Infinite-dimensional stochastic differential equations arising from random matrix theory

2014/8/22/Fri Kyoto

Interface fluctuations and KPZ universality class

- unifying mathematical, theoretical, and experimental approaches

Workshop dates: 2014-08-20 – 2014-08-23

Outline:

- Dynamical soft edge scaling limit: Airy_β RPFs ($\beta = 1, 2, 4$)
- Dynamical bulk scaling limit: Sine RPFs and an SDE gap
- Ginbre and Bessel RPFs

Geometric scaling limit

Geometric soft edge/bulk scaling limits
of Gaussian ensembles

- The distribution of eigen values of the G(O/U/S)E Random Matrices are given by ($\beta = 1, 2, 4$)

$$m_\beta^N(d\mathbf{x}_N) = \frac{1}{Z}\prod_{i<j}^N |x_i - x_j|^\beta e^{-\frac{\beta}{4}\sum_{i=1}^N |x_i|^2} d\mathbf{x}_N, \qquad (1)$$

- The distribution of

$$\frac{1}{N}\sum_{i=1}^N \delta_{x_i/\sqrt{N}} \qquad \text{under } m_\beta^N$$

converges to the semi-circle law

$$\varsigma(x)dx = \frac{1}{2\pi}\sqrt{4 - x^2}dx \qquad (2)$$

Bulk/Soft edge scaling

$$m_\beta^N(d\mathbf{x}_N) = \frac{1}{Z}\prod_{i<j}^{N}|x_i - x_j|^\beta e^{-\frac{\beta}{4}\sum_{i=1}^N |x_i|^2} d\mathbf{x}_N, \qquad (1)$$

$$\varsigma(x)dx = \frac{1}{2\pi}\sqrt{4 - x^2}dx$$

- Bulk scaling: For $-2 < \theta < 2$ take $x_i = (s_i - \theta)/\sqrt{N}$ in (1):

$$\mu_{\mathrm{sin},\beta,\theta}^N(d\mathbf{s}_N) = \frac{1}{Z}\prod_{i<j}^{N}|s_i - s_j|^\beta \prod_{k=1}^{N} e^{-\beta|s_k-\theta|^2/4N} d\mathbf{s}_N \qquad (3)$$

- Soft edge scaling: Take $x_i \mapsto 2\sqrt{N} + s_i N^{-1/6}$ in (1):

$$\mu_{\mathrm{Ai},\beta}^N(d\mathbf{s}_N) = \frac{1}{Z}\prod_{i<j}^{N}|s_i - s_j|^\beta e^{-\frac{\beta}{4}\sum_{i=1}^N |2\sqrt{N}+N^{-1/6}s_i|^2} d\mathbf{s}_N.$$

Soft edge scaling limit

Airy RPF: $\mu_{\mathsf{Ai},\beta}$ $(\beta = 1, 2, 4)$

- Take the scaling $x_i \mapsto 2\sqrt{N} + s_i N^{-1/6}$ in

$$m_\beta^N(d\mathbf{x}_N) = \frac{1}{Z}\prod_{i<j}^{N} |x_i - x_j|^\beta e^{-\frac{\beta}{4}\sum_{i=1}^N |x_i|^2} d\mathbf{x}_N$$

and set

$$\mu_{\mathsf{Ai},\beta}^N(d\mathbf{s}_N) = \frac{1}{Z}\prod_{i<j}^{N} |s_i - s_j|^\beta e^{-\frac{\beta}{4}\sum_{i=1}^N |2\sqrt{N}+N^{-1/6}s_i|^2} d\mathbf{s}_N.$$

Then $\mu_{\mathsf{Ai},\beta}^N$ converge to Airy RPF $\mu_{\mathsf{Ai},\beta}$:

$$\lim_{N\to\infty} \mu_{\mathsf{Ai},\beta}^N = \mu_{\mathsf{Ai},\beta}$$

Airy RPF – Soft edge scaling limit

- $\beta = 2 \Rightarrow \mu_{\mathsf{Ai},\beta}$ is a determinantal RPF given by (K_{Ai}, dx):

$$K_{\mathsf{Ai}}(x, y) = \frac{\mathsf{Ai}(x)\mathsf{Ai}'(y) - \mathsf{Ai}'(x)\mathsf{Ai}(y)}{x - y}$$

Here $\mathsf{Ai}(\cdot)$ is the Airy function.
The correlation function ρ^n_{Ai} is defined as

$$\rho^n_{\mathsf{Ai}}(\mathbf{x}) = \det[K_{\mathsf{Ai}}(x_i, x_j)]^n_{i,j=1}.$$

- If $\beta = 1, 4$, the correlation func of $\mu_{\mathsf{Ai},\beta}$ are given by similar formula of quaternion determinant.

- We discuss a dynamical counter part of this scaling.

Airy RPF – Dynamical soft edge scaling limit

- From

$$\mu^N_{\mathrm{Ai},\beta}(d\mathbf{s}_N) = \frac{1}{Z}\prod_{i<j}|s_i - s_j|^\beta e^{-\frac{\beta}{4}\sum_{i=1}^N |2\sqrt{N}+N^{-1/6}s_i|^2} d\mathbf{s}_N$$

we deduce the SDE of the N particle system:

$$dX_t^{N,i} = dB_t^i + \frac{\beta}{2}\sum_{j=1, j\neq i}^N \frac{1}{X_t^{N,i} - X_t^{N,j}}dt - \frac{\beta}{2}\{N^{1/3} + \frac{1}{2N^{1/3}}X_t^{N,i}\}dt$$

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- From

$$\mu^N_{\mathsf{Ai},\beta}(d\mathbf{s}_N) = \frac{1}{Z}\prod_{i<j}|s_i - s_j|^\beta e^{-\frac{\beta}{4}\sum_{i=1}^N |2\sqrt{N}+N^{-1/6}s_i|^2} d\mathbf{s}_N$$

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- Indeed, $X_t^{N,i}$ are associated with the Dirichlet form:

$$\mathcal{E}^{\mu^N_{\mathsf{Ai},\beta}}(f,g) = \int_{\mathbb{R}^N}\frac{1}{2}\sum_i^N \frac{\partial f}{\partial s_i}\frac{\partial g}{\partial s_i}\mu^N_{\mathsf{Ai},\beta}(d\mathbf{s}_N) \text{ on } L^2(\mathbb{R}^N, \mu^N_{\mathsf{Ai},\beta}).$$

Then, by integration by parts, the generator is

$$-L^N = \frac{1}{2}\Delta_N + \frac{\beta}{2}\sum_{i=1}^N[\sum_{j\neq i}^N \frac{1}{s_i - s_j} - \frac{\beta}{2}\{N^{1/3} + \frac{s_i}{2N^{1/3}}\}]\frac{\partial}{\partial s_i}$$

Airy RPF – Dynamical soft edge scaling limit

- The SDE of the N particle system:

$$dX_t^{N,i} = dB_t^i + \frac{\beta}{2} \sum_{j=1, j \neq i}^{N} \frac{1}{X_t^{N,i} - X_t^{N,j}} dt - \frac{\beta}{2} \{ N^{1/3} + \frac{1}{2N^{1/3}} X_t^{N,i} \} dt$$

- The dynamics are also given by the space-time correlation functions.

- Problem: What SDE does the limit $\mathbf{X}_t = \lim_{N \to \infty} \mathbf{X}_t^N$ satisfy?

$$\text{Does } \lim_{N \to \infty} \{ \sum_{j=1, j \neq i}^{N} \frac{1}{X_t^{N,i} - X_t^{N,j}} - N^{1/3} \} \quad \text{converge ?}$$

How to solve the limit ISDE?

Airy RPF – Dynamical soft edge scaling limit

For a configuration $\mathsf{s} = \sum_i \delta_{s_i}$, let $\ell(\mathsf{s}) = (s_1, s_2, \dots,) = \mathbf{s} \in \mathbb{R}^{\mathbb{N}}$ be a label such that $s_1 > s_2 > \cdots$, which is well defined for $\mu^{\ell}_{\mathsf{Ai},\beta}$-a.s..

Thm 1 (O.-Tanemura '14). [Existence of strong solutions]
Let $\beta = 1, 2, 4$. Define ISDE (4) of $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$ as

$$dX^i_t = dB^i_t + \frac{\beta}{2} \lim_{r\to\infty} \{ (\sum_{j\neq i,\ |X^j_t|<r} \frac{1}{X^i_t - X^j_t}) - \int_{|x|<r} \frac{\varrho(x)}{-x} dx \} dt \qquad (4)$$

$$\varrho(x) = \frac{\sqrt{-x}}{\pi} 1_{(-\infty,0]}(x).$$

- *For $\mu^{\ell}_{\mathsf{Ai},\beta}$-a.s. $\mathbf{s}$, ISDE (4) has a strong solution with $\mathbf{X}_0 = \mathbf{s}$.*
- *The associated unlabeled dynamics $\mathsf{X}_t = \sum_{i=1}^{\infty} \delta_{X^i_t}$ is $\mu_{\mathsf{Ai},\beta}$-reversible.*
- *If $\beta = 2$ and $\mathbf{X}_0 \sim \mu^{\ell}_{\mathsf{Ai},2}$, then $X^1_t \sim F_2$. Here F_2 is the Tracy-Widom distribution and X^1_t is the Airy process $\mathcal{A}(t)$.*

Remarks:

- The key idea to derive the limit ISDE is to take the rescaled semi-circle law ς^N:

$$\varsigma^N(x) := N^{1/3}\varsigma(\frac{x}{N^{2/3}} + 2)$$
$$= \frac{1_{(-4N^{2/3},0)}}{\pi}\sqrt{-x(1 + \frac{x}{4N^{2/3}})}$$

as the first approximation of the 1-correlation fun $\rho^{N,1}_{\mathrm{Ai},\beta}$.

- We expect that our method can be applied to other soft edge scaling.
- The SDE gives a kind of Girsanov formula. This yields that finite particles $(X^1_t, \dots, X^M_t)$ are absolutely continuous with respect to M-dimensional Brownian motion under the distribution conditioned $(X^{M+1}_t, \dots)$. From this we can solve the conjecture of Johansson. This conjecture has been already solved by Corwin-Hammond ('14) and Hägg ('08) by a different method.

Airy RPF – Dynamical soft edge scaling limit

Thm 2 (O.-Tanemura '14). [Pathwise uniqueness]

Let $\beta = 1, 2, 4$. *Then:*

- *Solutions of ISDE* (4) *of* $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$ *starting at* s

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r\to\infty} \{ (\sum_{j\neq i,\ |X_t^j|<r} \frac{1}{X_t^i - X_t^j}) - \int_{|x|<r} \frac{\varrho(x)}{-x} dx \} dt \qquad (4)$$

satisfying abs cont cond (5) *are pathwise unique for* $\mu_{\mathsf{Ai},\beta}^{\ell}$*-a.s.* s*.*

$$\mu_{\mathsf{Ai},\beta,\mathsf{t}} \circ \mathsf{X}_t^{-1} \prec \mu_{\mathsf{Ai},\beta,\mathsf{t}} \qquad \text{for } \mu_{\mathsf{Ai},\beta}\text{-a.s. } \mathsf{t}. \qquad (5)$$

Here $\mu_{\mathsf{Ai},\beta,\mathsf{t}}$ *is a regular conditional probability w.r.t. to the tail* σ*-field* $\mathcal{T}$ *of the configuration space.*

- *If* $\beta = 2$*, then* $\mathcal{T}$ *is* $\mu_{\mathsf{Ai},\beta}$*-trivial. Hence the uniqueness holds.*
- *The solutions in Thm 1 satisfy* (5). *Hence tail preserving solutions exist uniquely.*
- *Weak solutions satisfying* (5) *are automatically unique strong solutions.*

Airy RPF – Dynamical soft edge scaling limit

If $\beta = 2$, then Johansson, Spohn, Katori-Tanemura & others show that there exist stochastic dynamics associated with $\mu_{\mathsf{Ai},2}$ given by the space-time correlation function given by the extended Airy kernel:

$$K_{\mathsf{Ai}}(s,x;t,y) = \begin{cases} \int_0^\infty du e^{-u(t-s)/2} \mathsf{Ai}(u+x)\mathsf{Ai}(u+y), & t \geq s \\ -\int_{-\infty}^0 du e^{-u(t-s)/2} \mathsf{Ai}(u+x)\mathsf{Ai}(u+y), & t < s \end{cases}$$

Thm 3 [O.-Tanemura, '14]
Let $\beta = 2$. Then these two dynamics are the same.

- This comes from the uniqueness of Dirichlet forms associated with these dynamics. To prove the uniqueness of Dirichlet forms, we use the uniqueness of weak solutions of the ISDE (4) .

Airy RPF – Dynamical soft edge scaling limit

Let $\mathbf{X}_t^N = (X_t^{N,i})_{i=1}^N$ be the N-particle system as before:

$$dX_t^{N,i} = dB_t^i + \frac{\beta}{2} \sum_{j=1, j\neq i}^{N} \frac{1}{X_t^{N,i} - X_t^{N,j}} dt - \frac{\beta}{2} \{N^{1/3} + \frac{1}{2N^{1/3}} X_t^{N,i}\} dt$$

Set $\mathbf{X}^{N,m}$ be the first m-component.

$$\mathbf{X}^{N,m} = (X_t^{N,1}, \dots, X_t^{N,m})$$

Thm 4 [O.-Tanemura, O.-Kawamoto] (Finite-particle approximation)
Let $\beta = 1, 2, 4$. Then for each $0 \leq \varphi \in L^1(\mu_{\mathrm{Ai},\beta}^{\ell})$ with $\int \varphi \mu_{\mathrm{Ai},\beta}^{\ell} = 1$, $\mathbf{X}^{N,m}$ with $\mathbf{X}_0^{N,m} \sim \varphi \mu_{\mathrm{Ai},\beta}^{\ell}$ converge to the first m-component $\mathbf{X}^m$ of the solution of the limit ISDE weakly in $C([0,\infty); \mathbb{R}^m)$.

- When $\beta = 2$, we have two proofs.

Bulk scaling limit & an SDE gap

- Bulk scaling: For $-2 < \theta < 2$ take $x_i = (s_i - \theta)/\sqrt{N}$ in (1):

$$\mu^N_{\text{sin},\beta,\theta}(d\mathbf{s}_N) = \frac{1}{Z}\prod_{i<j}^{N} |s_i - s_j|^\beta \prod_{k=1}^{N} e^{-\beta|s_k-\theta|^2/4N} d\mathbf{s}_N \qquad (6)$$

Then $\mu^N_{\text{sin},\beta,\theta}$ converge to sine_β RPF:

$$\lim_{N\to\infty} \mu^N_{\text{Sine},\beta,\theta} = \mu_{\text{Sine},\beta}$$

The right-hand side is independent of θ up to constant scaling.
If $\beta = 2$, then $\mu_{\text{Sine},\beta}$ is determinantal with kernel

$$\mathsf{K}_2(x,y) = \sqrt{1 - (\frac{\theta}{2})^2}\,\frac{\sin(x-y)}{\pi(x-y)}$$

Bulk scaling

- The associated N particle system is given by the SDE:

$$dX_t^{N,i} = dB_t^i + \frac{\beta}{2}\sum_{j\neq i}^{N} \frac{1}{X_t^{N,i} - X_t^{N,j}} dt - \frac{\beta}{4N} X_t^{N,i} dt + \frac{\beta\theta}{4} dt \tag{7}$$

- Very loosely, the associated ∞ particle system is given by

$$dX_t^i = dB_t^i + \frac{\beta}{2}\sum_{j\neq i}^{\infty} \frac{1}{X_t^i - X_t^j} dt + \frac{\beta\theta}{4} dt.$$

This is not the case for $\theta \neq 0$.

The limit ISDE is for all θ

$$dX_t^i = dB_t^i + \frac{\beta}{2}\lim_{r\to\infty}\{\sum_{j\neq i,\ |X_t^j|<r} \frac{1}{X_t^i - X_t^j}\} dt \tag{8}$$

Bulk scaling: limit ISDE

Thm 5 [O.-Tanemura '14] [Existence of strong solutions]
Let $\beta = 1, 2, 4$. Define ISDE (4) of $\mathbf{X} = (X^i)_{i \in \mathbb{N}}$ as

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r \to \infty} \{ \sum_{j \neq i,\ |X_t^j| < r} \frac{1}{X_t^i - X_t^j} \} dt \tag{8}$$

- For $\mu_{\mathsf{Sine},\beta}^{\ell}$-a.s. s, ISDE (8) has a strong solution with $\mathbf{X}_0 = \mathsf{s}$.
- The associated unlabeled dynamics $\mathsf{X}_t = \sum_{i=1}^{\infty} \delta_{X_t^i}$ is $\mu_{\mathsf{Sine},\beta}$-reversible.

Thm 6 [O.-Tanemura '14] [Pathwise uniqueness]

Let $\beta = 1, 2, 4$. Then:

• Solutions of ISDE (8) of $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$ starting at s

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r\to\infty} \{ \sum_{j\neq i,\ |X_t^j|<r} \frac{1}{X_t^i - X_t^j} \} dt \tag{8}$$

satisfying abs cont cond (9) are pathwise unique for $\mu^{\ell}_{\mathsf{Sine},\beta}$-a.s. s.

$$\mu_{\mathsf{Sine},\beta,\mathsf{t}} \circ \mathsf{X}_t^{-1} \prec \mu_{\mathsf{Sine},\beta,\mathsf{t}} \quad \text{for } \mu_{\mathsf{Sine},\beta}\text{-a.s. t.} \tag{9}$$

• If $\beta = 2$, then $\mathcal{T}$ is $\mu_{\mathsf{Sine},\beta}$-trivial. Hence the uniqueness holds.

• The solutions in Thm satisfy (9). Hence tail preserving solutions exist uniquely.

• Weak solutions satisfying (9) are automatically unique strong solutions.

• If $\beta = 2$, the solution equal to the stochastic dynamics given by space-time correlation functions (extended Sine kernels).

Sine RPF - Dynamical bulk scaling limit

Let $\mathbf{X}_t^N = (X_t^{N,i})_{i=1}^N$ be the N-particle system as before:

$$dX_t^{N,i} = dB_t^i + \frac{\beta}{2}\sum_{j\neq i}^{N} \frac{1}{X_t^{N,i} - X_t^{N,j}}dt - \frac{\beta}{4N}X_t^{N,i}dt + \frac{\beta\theta}{4}dt \tag{7}$$

Thm 7 [O.-Tanemura, O.-Kawamoto] (Finite-particle approxim) Let $\beta = 1, 2, 4$. Then for each $0 \le \varphi \in L^1(\mu_{\mathrm{Ai},\beta}^{\ell})$ with $\int \varphi\mu_{\mathrm{Ai},\beta}^{\ell} = 1$, $\mathbf{X}^{N,m}$ with $\mathbf{X}_0^{N,m} \sim \varphi\mu_{\mathrm{Ai},\beta}^{\ell}$ converge to the first m-component $\mathbf{X}^m$ of the solution of the limit ISDE

$$dX_t^i = dB_t^i + \frac{\beta}{2}\lim_{r\to\infty}\{\sum_{j\neq i,\ |X_t^j|<r} \frac{1}{X_t^i - X_t^j}\}dt \tag{8}$$

weakly in $C([0,\infty);\mathbb{R}^m)$.

- The limit ISDE (8) is independent of θ. In this sense, an SDE gap occurs.

Bessel RPF & a hard edge scaling

Thm 8 [O.-Honda, '14] Let $a > 1$ and $\beta = 2$. Let $\mu^{a}_{\text{bes},2}$ be the Bessel^{a}_{2} RPF. Then the associated ISDE is given by the following, and has a unique strong solution as in the same meaning of the previous theorems.

$$dX^i_t = dB^i_t + \frac{a}{2X^i_t}dt + \frac{\beta}{2}\sum_{j=1, j\neq i}^{\infty} \frac{1}{X^i_t - X^j_t}dt.$$

Thm 9 The associated N-particle system $\mathbf{X}^N_t = (X^{N,1}_t, \dots, X^{N,N}_t)$ converge to the limit $\mathbf{X}_t = (X^i_t)_{i\in\mathbb{N}}$ in the same sense as before.

- $\beta = 1, 4$ is in progress.

Ginibre RPF : Non-hermitian Gaussian random matrixes

Ginibre RPF is a determinantal RPF on $\mathbb{C}$ with exponential kernel.

Thm 11 [O.,'13, O.-Tanemura '14] Let μ_{gin} be a Ginibre RPF. Then the associated ISDE is given by the following, and has a unique strong solution as in the same meaning of the previous theorems.

$$dX_t^i = dB_t^i + \lim_{r\to\infty} \sum_{\substack{|X_t^i - X_t^j| < r \\ j \neq i}} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} dt.$$

The solution also satisfy the following ISDEs for all $a \in \mathbb{C}$:

$$dX_t^i = dB_t^i - (X_t^i - a)dt + \lim_{r\to\infty} \sum_{\substack{|a - X_t^j| < r \\ j \neq i}} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} dt$$

The associated N-particle system converge to the limit in the same sense as before.

Idea of "weak/strong solutions of ISDEs"

- General theory to construct unique, strong solutions of infinite-dimensional stochastic differential equations

- Weak solution: (O. JPSJ 10, PTRF 12, AOP 13, SPA 13)
- logarithmic derivative d^{μ}: Very informally,

$$\mathsf{d}^{\mu}(x, \mathsf{y}) = \nabla_x \log \mu^{[1]}$$

Here $\mu^{[1]}$ is a 1-Campbell measure of μ.

- μ is quasi-Gibbs
- mariginal assumptions

Then ISDE has a weak solution $(\mathbf{X}, \mathbf{B})$:

$$dX_t^i = dB_t^i + \frac{1}{2}\mathsf{d}^{\mu}(X_t^i, \sum_{j \neq i}^{\infty} \delta_{X_t^j}) \quad (i \in \mathbb{N})$$

- Strong solutions and uniqueness:
- IFC solutions, tail analysis.

Strong solutions of ISDE: Non Markov type

$S = \mathbb{R}^d, [0,\infty), \mathbb{C}$

$W(S^{\mathbb{N}}) = C([0,T); S^{\mathbb{N}})$, $(0 < T < \infty)$ labeled path sp.

- a quadruplet $(\{\sigma^i\}, \{b^i\}, W_{\mathsf{sol}}, \mathbf{S_0})$

W_{sol} : a Borel subset of $W(S^{\mathbb{N}})$ sp of solutions of ISDE

$\sigma^i, b^i : W_{\mathsf{sol}} \to W(S^{\mathbb{N}})$ coefficients of ISDE

$\mathbf{S_0}$ be a Borel subset of $S^{\mathbb{N}}$ initial starting points of ISDE

- the ISDE on $S^{\mathbb{N}}$ of the form

$$dX_t^i = \sigma^i(\mathbf{X})_t dB_t^i + b^i(\mathbf{X})_t dt \quad (i \in \mathbb{N}) \tag{10}$$

$$\mathbf{X}_0 = \mathbf{s} = (s_i)_{i\in\mathbb{N}} \in \mathbf{S_0} \tag{11}$$

$$\mathbf{X} \in W_{\mathsf{sol}}. \tag{12}$$

- $\mathbf{X} = \{(X_t^i)_{i\in\mathbb{N}}\}_{t\in[0,T)} \in W_{\mathsf{sol}}$
- $\mathbf{B} = (B^i)$ $(i \in \mathbb{N})$ is the $S^{\mathbb{N}}$-valued standard Br motion.

Strong solutions of ISDE: Assump (P1)

$$dX_t^i = \sigma^i(\mathbf{X})_t dB_t^i + b^i(\mathbf{X})_t dt \quad (i \in \mathbb{N})$$
$$\mathbf{X}_0 = \mathbf{s} = (s_i)_{i\in\mathbb{N}} \in \mathbf{S_0}$$
$$\mathbf{X} \in W_{\text{sol}}.$$

(P1) ISDE (10) has a solution $(\mathbf{X}, \mathbf{B})$. (not a strong sol!)

Here $\mathbf{B} = (B^i)_{i\in\mathbb{N}}$ is the Brownian motion on $S^{\mathbb{N}}$

Problem: Prove that $\mathbf{X}$ is a functional of the Br $\mathbf{B}$

Idea:
Strong solutions of Infinite-dimensional SDE
$\Leftrightarrow$
Infinite-many, finite-dimensional SDEs with consistency
$+$
Triviality of Tail σ-field of label paths

Assump (P2) infinite-many, finite-dimensional SDEs with consistency

- $\bar{P}_{\mathbf{s}}$: a prob meas on $W(S^{\mathbb{N}}) \times W^{\mathbf{0}}(S^{\mathbb{N}})$
- $\bar{P}_{\mathbf{s},\mathbf{B}} = \bar{P}_{\mathbf{s}}(\mathbf{X} \in \cdot|\mathbf{B})$: the regular conditional prob
- $\mathbf{P}_{\mathbf{s}} = \bar{P}_{\mathbf{s}}(\mathbf{X} \in \cdot)$, $\quad P_{\mathsf{Br}}^{\infty} = \bar{P}_{\mathbf{s}}(\mathbf{B} \in \cdot)$

For $\mathbf{X} \in W_{\mathsf{sol}}$, $\mathbf{s} \in \mathbf{S_0}$, and $m \in \mathbb{N}$,
we introduce a new SDE (15) on $\mathbf{Y}^m = (Y_t^1, \dots, Y_t^m)$.

$$dY_t^i = \sigma^i(\mathbf{Y}^m + \mathbf{X}^{m*})_t dB_t^i + b^i(\mathbf{Y}^m + \mathbf{X}^{m*})_t dt \tag{13}$$

$$\mathbf{Y}_0^m = (s_1, \dots, s_m) \in \mathsf{S}^m, \quad \text{where } \mathbf{s} = (s_i)_{i=1}^{\infty},$$

$$\mathbf{Y}^m + \mathbf{X}^{m*} \in W_{\mathsf{sol}}.$$

Here $\mathbf{X}^{m*} = (0, \dots, 0, X_t^{m+1}, X_t^{m+2}, \dots)$ and we set

$$\mathbf{Y}^m + \mathbf{X}^{m*} = (Y_t^1, \dots, Y_t^m, X_t^{m+1}, X_t^{m+2}, \dots). \tag{14}$$

$\mathbf{X}^{m*}$ is interpreted as a part of the coefficients of the SDE (15).

Strong solutions of ISDE: (P2) seq of finite-dim SDEs with consistecy

$$dY_t^i = \sigma^i(\mathbf{Y}^m + \mathbf{X}^{m*})_t dB_t^i + b^i(\mathbf{Y}^m + \mathbf{X}^{m*})_t dt \tag{15}$$
$$\mathbf{Y}_0^m = (s_1, \ldots, s_m) \in \mathsf{S}^m,$$
$$\mathbf{Y}^m + \mathbf{X}^{m*} \in W_{\mathsf{sol}}.$$

(P2) The SDE (15) has a unique, strong solution for each $\mathbf{s} \in \mathbf{S_0}$, $\mathbf{X} \in W_{\mathsf{sol}}^{\mathbf{s}}$, and $m \in \mathbb{N}$.

Strong solutions of ISDE: (P3) Tail triviality

Let $\mathit{Tail}\,(W(S^{\mathbb{N}}))$ be the tail σ-field of $W(S^{\mathbb{N}})$; we set

$$\mathit{Tail}\,(W(S^{\mathbb{N}})) = \bigcap_{m=1}^{\infty} \sigma[\mathbf{X}^{m*}]. \tag{16}$$

Here $\mathbf{P}$ is a probability measure on $W(S^{\mathbb{N}})$.

(P3) $\mathit{Tail}\,(W(S^{\mathbb{N}}))$ is $\mathbf{P}_{\mathbf{s}}$-trivial for each $\mathbf{s} \in \mathbf{S_0}$.

Strong solutions of ISDE: Main Theorem 1

(P1) ISDE (10) has a solution $(\mathbf{X}, \mathbf{B})$.
(P2) SDE (15) has a unique, strong solution for all $\mathbf{s}, \mathbf{X}, m$.
(P3) $\mathit{Tail}\,(W(S^{\mathbb{N}}))$ is $\mathbf{P}_{\mathbf{s}}$-trivial for each $\mathbf{s} \in \mathbf{S_0}$.

Thm 3. *Assume* (P1)–(P3). *Then*
(1) *ISDE* (10)–(12) *has a strong solution for each* $\mathbf{s} \in \mathbf{S}_0$.
(2) *Let* $\mathbf{Y}_{\mathbf{s}}$ *and* $\mathbf{Y}'_{\mathbf{s}}$ *be strong solutions of ISDE* (10)–(12) *starting at* $\mathbf{s} \in \mathbf{S}_0$ *defined on the same space of Brownian motions* $\mathbf{B}$. *Then* $\mathbf{Y}_{\mathbf{s}} = \mathbf{Y}'_{\mathbf{s}}$ *a.s. if and only if*

$$\mathit{Tail}^{[1]}(\mathrm{Law}(\mathbf{Y}_{\mathbf{s}})) = \mathit{Tail}^{[1]}(\mathrm{Law}(\mathbf{Y}'_{\mathbf{s}})). \tag{17}$$

Here

$$\mathit{Tail}^{[1]}(\mathbf{P}) = \{A \in \mathit{Tail}\,(W(S^{\mathbb{N}}))\,;\, \mathbf{P}(A) = 1\}$$

• Thus the tail σ-field of the labeled path can be regarded as a boundary condition of ISDEs.

Strong solutions of ISDE: Idea of Main Theorem 1 (1)

(P1) ISDE (10) has a solution $(\mathbf{X}, \mathbf{B})$.

(P2) SDE (15) has a unique, strong solution for all $\mathsf{s}, \mathbf{X}, m$.

(P3) $\mathit{Tail}\,(W(S^{\mathbb{N}}))$ is $\mathbf{P}_{\mathsf{s}}$-trivial for each $\mathsf{s} \in \mathbf{S_0}$.

- $(\mathbf{X}, \mathbf{B})$: sol of ISDE by (P1). Let $(\mathbf{X}, \mathbf{B})$ be fixed.
- $\mathbf{Y}^m$ is a unique strong sol of SDE(14) by (P2)
- $\mathbf{Y}^m$ is $\sigma[\mathbf{B}] \bigvee \sigma[\mathbf{X}^{m*}]$-m'ble. $\mathbf{X}^{m*} = (X^n)_{m<n<\infty}$.
- $\mathbf{Y}^m = (X^1, \ldots, X^m)$. by (P2)
- $\mathbf{X}$ is $\sigma[\mathbf{B}] \bigvee \mathit{Tail}\,(W(S^{\mathbb{N}}))$-m'ble by $m \to \infty$.
- $\mathit{Tail}\,(W(S^{\mathbb{N}}))$ is trivial by (P3) $\Rightarrow$ $\mathbf{X}$ is a strong solution.

Strong solutions of ISDE: How to prove (P1)–(P3)

(P1) ISDE (10) has a solution $(\mathbf{X}, \mathbf{B})$.
(P2) SDE (15) has a unique, strong solution for all $\mathsf{s}, \mathbf{X}, m$.
(P3) $\mathit{Tail}\,(W(S^{\mathbb{N}}))$ is $\mathbf{P}_{\mathsf{s}}$-trivial for each $\mathsf{s} \in \mathbf{S_0}$.

- (P1) follows from a general theory of O..
- (P2) is classical.
- How to prove (P3) **?** $\Rightarrow$ Tail Theorems.

- We prove (P3) for ISDE of the form

$$dX_t^i = \sigma(X_t^i, \sum_{j\neq i} \delta_{X_t^j}) dB_t^i + b(X_t^i, \sum_{j\neq i} \delta_{X_t^j}) dt$$

Here $a = \sigma^t \sigma$ and

$$b(x, \mathsf{y}) = \frac{1}{2}\{\nabla a(x, \mathsf{y}) + a(x, \mathsf{y})\mathsf{d}^{\mu}(x, \mathsf{y})\}dt$$

$\mathsf{d}^{\mu}(x, \mathsf{y})$ is the logarithmic derivative (informally) defined as

$$\nabla_x \log \mu^{[1]}$$

with 1-Campbel measure $\mu^{[1]}$ of μ.

Strong solutions of ISDE: How to prove (P1)–(P3)

(Q1) μ is tail trivial.

(Q2) $P_\mu \circ \mathsf{X}_t^{-1} \prec \mu$ for all t.

Let $S_r = \{|x| < r\}$, $\mathsf{X}_t = \sum_{i\in\mathbb{N}} \delta_{X_t^i}$, $X^i = \{X_t^i\}$.

$$\mathsf{m}_r = \inf\{m \in \mathbb{N}; X^i \in C([0,T]; S_r^c) \text{ for } m < \forall i \in \mathbb{N}\}.$$

(Q3) $P_\mu(\cap_{r=1}^{\infty}\{\mathsf{m}_r(\mathsf{X}) < \infty\}) = 1$.

Thm 4. *Assume* (Q1)–(Q3). *Then* (P3) *holds.*

(P3) $\mathit{Tail}\,(W(S^{\mathbb{N}}))$ *is* $\mathbf{P_s}$*-trivial for each* $\mathbf{s} \in \mathbf{S_0}$*.*

- All determinantal measures satisfy (Q1). Quasi-Gibbs measures have a decomposition withe respect to tail σ-field such that each components are tail trivial.
- (Q2) follows from the μ-reversibility of the unlabeled diffusion X_t.
- (Q3) holds if $\sigma = 1$ or bounded from above.

END