3})$.

Let f_α denote the α-percentile for $X \sim F_1$,

$$F_1(f_\alpha) = \alpha$$

and similarly θ_α the α-percentile of $\theta(p, m, n)$, then corollary of above is

$$\theta_\alpha \approx \frac{\exp(\mu + f_\alpha \sigma)}{1 + \exp(\mu + f_\alpha \sigma)}$$

Tables use variables

$$s = \min(n, p), \;\; \mathfrak{m} = (|n - p| - 1)/2, \;\; \mathfrak{n} = (m - p - 1)$$

From Iain Johnstone

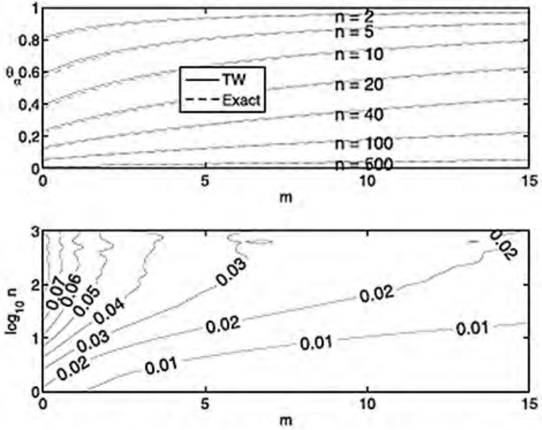

FIG. 2. *Comparison of exact and approximate 95th percentiles for* s $=2$. *Top panel: solid line is the Tracy–Widom approximation* $\theta_\alpha^{TW}(2, \mathsf{m}, \mathsf{n})$ *plotted as a function of* m *for values of* n *shown. Dashed lines are the exact percentiles* $\theta_\alpha(2, \mathsf{m}, \mathsf{n})$ *from Chen's tables. Bottom panel: Contour plots of relative error* $r = (\theta_\alpha^{TW}/\theta_\alpha) - 1$. *Horizontal axis is* m, *vertical axis is* $\log_{10}$ n, *thus covering the range from* n $= 1$ *to* 1000.

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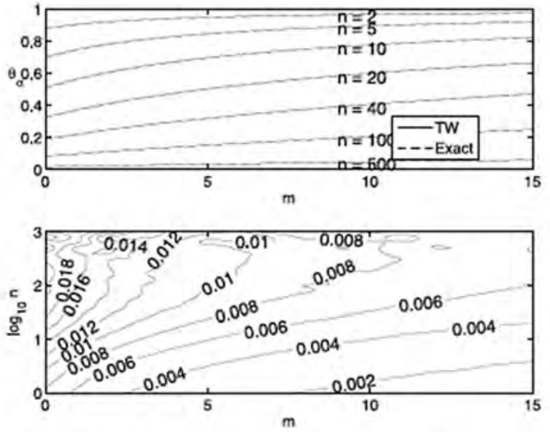

FIG. 3. *Comparison of exact and approximate 90th percentiles for* $s = 4$. *Top panel: solid line is the Tracy–Widom approximation* $\theta_\alpha^{TW}(4, m, n)$ *plotted as a function of* m *for values of* n *shown. Dashed lines are the exact percentiles* $\theta_\alpha(4, m, n)$ *from Chen's tables. Bottom panel: Contour plots of relative error* $r = (\theta_\alpha^{TW}/\theta_\alpha) - 1$. *Horizontal axis is* m, *vertical axis is* $\log_{10} n$, *thus covering the range from* $n = 1$ *to* 1000.

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