A few caveats and exact solutions for the 1D KPZ equation

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Plan of the talk

- The KPZ equation was introduced by Kardar, Parisi, Zhang
- The scaling properties were studied by Halpin-Healy, Krug, Spohn etc.
- The Tracy-Widom distributions were introduced by Tracy, Widom. It appeared in problems of zero-temperature directed polymer and determinantal growth processes (PNG, TASEP$\cdots$).
- What about the KPZ equation itself?

  $\Rightarrow$ One can solve the KPZ equation exactly!

Explanation about a few subtle (but important) points and exact solutions for the 1D KPZ equation. The goal is to understand the meanings of the sentences:

"The KPZ equation is not really well-defined."

"The 1D KPZ equation has been solved exactly. The height distribution can be written in terms of a Fredholm determinant"

"The KPZ universality and the universality of the KPZ equation are different."

1. Well-definedness of the KPZ equation

$h(x, t)$: height at position $x \in \mathbb{R}$ and at time $t \geq 0$

1986 Kardar Parisi Zhang

$$\partial_t h(x,t) = \frac{1}{2}\lambda(\partial_x h(x,t))^2 + \nu \partial_x^2 h(x,t) + \sqrt{D}\eta(x,t)$$

where η is the Gaussian noise with mean zero and covariance $\langle \eta(x,t)\eta(x',t')\rangle = \delta(x-x')\delta(t-t')$

By a simple scaling we can and will do set $\nu = \frac{1}{2}, \lambda = D = 1$.

The KPZ equation now looks like

$$\partial_t h(x,t) = \frac{1}{2}(\partial_x h(x,t))^2 + \frac{1}{2}\partial_x^2 h(x,t) + \eta(x,t)$$

Cole-Hopf transformation

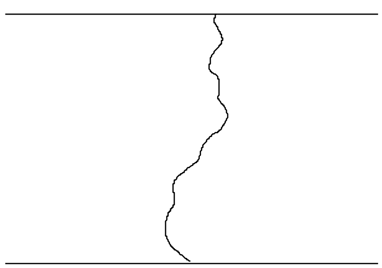
If we set

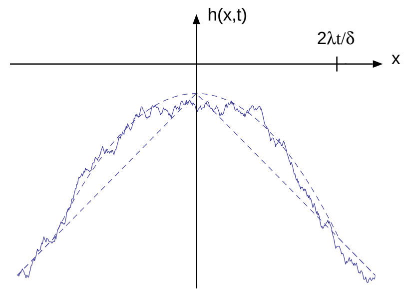
$$\boldsymbol{Z(x,t) = \exp\left(h(x,t)\right)}$$

this quantity (formally) satisfies

$$\boldsymbol{\frac{\partial}{\partial t} Z(x,t) = \frac{1}{2}\frac{\partial^2 Z(x,t)}{\partial x^2} + \eta(x,t) Z(x,t)}$$

This can be interpreted as a (random) partition function for a directed polymer in random environment $\boldsymbol{\eta}$.





The polymer from the origin: $\boldsymbol{Z(x,0) = \delta(x) = \lim_{\delta \to 0} c_\delta e^{-|x|/\delta}}$ corresponds to narrow wedge for KPZ.

The KPZ equation is not well-defined

- With $\eta(x,t)$" $=$ "$dB(x,t)/dt$, the equation for Z can be written as (Stochastic heat equation)

$$dZ(x,t) = \frac{1}{2}\frac{\partial^2 Z(x,t)}{\partial x^2}dt + Z(x,t) \times dB(x,t)$$

Here $B(x,t)$ is the cylindrical Brownian motion with covariance $dB(x,t)dB(x',t) = \delta(x-x')dt$.

- Interpretation of the product $Z(x,t) \times dB(x,t)$ should be Stratonovich $Z(x,t) \circ dB(x,t)$ since we used usual calculus. Switching to Ito by $Z(x,t) \circ dB(x,t) = Z(x,t)dB(x,t) + \frac{1}{2}dZ(x,t)dB(x,t)$, we encounter $\delta(0)$.

- On the other hand SHE with Ito interpretation from the beginning

$$dZ(x,t) = \frac{1}{2}\frac{\partial^2 Z(x,t)}{\partial x^2}dt + Z(x,t)dB(x,t)$$

is well-defined. For this Z one can define the "Cole-Hopf" solution of the KPZ equation by $h = \log Z$.

So the well-defined version of the KPZ equation may be written as

$$\partial_t h(x,t) = \tfrac{1}{2}(\partial_x h(x,t))^2 + \tfrac{1}{2}\partial_x^2 h(x,t) - \infty + \eta(x,t)$$

- Hairer [who got the Fields medal last week!] found a way to define the KPZ equation without but equivalent to Cole-Hopf (using ideas from rough path and renormalization).

2. Exact solution for the height distribution

Thm(2010 TS Spohn, Amir Corwin Quastel)

$$h(x,t) = -x^2/2t - \tfrac{1}{12}\gamma_t^3 + \gamma_t \xi_t$$

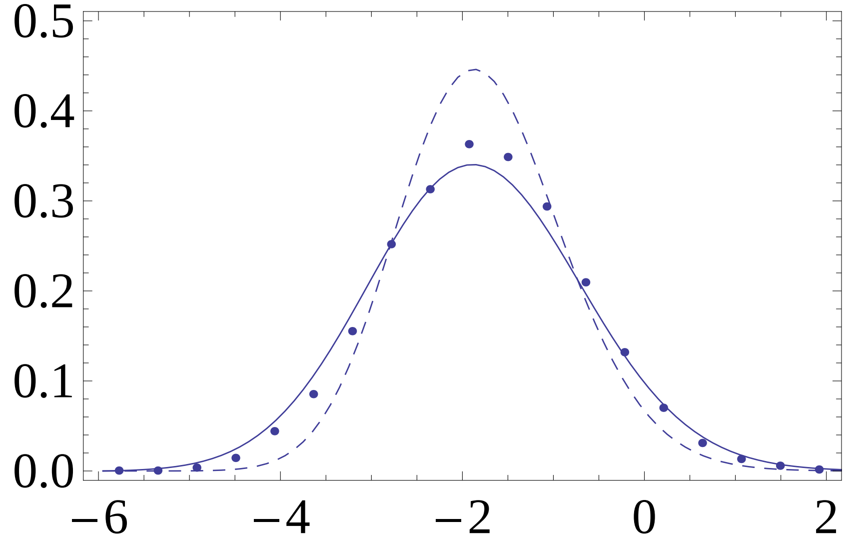
where $\gamma_t = (t/2)^{1/3}$. The distribution function of ξ_t is

$$F_t(s) = \mathbb{P}[\xi_t \le s] = 1 - \int_{-\infty}^{\infty} \exp\big[-\mathrm{e}^{\gamma_t(s-u)}\big]$$
$$\times \big(\det(1 - P_u(B_t - P_{\mathrm{Ai}})P_u) - \det(1 - P_u B_t P_u)\big)\mathrm{d}u$$

where $P_{\mathrm{Ai}}(x,y) = \mathrm{Ai}(x)\mathrm{Ai}(y)$, P_u is the projection onto $[u,\infty)$ and the kernel B_t is

$$B_t(x,y) = \int_{-\infty}^{\infty} \mathrm{d}\lambda \frac{\mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda)}{e^{\gamma_t \lambda} - 1}$$

Finite time KPZ distribution and TW



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—: exact KPZ density $F_t'(s)$ at $\gamma_t = 0.94$

− −: Tracy-Widom density

- In the large t limit, F_t tends to the GUE Tracy-Widom distribution F_2 from random matrix theory.

Fredholm determinant formula for generating function

The formula for the height distribution is equivalent to the following formula for the generating function.

For the initial condition $Z(x,0) = \delta(x)$ (narrow wedge for KPZ)

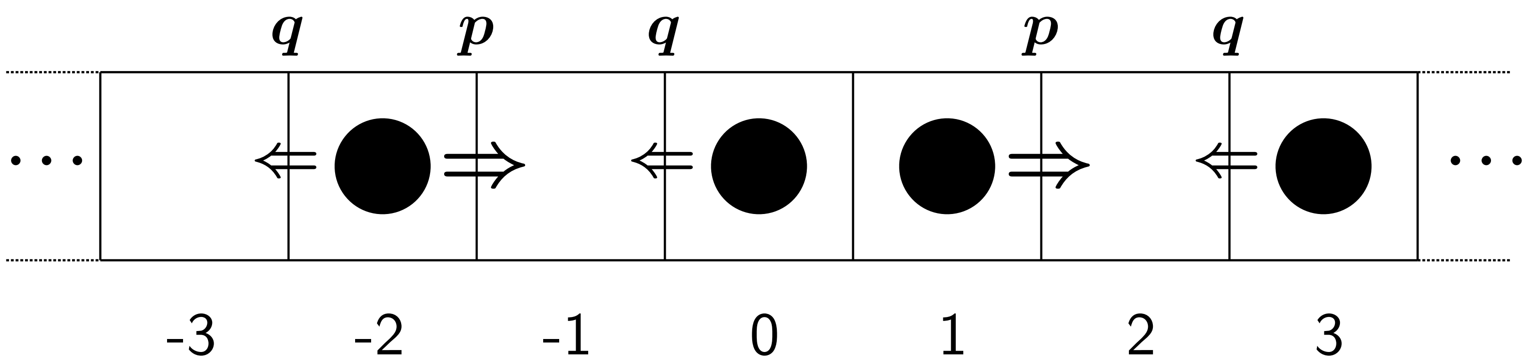
$$\langle e^{-e^{h(0,t)+\frac{t}{24}-\gamma_t s}} \rangle = \det(1 - K_{s,t})$$

where $\gamma_t = (t/2)^{1/3}$ and $K_{s,t}$ is

$$K_{s,t}(x,y) = \int_{-\infty}^{\infty} \mathrm{d}\lambda \frac{\mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda)}{e^{\gamma_t(s-\lambda)}+1}$$

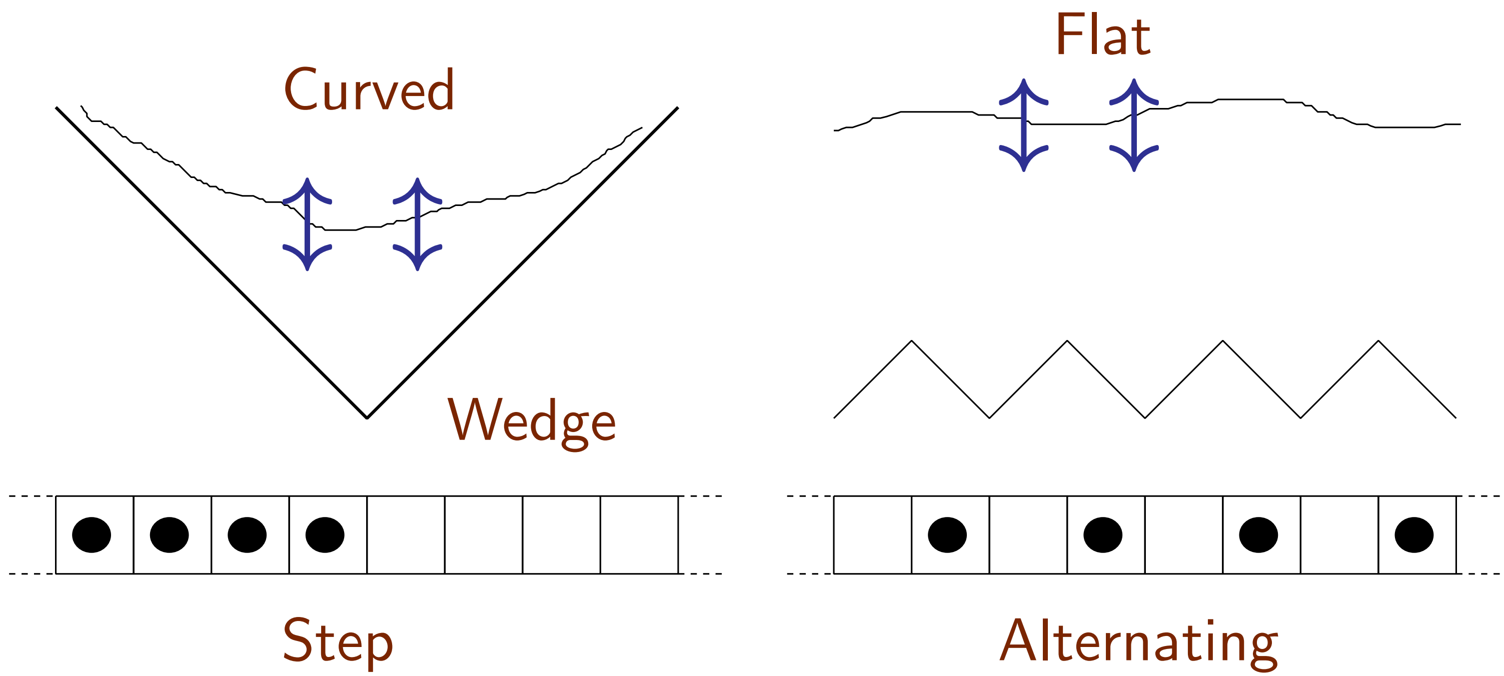
3. The first derivation through ASEP

ASEP = asymmetric simple exclusion process



- TASEP(Totally ASEP, $p = 0$ or $q = 0$)
- $N(x, t)$: Integrated current at $(x, x + 1)$ upto time t
- $\tau = p/q$

Mapping to surface growth



Integrated current $N(x,t)$ in ASEP

$\Leftrightarrow$ Height $h(x,t)$ in surface growth

This surface growth model is in the KPZ universality class.

KPZ equation as WASEP limit

1988 Gärtner, 1997 Bertini Giacomin

Scaling (ε: small) in ASEP

space: $\varepsilon^{-1}x$, time: $\varepsilon^{-2}t$

asymmetry: $q - p = \sqrt{\varepsilon}$ (Weakly ASEP, WASEP)

In this limit, the ASEP becomes the KPZ equation!

Gärtner transformation

For ASEP, we can introduce a lattice version of the Cole-Hopf transformation (Gärtner transformation)

$$Z_\varepsilon(x,t) = \frac{1}{2\sqrt{\varepsilon}}\tau^{N(x,t)+a_\varepsilon t}$$

which satisfies a discrete version of the SHE,

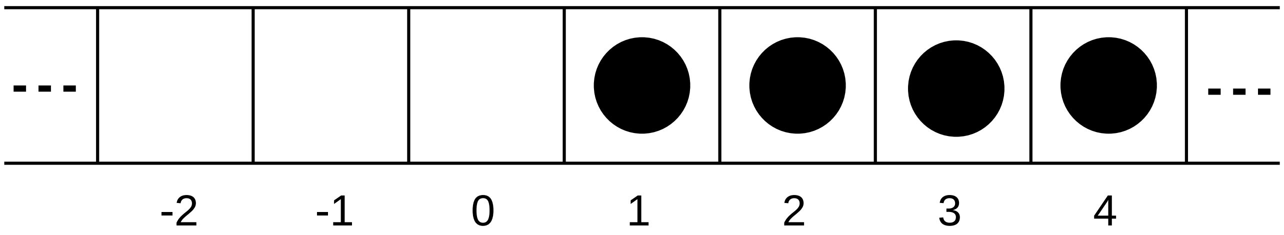
$$dZ_\varepsilon = \frac{1}{2}b_\varepsilon \Delta_\varepsilon Z_\varepsilon + Z_\varepsilon dM_\varepsilon,$$

where Δ_ε is lattice Laplacian and M_ε is a martingale.

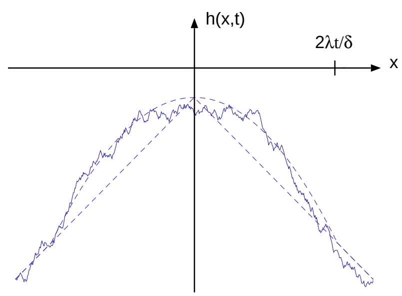
In the limit $\varepsilon \to 0$, this goes to the continuous SHE.

The initial condition

The step initial condition for ASEP



corresponds to the narrow wedge initial condition
$Z(x,0) = e^{h(x,0)} = \delta(x)$ for the KPZ equation.



Tracy-Widom formula for ASEP

$x_m(t)$: the position of the mth particle from left at time t

2008 Tracy Widom

$$\mathbb{P}\big(x_m(t/(q-p)) \le x\big) = \int_{\mathcal{C}_0} \prod_{k=0}^{\infty}(1-\mu\tau^k)\,\det(1+J(\mu))\frac{\mathrm{d}\mu}{\mu}$$

where

$$J(\mu;\eta,\eta') = \int_{\mathcal{C}_1} \frac{\varphi_\infty(\zeta)}{\varphi_\infty(\eta')}\frac{\zeta^m}{(\eta')^{m+1}}\frac{\mu f(\mu,\zeta/\eta')}{\zeta-\eta}\mathrm{d}\zeta$$

$$\varphi_\infty(\eta) = (1-\eta)^{-x}\mathrm{e}^{t(\eta/(1-\eta))}$$

$$f(\mu,z) = \sum_{k=-\infty}^{\infty}\frac{\tau^k}{1-\mu\tau^k}z^k$$

This is obtained by Bethe ansatz and ingenious calculations.

Ramanujan summation formula

$$\sum_{n=-\infty}^{\infty} \frac{(a;q)_n}{(b;q)_n} x^n = \frac{(ax;q)_\infty (q/ax;q)_\infty (q;q)_\infty (b/a;q)_\infty}{(x;q)_\infty (b/ax;q)_\infty (b;q)_\infty (q/a;q)_\infty}$$

where

$$(a;q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n)$$

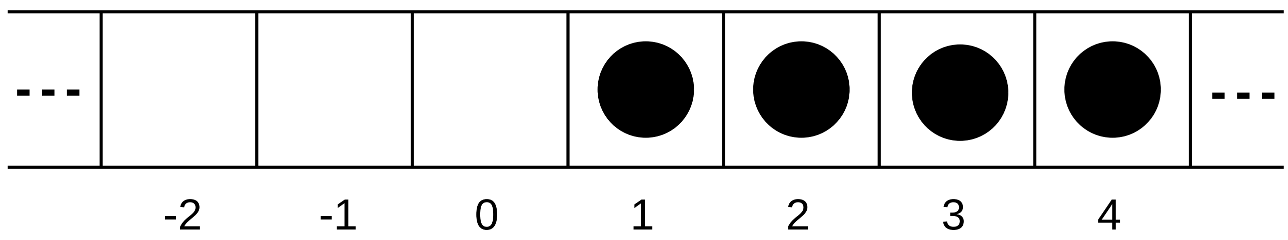
$$(a;q)_n = (a;q)_\infty / (aq^n;q)_\infty$$

Using this one finds

$$f(\mu, z) = \frac{(\mu\tau z;\tau)_\infty (1/\mu z;\tau)_\infty (\tau;\tau)_\infty (\tau;\tau)_\infty}{(\tau z;\tau)_\infty (1/z;\tau)_\infty (\mu;\tau)_\infty (\tau/\mu;\tau)_\infty}$$

The distribution of $N(x=0,t)$ in WASEP

$N(0,t) =$ integrated current up to time t at the bond $(0,1)$



$$\lim_{\varepsilon \to 0} \mathbb{P}\left[\frac{N(0,\varepsilon^{-2}t) - \frac{1}{4}t\varepsilon^{-3/2} + \frac{\log(2\sqrt{\varepsilon})}{2\sqrt{\varepsilon}}}{\gamma_t} \geq s \right] = \tilde{F}_t(s)$$

where

$$\gamma_t = 2^{-1/3} t^{1/3}$$

The distribution for finite t

In the limit $\epsilon \to 0$, we get the distribution

$$\tilde{F}_t(s) = \int_{\Gamma_\mu} \mathrm{e}^{-\mu} \det(1 + I(\mu)) \frac{1}{\mu} \mathrm{d}\mu$$

where the kernel is

$$I(\mu;\eta,\eta') = \int_{\Gamma_\zeta} \exp\big[-\tfrac{1}{3}\zeta^3 + \tfrac{1}{3}(\eta')^3 + s(\zeta - \eta')\big] \frac{1}{\zeta - \eta'} \times \frac{\pi}{\sin(\gamma_t^{-1}\pi(\eta' - \zeta))} \mathrm{e}^{\gamma_t^{-1}(\eta'-\zeta)\log(-\mu)} \gamma_t^{-1} \mathrm{d}\zeta$$

In fact $\tilde{F}_t(s) = F_t(s)$ by a manipulation of determinants.

3. "Universalities" associated with the KPZ equation

"Usual" universality class for equilibrium systems

- In equilibrium statistical mechanics, systems at a critical temperature shows scale-invariant universal features which are not system-dependent.
- These "universality classes" are associated with a few scaling exponents. "Usually" a universality class (and hence its all correlation functions) is determined only by its exponents. [cf. renormalization group, CFT]

For the KPZ equation

- We have seen an exact solution for the KPZ equation for finite time t.
- The KPZ universality class in the "usual" sense appears only in the $t \to \infty$ limit. For the height distribution for the wedge case, the universal function is the GUE TW distribution.
- It seems that systems in the KPZ class share the exponents (like $1/3$) but the universal function depends on initial and boundary conditions. The KPZ universality class is subdivided into a few subclasses.
- The KPZ equation itself has some "universal" features: it is expected to appear for many surface growth models with weak drive (universal crossover from Edwards-Anderson to KPZ) .

4. Developments: 4.1. Derivation by replica approach

Dotsenko, Le Doussal, Calabrese

Feynmann-Kac expression for the partition function,

$$Z(x,t) = \mathbb{E}_x \left(e^{\int_0^t \eta(b(s),t-s)ds} Z(b(t),0) \right)$$

Because η is a Gaussian variable, one can take the average over the noise η to see that the replica partition function can be written as (for narrow wedge case)

$$\langle Z^N(x,t) \rangle = \langle x | e^{-H_N t} | 0 \rangle$$

where H_N is the Hamiltonian of the (attractive) δ-Bose gas,

$$H_N = -\frac{1}{2} \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} - \frac{1}{2} \sum_{j \neq k}^{N} \delta(x_j - x_k).$$

We are interested not only in the average $\langle h \rangle$ but the full distribution of h. We expand the quantity of our interest as

$$\langle e^{-e^{h(0,t)+\frac{t}{24}-\gamma_t s}} \rangle = \sum_{N=0}^{\infty} \frac{\left(-e^{-\gamma_t s}\right)^N}{N!} \left\langle Z^N(0,t) \right\rangle e^{N\frac{\gamma_t^3}{12}}$$

Using the integrability (Bethe ansatz) of the δ-Bose gas, one gets explicit expressions for the moment $\langle Z^n \rangle$ and see that the generating function can be written as a Fredholm determinant. But for the KPZ, $\langle Z^N \rangle \sim e^{N^3}$ ($\Rightarrow$ rigorous version for lattice models like ASEP)

- Various generalizations for flat surface, half-space etc have been achieved ($\Rightarrow$ talk by Le Doussal).

4.2 Stationary 2pt correlation

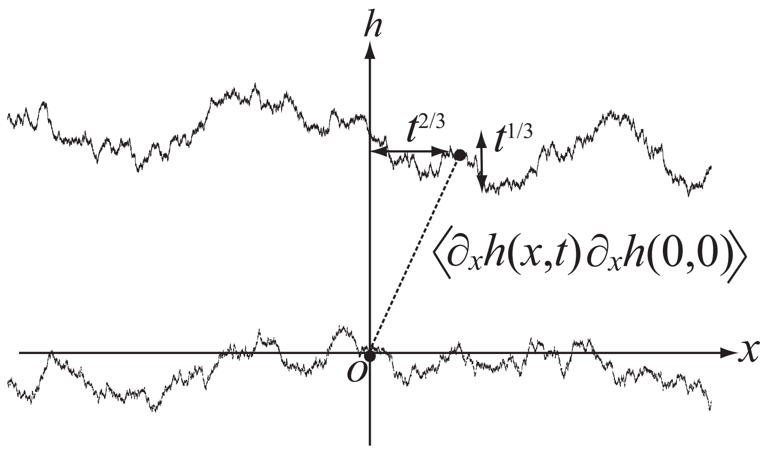
Not only the height/current distributions but correlation functions show universal behaviors.

- For the KPZ equation, the Brownian motion is stationary.

$$h(x, 0) = B(x)$$

  where $B(x), x \in \mathbb{R}$ is the two sided BM.

- Two point correlation



Scaling limit

- The limiting two-point correlation function was first computed (for PNG) by Prähofer Spohn (2002).
- For TASEP (Ferrari Spohn (2002)) with density 1/2,

$$\begin{aligned} S(j,t) &= \langle \eta(j,t)\eta(0,0)\rangle - \frac{1}{4} \\ &\sim C_1 t^{-2/3} g''(C_2 j/t^{2/3}) \end{aligned}$$

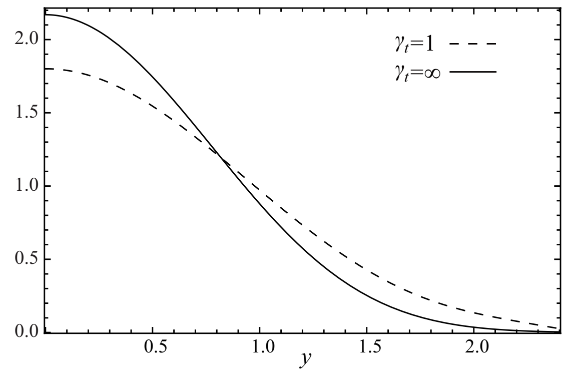
- The KPZ equation case was studied by Imamura TS (2012).

$$\langle \partial_x h(x,t) \partial_x h(0,0)\rangle = \frac{1}{2}(2t)^{-2/3} g_t''(x/(2t)^{2/3})$$

$$\lim_{t\to\infty} g_t''(x) = g''(x)$$

Scaled KPZ 2-pt function

Figure from exact formula



Stationary 2pt correlation function $g_t''(y)$ for $\gamma_t := (\frac{t}{2})^{\frac{1}{3}} = 1$.
The solid curve is the scaling limit $g''(y)$.

- This scaled KPZ 2-pt function is expected to appear universally in systems in the KPZ class.

4.3 "Stochastic Integrability

- What is the underlying mechanism for the exact calculations?
- Basically the answer is the integrability.
- The models in the KPZ class are often related to "quantum integrable" systems, but there are extra nice features.
- There would be unifying notions and frameworks for integrable stochastic models (Stochastic Integrability)
- $\Rightarrow$ Talks by Borodin, Corwin

Finite T polymer and quantum Toda

O'Connell 2010

Partition function of the semi-discrete directed polymer

$$Z_t^N(\beta) = \int_{0<t_1<\ldots<t_{N-1}<t} \exp \beta \left(\sum_{i=1}^{N} (B_i(t_i) - B_i(t_{i-1}) \right)$$

Toda lattice

$$H_{\text{Toda}} = \sum_{j=1}^{N} \frac{1}{2} p_j^2 + \sum_{j=1}^{N-1} e^{q_{j+1} - q_j}$$

Th. $Z_t^N(1)$ is the same (in distribution) as the left-most particle in N particle quantum Toda lattice

Summary

- There are well-defined notions for (a solution of) the KPZ equation.
- Exact solutions have been obtained for the height distribution for the 1D KPZ equation.
- The first derivation was through the weakly asymmetric limit of the ASEP, for which the Tracy-Widom found a current formula.
- There are so many interesting further developments, as will be explained in the rest of this workshop!