divergence-free WKB method

divergence-free saddle point method

Tadanori Hyouguchi (Post Doc. YITP)

Main co-worker of this work

Satoshi Adachi (Tokyo institute of technology)

Asymptotic expansion theory

WKB method for ODE
saddle point method (SPM) for integrals

Significance of these methods

$$\left.\begin{array}{r}\textbf{ODE}\\ \textbf{integral}\end{array}\right\} \longrightarrow\!(\epsilon \to 0)\!\longrightarrow \text{“}\textbf{essence}\text{”}$$

WKB		**SPM**
ODE : $\epsilon^2\Psi'' + K(x)\Psi = 0$	**applied to**	$I = \int_C ds\, e^{-f(s)/\epsilon}$
$\Psi \sim e^{\int dx\, q_0/\epsilon} \cdot e^{\int dx\, q_1}$	**evaluation**	$I \sim e^{-f(s_0)/\epsilon}\sqrt{\frac{2\pi\epsilon}{f''(s_0)}}$
AE : $q_0^2 + K(x) = 0$	**“essence”**	**SP :** $f'(s_0) = 0$
turning point $K(x) = 0$	**breakdown**	**caustic** $f''(s_0) = 0$

equivalent for Laplace-type

Applications to physics

	limit	"essence"	breakdown
quantum mechanics	$\hbar \to 0$	**classical path**	**turning point** **caustic**
statistical mechanics	$N \to \infty$	**mean field**	**transition point**
optics	$\lambda \to 0$	**ray**	**caustic**

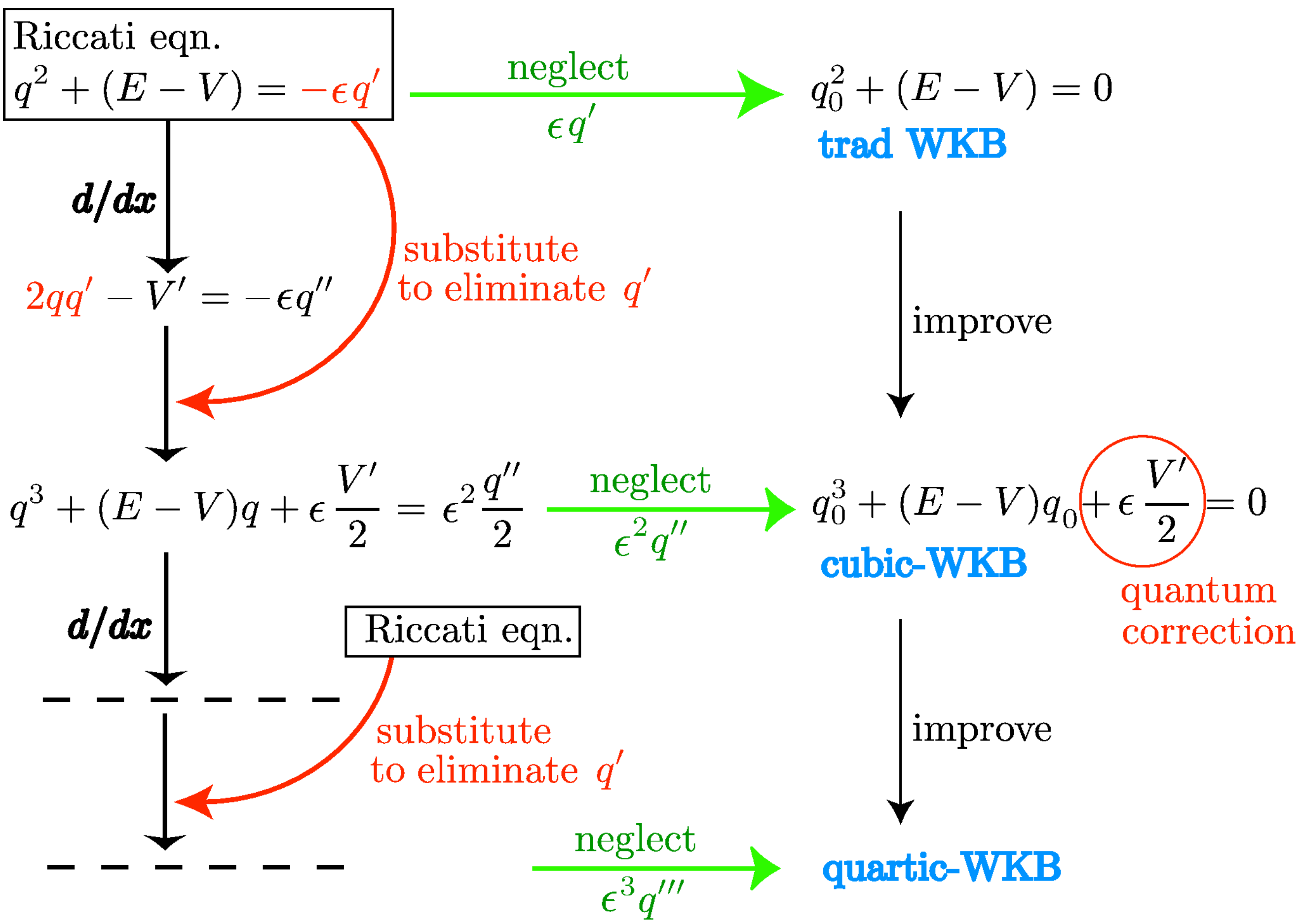
(purpose of my work)
Can we remove the breakdown from asymptotic theory with shifting the essence by incorporating non-perturbative effects of small parameter $(\hbar, N^{-1}, \lambda)$ **?**

cubic-WKB method

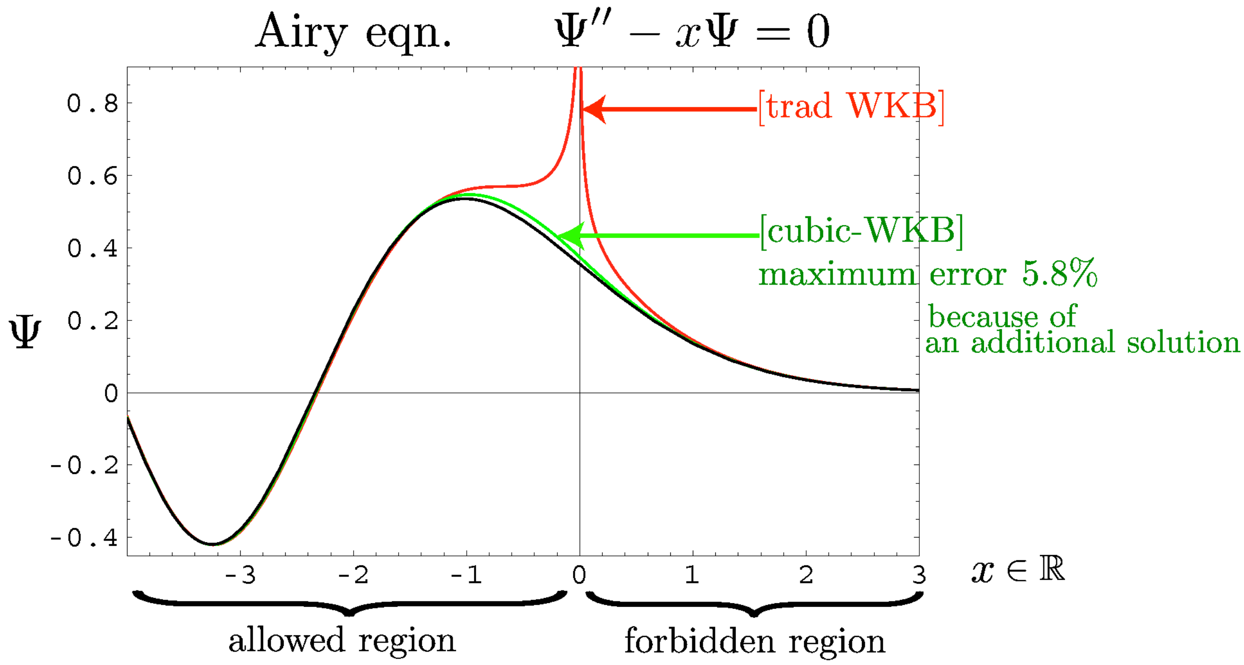
Phys.Rev.Lett. (2002) 88 p170404
Ann.Phys. (2004) 312 p177

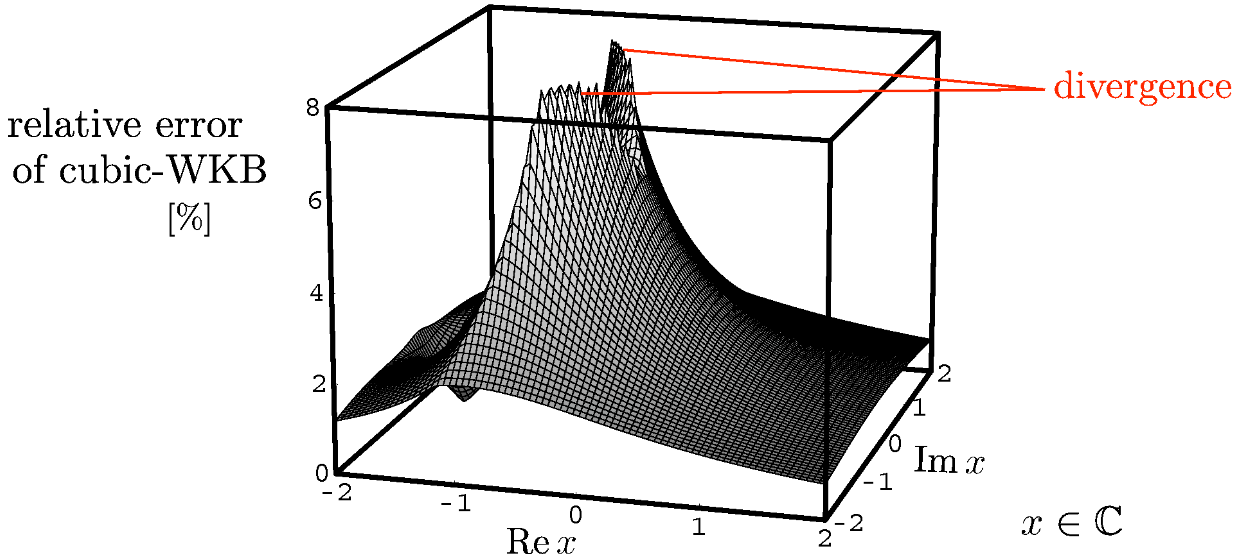
WKB ansatz $\Psi = e^{\int dx\, q(x)/\epsilon}$ $\quad \epsilon \equiv \frac{\hbar}{\sqrt{2m}}$

Schrödinger eqn. $\epsilon^2 \Psi'' + (E - V)\Psi = 0$



Main features of cubic-WKB method





(truly) divergence-free WKB method

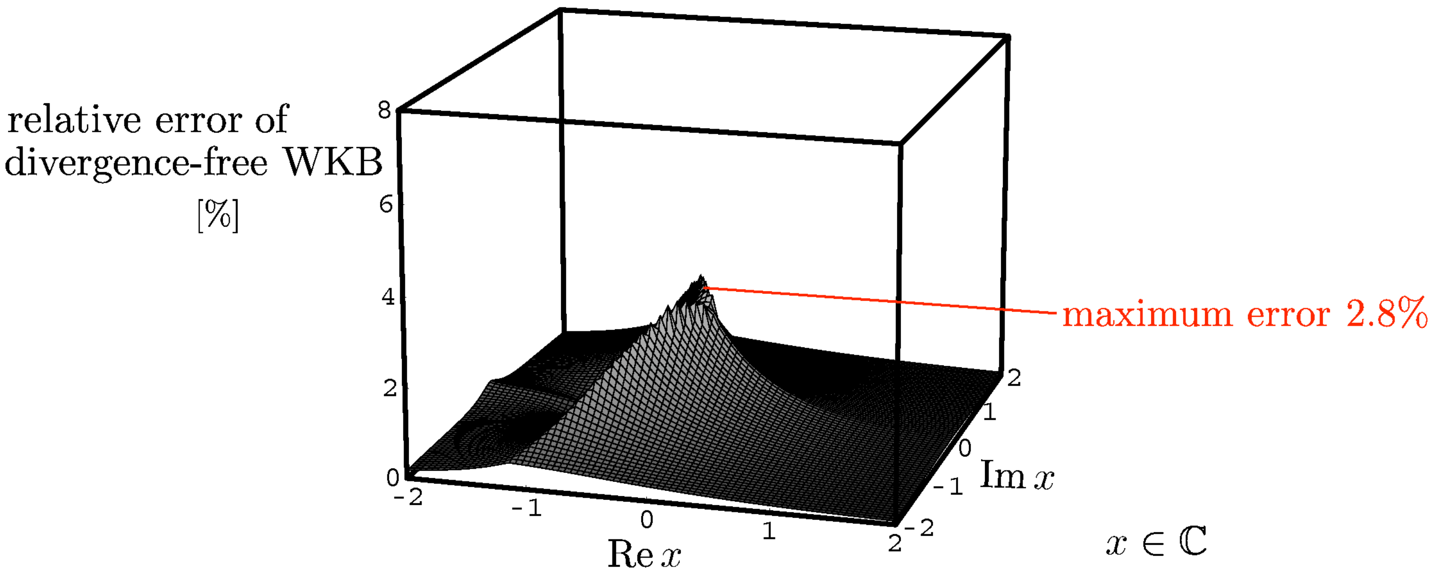
WKB ansatz $\Psi = e^{\int dx\, q(x)/\epsilon} e^{G(q(x))}$ $\epsilon \equiv \frac{\hbar}{\sqrt{2m}}$

Schrödinger eqn. $\epsilon^2 \Psi'' + (E - V)\Psi = 0$

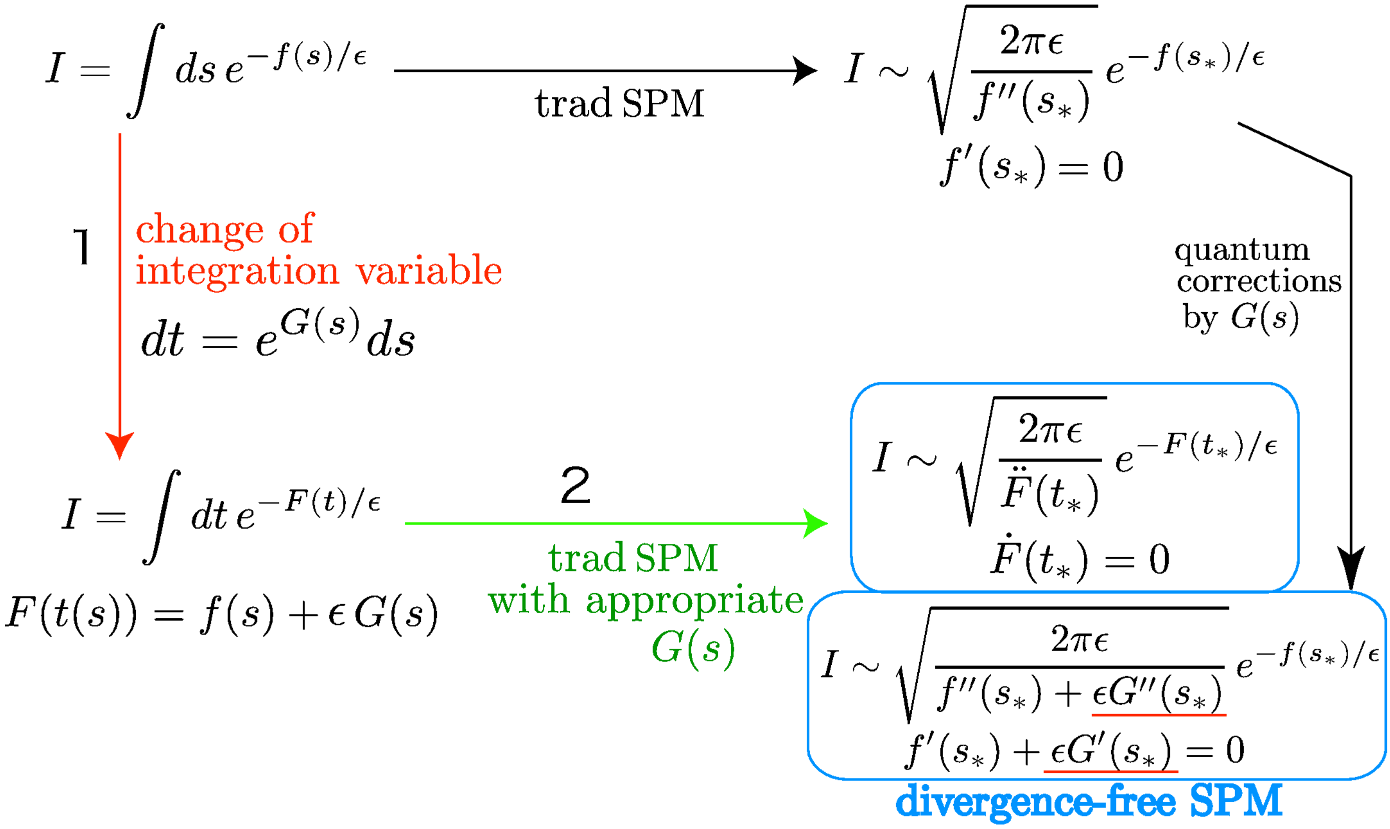
$G(q)$ **: control function (determined later appropriately)**

- **Substitute the ansatz into the Schrödinger eqn.**
- **Use the technique of cubic-WKB to obtain cubic-WKB-like wavefunction including** $G(q)$
- **Determine** $G(q)$ **in order that it becomes most accurate**

$G(q) = \pm i \frac{1}{\sqrt{6}} \ln q$ **(in the application to Airy equation)**



divergence-free SPM $\leftarrow$ transl. of divergence-free WKB



What is the ultimate choice of integration variable for SPM ?

(present) appropriate $G(s)$ is obtained only for single integral
(next task) extend this method to the case of multiple integrals

Future works

Apply the divergence-free SPM to ...

- **evaluate Feynman path integral in quantum mechanics**

⇔ ordinary semiclassical theory
(Van Vleck propagator, Gutzwiller trace formula, instanton)

- **evaluate partition function in statistical mechanics**
(phase transition)

⇔ Mean field theory, Renormalization group theory

- **evaluate scattering amplitude in optics (caustic, diffraction)**

⇔ Theory of complex angular momentum (Nussenzveig)
Geometrical theory of diffraction (Keller)

Thank you for giving me an opportunity to introduce my work