Spectral Problem in N=4 Super Yang-Mills Theory and Integrable Spin-Chains

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N=4 SYM

- Interesting field theory
- The main motivation is that it plays an important role in gauge/string duality (AdS/CFT correspondence)
- Maximally supersymmetric field theory in 4D
- Conformally invariant even at quantum level
- Conformal dimensions are fundamental quantities

$$\langle \mathcal{O}_n(x)\mathcal{O}_m(y)\rangle = \frac{c_n \delta_{nm}}{|x-y|^{2\Delta_n}}$$

conformal dim.

- Integrable structure appears in computing conformal dimensions (today's talk)

- **Action** $(\mu,\nu=1,\ldots,4;\ i,j=1,\ldots,6)$

$$S=\frac{1}{2g_{\rm YM}^2}\int d^4x\,{\rm Tr}\Big[-\frac{1}{2}(F_{\mu\nu})^2+(D_\mu\Phi_i)^2+\frac{1}{2}[\Phi_i,\Phi_j]^2$$
$$+i\bar\psi\Gamma^\mu D_\mu\psi+\bar\psi\Gamma^i[\Phi_i,\psi]\Big]$$

- **All appearing fields belong to the ajoint representation of the gauge group SU(*N*)**
- **The large *N* limit is taken: $N\to\infty$ with $\lambda\equiv g_{\rm YM}^2N$ fixed**

  ➡ **Only planar diagrams contribute** **'t Hooft coupling**
- **Conformal dims. are eigenvalues of the dilatation operator**
- **The dilatation operator is expanded in λ perturbatively**

$$D=D_0+g^2D_1+g^4D_2+g^6D_3+\cdots,\quad (g\equiv\sqrt{\lambda}/4\pi)$$

- **Generic gauge inv. op. is not an eigenstate of *D***

  (operator mixing problem)
- **We have to diagonalize it to know conformal dimensions**

Integrable Spin-Chain

- **Minahan and Zarembo found that the one-loop dilatation operator is identified with the Hamiltonian of certain integrable spin-chain**

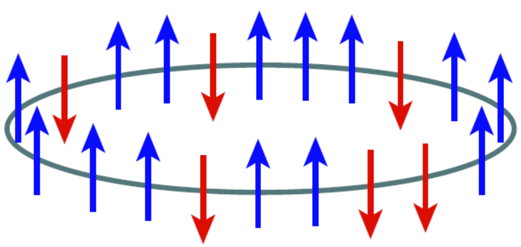
Minahan, Zarembo '02

$$D_1 = 2\sum_{k=1}^{L}(I_{k,k+1} - P_{k,k+1}) \longleftarrow$$ **Hamiltonian of $\mathrm{XXX}_{1/2}$ spin-chain**

- **Operator-state mapping**

$$Z = \Phi_1 + i\Phi_2$$
$$W = \Phi_3 + i\Phi_4$$

$$\mathrm{Tr}(ZZZZWZZWWZZZWZZZWZZW) \longleftrightarrow$$



- **This Hamiltonian is diagonalized by the Bethe ansatz method**

$$e^{ip_j L} = \prod_{k(\neq j)}^{M} \boxed{\frac{u(p_j) - u(p_k) - i}{u(p_j) - u(p_k) + i}}$$

two-body S-matrix

$$\Delta = L + \sum_{j=1}^{M} \boxed{8g^2 \sin^2\left(\frac{p_j}{2}\right)}$$

dispersion relation

$$u(p) \equiv \frac{1}{2}\cot\left(\frac{p}{2}\right)$$

- **Dilatation operators at higher loops are also mapped to integrable spin-chains with long-range interactions**

$$D_2 = -8\{\} + 12\{1\} - 2(\{1,2\} + \{2,1\}),$$

$$D_3 = 60\{\} - 104\{1\} + 4\{1,3\} + 24(\{1,2\} + \{2,1\}) - 4i\epsilon_2(\{1,3,2\} - \{2,1,3\}) - 4(\{1,2,3\} + \{3,2,1\}),$$

$$\{a,b,c,\ldots\} \equiv \sum_{r=0}^{L-1} P_{a+r,a+r+1} P_{b+r,b+r+1} P_{c+r,c+r+1} \cdots .$$

- **Higher-loop operators have very complicated form, so it is very hard (but possible in principle) to compute them**
- **If we assume that these operators are also diagonalized by Bethe ansatz method, the problem is to determine the exact S-matrix and dispersion relation**

$$e^{ip_j L} = \prod_{k(\neq j)}^{M} S(p_j, p_k; \lambda) \qquad \Delta = L - M + \sum_{j=1}^{M} \varepsilon(p_j; \lambda)$$

including higher-loop corrections

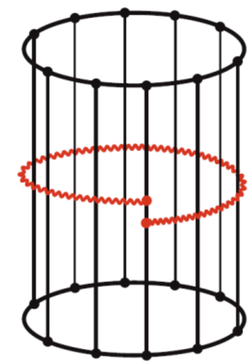
- **This problem has been solved**

Beisert '05
Beisert, Eden, Staudacher '06

Finite-Size Problem (Wrapping Problem)

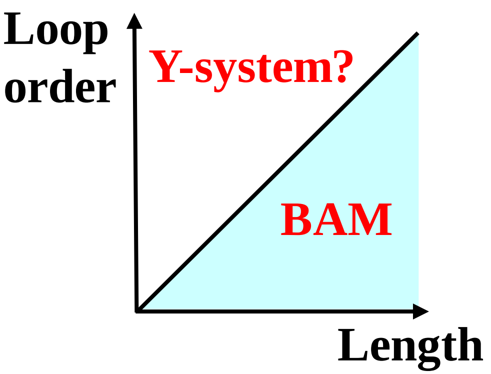
- Recall that higher-loop dilatation ops. contain long-range int.
- If (loop order) > (length of spin-chain), interactions 'wrap' the spin-chain
- In this situation, we cannot define asymptotic states and S-matrix

  ➡ cannot use Bethe ansatz method (wrapping problem)



- We need a new approach to resolve this problem
- Recently a resolution was proposed by using AdS/CFT Gromov, Kazakov, Vieira '09

  ➡ AdS/CFT Y-system



Good Example: Konishi Operator

- **Shortest operator with the non-trivial anomolous dim.**

$$\mathcal{O}_{\mathrm{K}} \equiv \mathrm{Tr}(Z^2W^2 - ZWZW)$$

- **Konishi operator has length four, so BA gives correct conformal dimension up to three loop order**

$$\Delta_{\mathrm{K}} = \underbrace{4 + 12g^2 - 48g^4 + 336g^6}_{\text{BA}} + \underbrace{\Delta^{(4)}g^8 + \Delta^{(5)}g^{10} + \dots}_{\cancel{\text{BA}}}$$

$$\Delta^{(4)} = -2496 + 576\zeta(3) - 1440\zeta(5)$$

via AdS/CFT **Bajnok, Janik '08**

Diagrammatic computations

Fiamberti, Santambrogio, Sieg, Zanon '07

Velizhanin '08

$$\Delta^{(5)} = 15168 + 6912\zeta(3) - 5184\zeta(3)^2 - 8640\zeta(5) + 30240\zeta(7)$$

via AdS/CFT **Bajnok, Janik, Hegedus, Lukowski '09**

No field theoretical computations

Konishi Dimension at Any Coupling

- **One result using AdS/CFT**

Gromov, Kazakov, Vieira, arXiv:0906.4240 [hep-th]

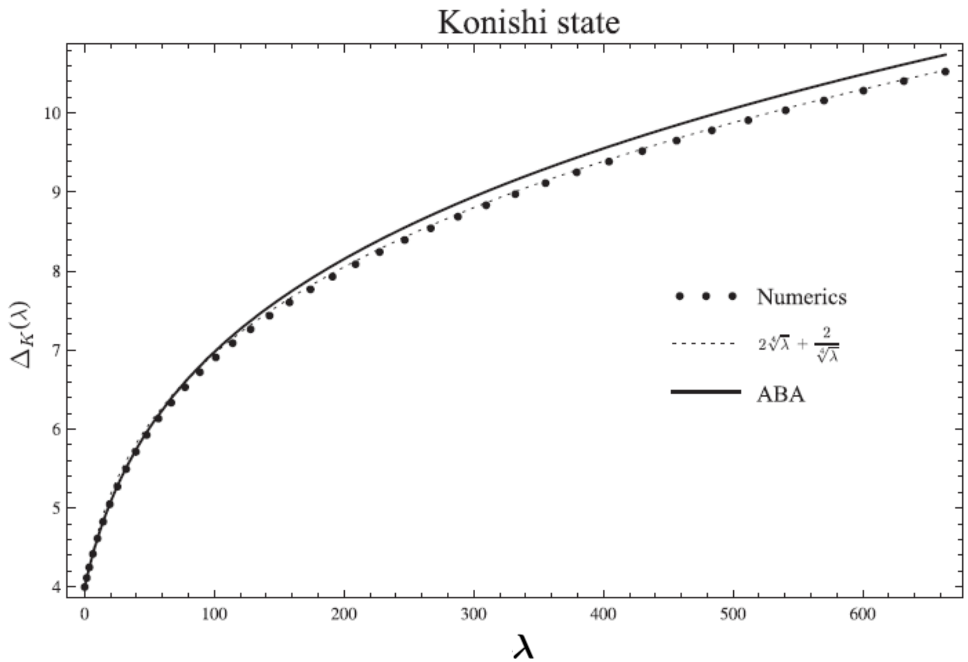


Figure 1: Numerical solution of exact finite size integral Y-system equations for the Konishi dimension $\Delta_K(\lambda)$ in a wide range of 't Hooft couplings λ, compared to the asymptotic Bethe ansatz curve and to the predicted large λ asymptotics $\Delta_K(\lambda) \simeq 2\lambda^{1/4} + 2/\lambda^{1/4}$ obtained by fit.

Summary

- **The dilatation operator is mapped to that of certain spin-chain Hamiltonian**
- **For long operators (or spin-chains), the conformal dims. are given by the Bethe ansatz method**
- **The Bethe ansatz method breaks down for finite-size ops.**
- **A resolution was proposed by using AdS/CFT**
- **Open problems**
  - **Derivation of Y-system purely in field theoretical framework (not using AdS/CFT)**
  - **Exact dimension for the Konishi operator?**