Kyoto, August 20-24th 2014

Tracy-Widom distributions in discrete models

Patrik L. Ferrari

universitätbonn iam

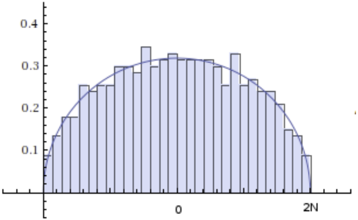
http://wt.iam.uni-bonn.de/∼ferrari

- Gaussian Unitary Ensemble (GUE) of random matrices
- Consider $N \times N$ hermitian matrices H with
  (a) random independent entries,
  (b) distribution invariant under unitary transformations

⇒ Probability density:

$$\text{const}\, e^{-\frac{1}{2N}\,\text{Tr}(H^2)}$$

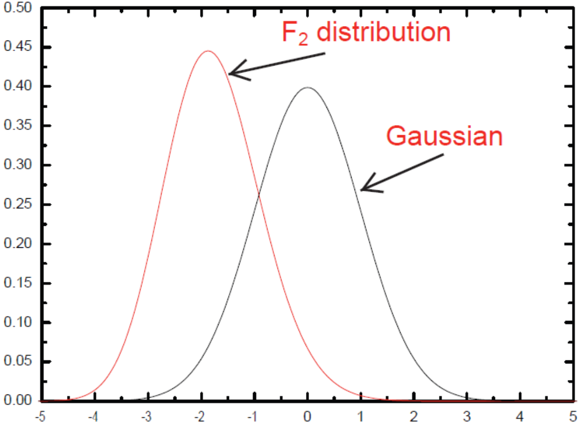
- Largest eigenvalue: $\lambda_{\max,N} \simeq 2N$ for large N



- Distribution of the largest eigenvalue: F_2 Tracy, Widom '94

$$\lambda_N \simeq 2N + \xi_2 N^{1/3}, \quad N \to \infty$$

where ξ_2 has the (GUE) Tracy-Widom distribution F_2.



Probability densities of the GUE Tracy-Widom and the normal distribution

- Let $\lambda = (\lambda_1, \ldots, \lambda_N)$ be the N eigenvalues of a GUE random matrix. The eigenvalues probability density $p(\lambda)$ is given by:

$$p(\lambda)d\lambda = \text{const}\, \Delta(\lambda)^2 \prod_{i=1}^{N} e^{-\lambda_i^2/2N} d\lambda_i$$

where $\Delta(\lambda) := \det(\lambda_i^{j-1})_{1 \leq i,j \leq N}$ is the Vandermonde determinant.

- The n-point correlation function $\rho^{(n)}(\lambda_1, \ldots, \lambda_n)$ is the probability density of observing an eigenvalue at each of the $\lambda_1, \ldots, \lambda_n$.
- For GUE, the correlation functions are determinantal, i.e., it exists a correlation kernel $K : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\rho^{(n)}(\lambda_1, \ldots, \lambda_n) = \det(K(\lambda_i, \lambda_j))_{1 \leq i,j \leq n}.$$

Biorthogonal ensembles

- The GUE eigenvalues measure is a special case of a measure of the form

$$\mathrm{const}\ \det(\Phi_i(\lambda_j))_{1\leq i,j\leq N} \det(\Psi_i(\lambda_j))_{1\leq i,j\leq N} \prod_{i=1}^{N} \mu(d\lambda_i)$$

  called biorthogonal ensemble.
- The correlation functions of biorthogonal ensembles are determinantal. Borodin '98
- If the families $\{\Phi_i, 1 \leq i \leq N\}$ and $\{\Psi_j, 1 \leq j \leq N\}$ are chosen such that $\int d\mu(\lambda)\Phi_i(\lambda)\Psi_j(\lambda) = \delta_{i,j}$, then the kernel is given by

$$K(x,y) = \sum_{k=1}^{N} \Psi_k(x)\Phi_k(y)$$

- For GUE, the Ψ_k's and Φ_k's are given in terms of Hermite polynomials

- Using the explicit determinantal structure of the n-point correlations functions one obtains

$$\mathbb{P}(\lambda_{N,\max} \le a) = \mathbb{P}(\cap_{i=1}^{N}\{\lambda_i \le a\})$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_a^{\infty} dx_1 \cdots \int_a^{\infty} dx_n \det(K(x_i, x_j))_{1\le i,j\le n}$$
$$\equiv \det(\mathbb{1} - K)_{L^2((a,\infty))}.$$

- Edge scaling: $\lambda_{N,\max} \simeq 2N + \xi_2 N^{1/3}$.
A change of variable and asymptotic analysis gives a formula for F_2 Tracy, Widom '94

$$F_2(s) := \lim_{N\to\infty} \mathbb{P}(\lambda_{N,\max} \le 2N + sN^{1/3}) = \det(\mathbb{1} - K_2)_{L^2((s,\infty))}$$

with the Airy kernel

$$K_2(x, y) = \int_0^{\infty} d\lambda \mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda).$$

- Consider a permutation $\sigma \in \mathcal{S}_N$

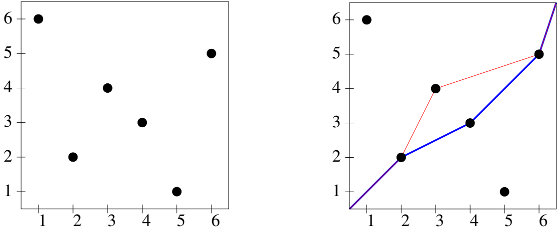
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \cdots & N \\ \sigma_1 & \sigma_2 & \sigma_3 & \cdots & \sigma_N \end{pmatrix}$$

and denote by $\ell_N(\sigma)$ the longest increasing subsequence in $\sigma = (\sigma_1, \ldots, \sigma_N)$.

- Example: $\ell_6(\sigma) = 3$ for

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 2 & 4 & 3 & 1 & 5 \end{pmatrix}$$

- Graphical representation: $k \mapsto \sigma_k$

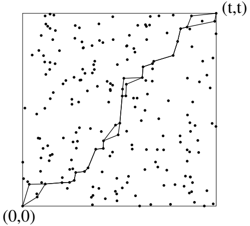


- Under uniform measure on $\mathcal{S}_N$ Baik, Deift, Johansson '99

$$\lim_{N\to\infty} \mathbb{P}(\ell_N \leq 2\sqrt{N} + sN^{1/6}) = F_2(s).$$

- For the proof one first studies a Poissonized version (a sort of "grand-canonical version" of the problem):

N is replaced by a random variable: $N \sim \text{Poisson}(t^2)$.

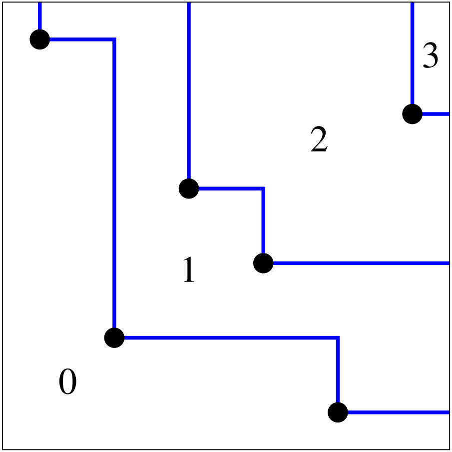


- Let L_t be the longest increasing subsequence in this setting, one first show

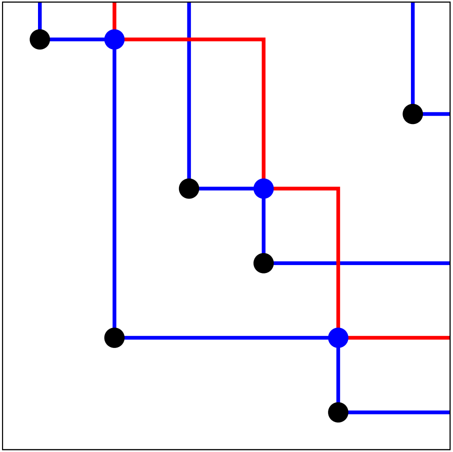
$$\lim_{t\to\infty} \mathbb{P}(L_t \leq 2t + st^{1/3}) = F_2(s).$$

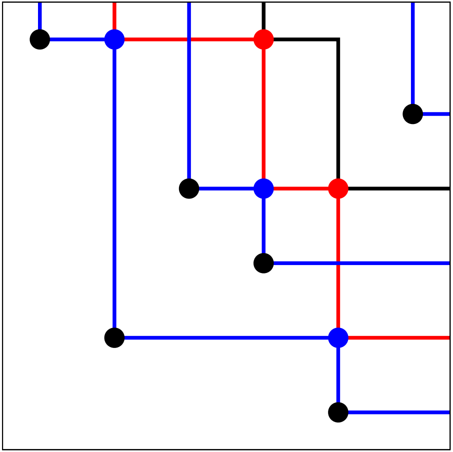
Why do the longest increasing subsequence shows the same fluctuation law as the largest eigenvalue of GUE matrices?

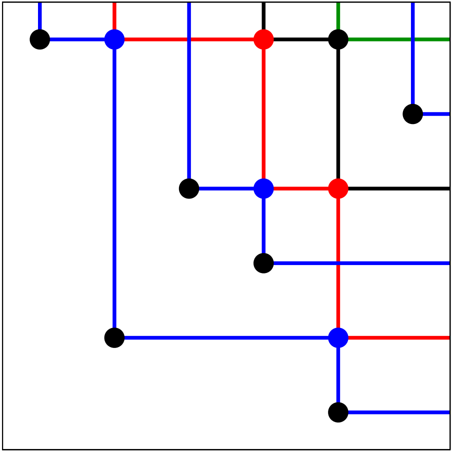
Line ensemble for LIS

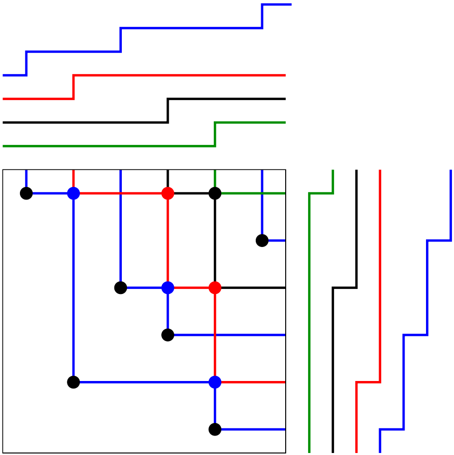


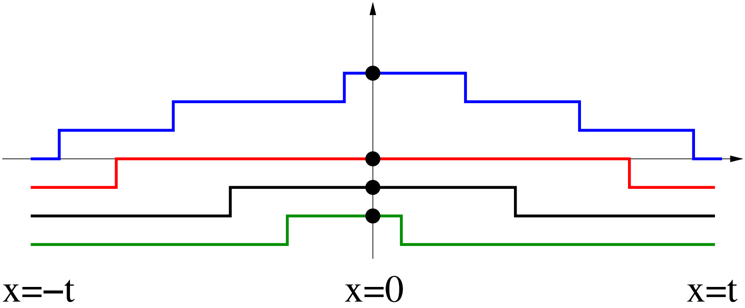
Line ensemble for LIS











Black dots at positions named $(X_1, X_2, X_3, X_4, \ldots)$;
$X_1 > X_2 > \ldots$

- For the Poissonized problem, the set of lines has the distribution as non-intersecting one-sided random walks starting and ending from fixed positions $0, -1, -2, \ldots$.
- For M non-intersecting lines, by the Karlin-Mc Gregor formula, the probability of seeing a configuration of black point $(X_1, X_2, \ldots, X_M)$ at $x = 0$ is given by

$$\text{const } \left[\det(p_t(-i, X_j)_{1\leq i,j\leq M}\right]^2$$

where $p_t(x, y) = e^{-t} t^{y-x}/(y-x)!$

$\Rightarrow$ The black dots have determinantal correlations.

- The biorthogonal ensemble has a kernel which, after $M \to \infty$ limit, becomes

$$K(x,y) = \sum_{\ell \geq 0} J_{\ell+x}(2t) J_{\ell+y}(2t)$$

with J the Bessel functions.

- Convergence to the Airy kernel K_2: under edge scaling

$$x = 2t + \xi t^{1/3}, \quad y = 2t + \zeta t^{1/3},$$

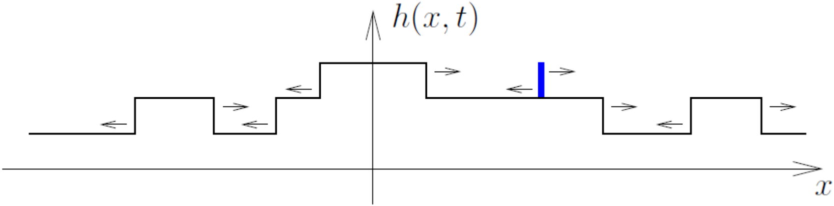
one has

$$t^{1/3} K(x,y) \to K_2(\xi,\zeta) \text{ as } t \to \infty$$

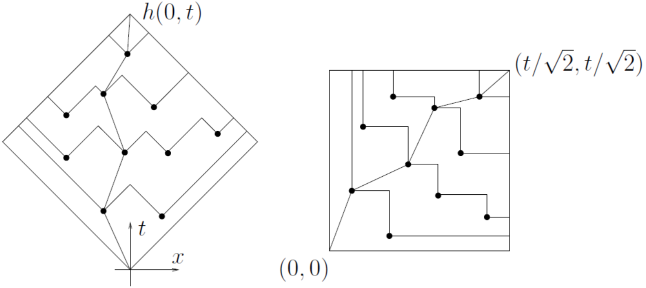
and

$$\lim_{t\to\infty} \mathbb{P}(X_1 \leq 2t + s t^{1/3}) = F_2(s).$$

- The polynuclear growth (PNG) model
- Height configurations: height function $x \mapsto h(x,t) \in \mathbb{Z}$, $x, t \in \mathbb{R}$.
- Dynamics, deterministic part: islands spread with unit speed, merges when touching
- Dynamics, stochastic part: nucleations (a spike of height 1) are added with intensity 2.
- PNG droplet: Nucleations restricted to the region $|x| \leq t$.

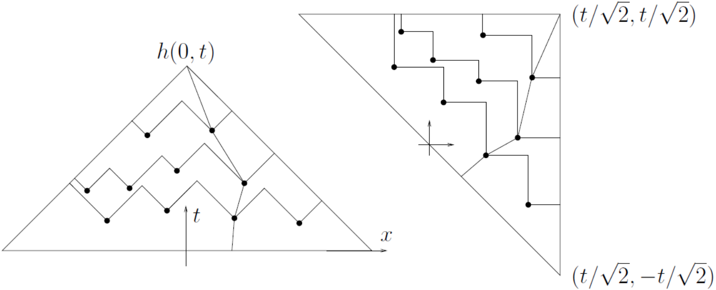


- PNG droplet: point-to-point problem



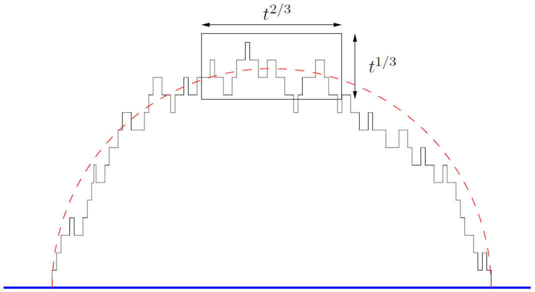
The lines are the space-time trajectories of the boundaries of the spreading islands

- Flat PNG: line-to-point problem



The lines are the space-time trajectories of the boundaries of the spreading islands

- For the PNG droplet, the line ensembles approach one can study also the top layer of the PNG multilayer.

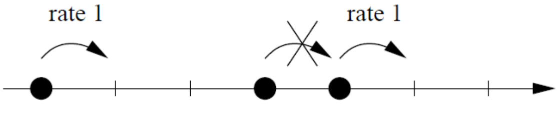


- The process of the fluctuations of the top layer is governed for large times by the Airy$_2$ process, $\mathcal{A}_2$ Prähofer, Spohn '02

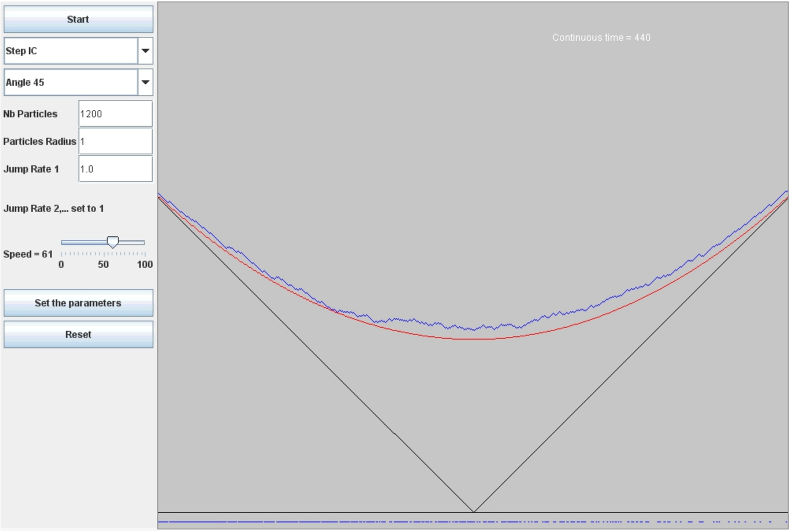
$$\lim_{t\to\infty} \frac{h(ut^{2/3},t) - 2t + u^2 t^{1/3}}{t^{1/3}} = \mathcal{A}_2(u)$$

in the sense of finite-dimensional distributions.

- TASEP: Totally Asymmetric Simple Exclusion Process
- Configurations: Particles are on $\mathbb{Z}$ and at most one particle for each site
- Dynamics: particles jumps to their right with rate 1 if the site is empty



- We use particle labels: $x_n(t) > x_{n+1}(t)$

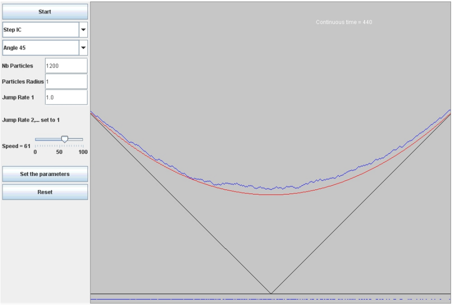


Start
Step IC
Angle 45
Nb Particles
1200
Particles Radius
1
Jump Rate 1
1.0
Jump Rate 2,... set to 1
Speed = 61
0
50
100
Set the parameters
Reset
Continuous time = 440

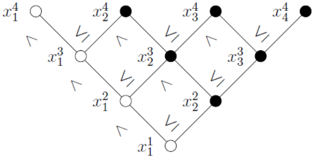
- Step initial condition: at time 0 particles occupy $\mathbb{Z}_-$
- For step IC, a multilayer approach gives Johansson'03

$$\lim_{t\to\infty} \mathbb{P}(x_{t/4}(t) \geq -s(t/2)^{1/3}) = F_2(s)$$

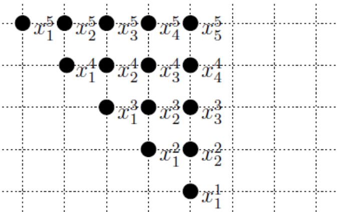
and joint distribution are governed by the Airy$_2$ process (this time one has Laguerre orthogonal polynomials).

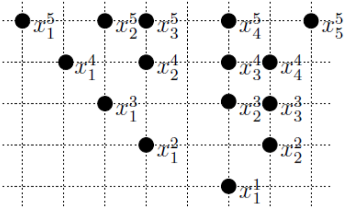


- An extension of TASEP dynamics on interlaced particles $\{x_k^n, 1 \leq k \leq n \leq N\}$: Borodin, Ferrari '08

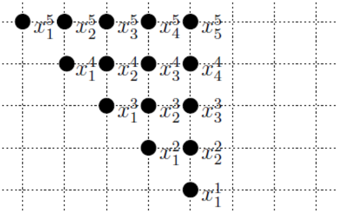


- Particles tries to jump to their right with rate 1
- Particles with smaller upper index have higher priority, so they block or push higher particles to satisfy interlacing





- For packed initial conditions

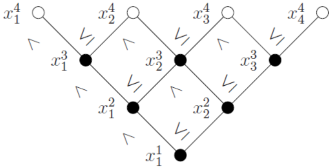


the particle system at any time $t \geq 0$ has determinantal correlations!

- The projection to the set

$$\{x_1^N, x_2^N, \dots, x_N^N\}$$

is still a Markov process (discrete analogue of the Dyson's Brownian Motion of random matrices)



- The measure on

$$\{x_1^N, x_2^N, \dots, x_N^N\}$$

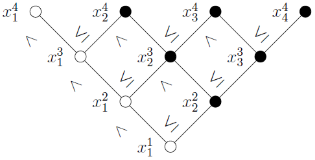
is a biorthogonal ensemble, similar to the GUE eigenvalues distributions (it arises under diffusion scalings)

- The kernel given in terms of Charlier orthogonal polynomials

- The projection to the set

$$\{x_1^1, x_1^2, \dots, x_1^N\}$$

is TASEP.



- In particular, the point x_1^N is common in both projections and when $N, t \to \infty$ has F_2 fluctuations.

- The interlacing structure was first obtained by Sasamoto '05: starting from a formula by Schütz '97 he extended the picture by adding "summation variables" (the x_k^n for $k \geq 2$)
- Algebraically one can think of the extended picture to have "determinantal correlations" although the measure is not anymore necessarily positive, i.e., it is not always a probability measure.
- Only the projection to $\{x_1^1, x_1^2, \ldots, x_1^N\}$ is ensured, a priori, to be a probability measure.

- The interlacing approach allowed to study the "flat" initial condition, where for TASEP particles starts from $2\mathbb{Z}$.
- The result is that the discovery of the Airy$_1$ process, the analogue of the Airy$_2$ process for flat interfaces in KPZ growth models.

  Sasamoto'05, Borodin, Ferrari, Prähofer, Sasamoto '06-'08
- A new determinantal formula for F_1 is obtained

  Sasamoto'05, Ferrari, Spohn'05

$$F_1(2s) = \det(\mathbb{1} - K_1)_{L^2((s,\infty))}$$

where $K_1(x,y) = \mathrm{Ai}(x+y)$.

- The approach with interlacing particles can be used to obtain the flat PNG height fluctuations / line-to-point problem in the Poisson point picture Borodin, Ferrari, Sasamoto '07
- It allows to study also transition processes from flat to curved interface Borodin, Ferrari, Sasamoto '08
- Results in a Fredholm determinant formula for the joint distributions of height fluctuations.

vs.

- The approach with interlacing particles can be used to obtain the flat PNG height fluctuations / line-to-point problem in the Poisson point picture Borodin, Ferrari, Sasamoto '07
- It allows to study also transition processes from flat to curved interface Borodin, Ferrari, Sasamoto '08
- Results in a Fredholm determinant formula for the joint distributions of height fluctuations.

vs.

- The multilayer version of the flat PNG leads to a Pfaffian correlation functions at a single position only Ferrari'04
- Its scaling limit for $t \to \infty$ leads to the analogue of F_2 for symmetric matrices, namely the GOE Tracy-Widom distribution, F_1.

$\Rightarrow$ Recovers Fredholm Pfaffian formula for F_1 by Tracy, Widom '96