Lunch Seminar @ YITP
17, November 2010

Arithmetic of Solvable Strings & Dualities

可解なストリング理論と双対性の算術

Masato Taki/ YITP, Kyoto University

YUKAWA INSTITUTE FOR THEORETICAL PHYSICS

String Theory has not yet established as fundamental understanding of Nature.

However string theory today is very fruitful because of its plenty applications.

String theory has had an impact upon many area of physics and mathematics, such as QCD, CMP, integrable system, topology, differential geometry, group theory and so forth.

The key idea that bridges the gap between these theories is Duality.

DUALITY : a “magic” that translates a physical system into a completely different one.

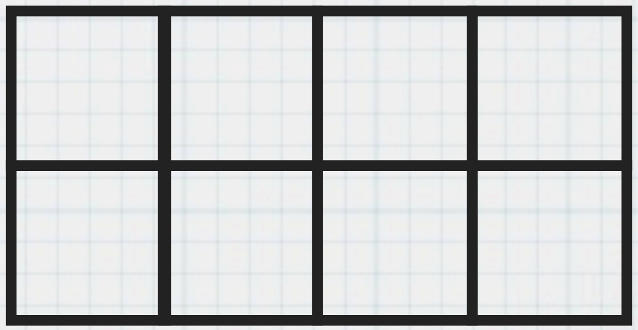
Example : Kramers–Wannier duality of 2D Ising model

high temp low temp

$$Z(K) \leftrightarrow Z(K^*)$$

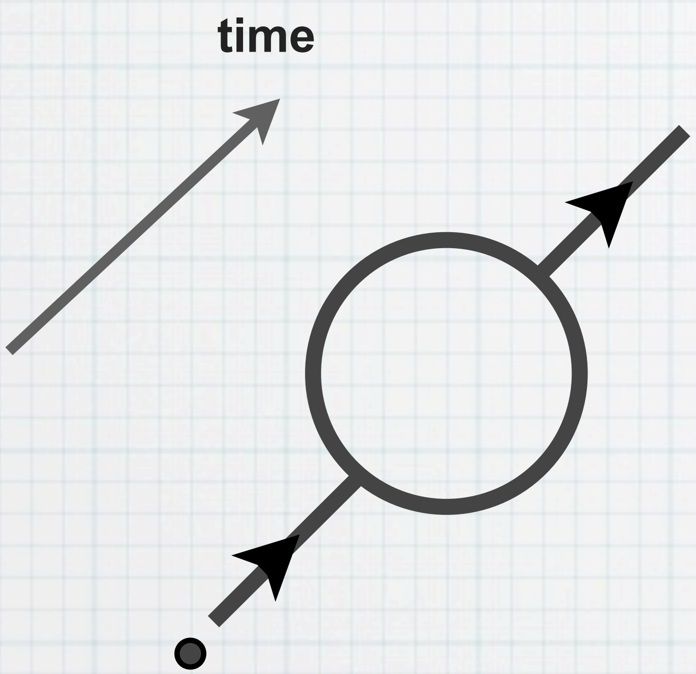
$$Z(K) = \sum_S e^{K \sum_{(i,j)} S_i S_j}$$

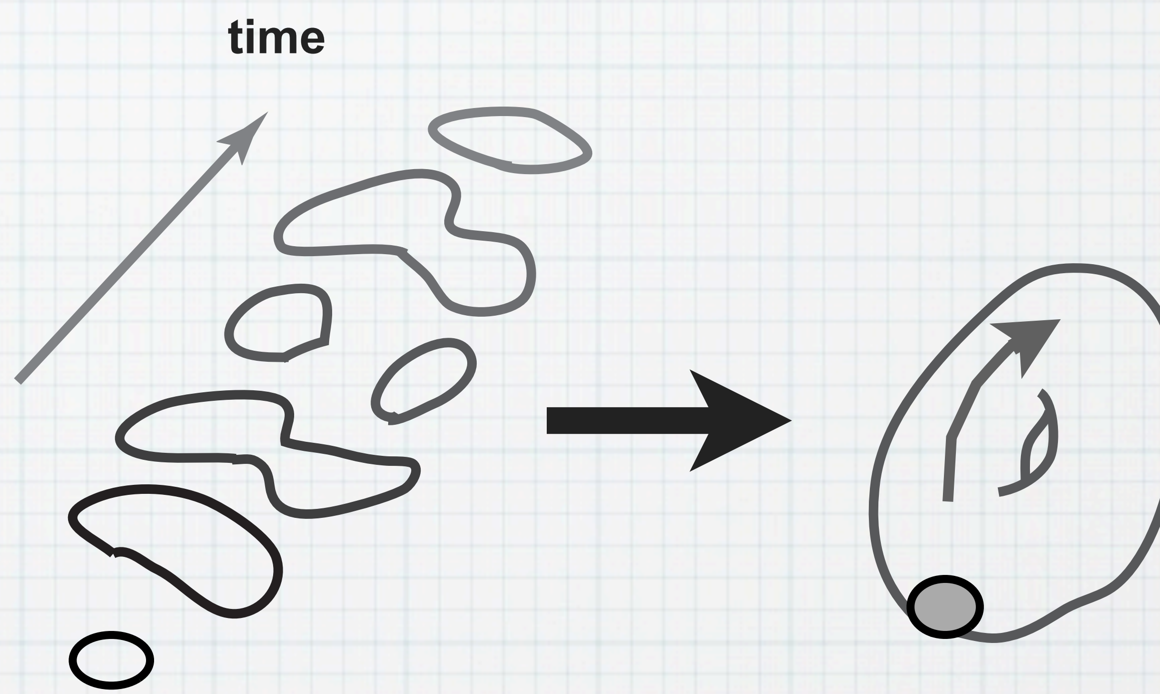
$$e^{-2K^*} = \tanh K$$



1. String Theory v.s. Melting Crystal

Today we will study topological string theory, which is a toy model of superstring.





Feynman diagram of a particle
(1-loop)

in first quantization

$$S = \int d\tau$$

Feynman diagram of a string
("1-loop")

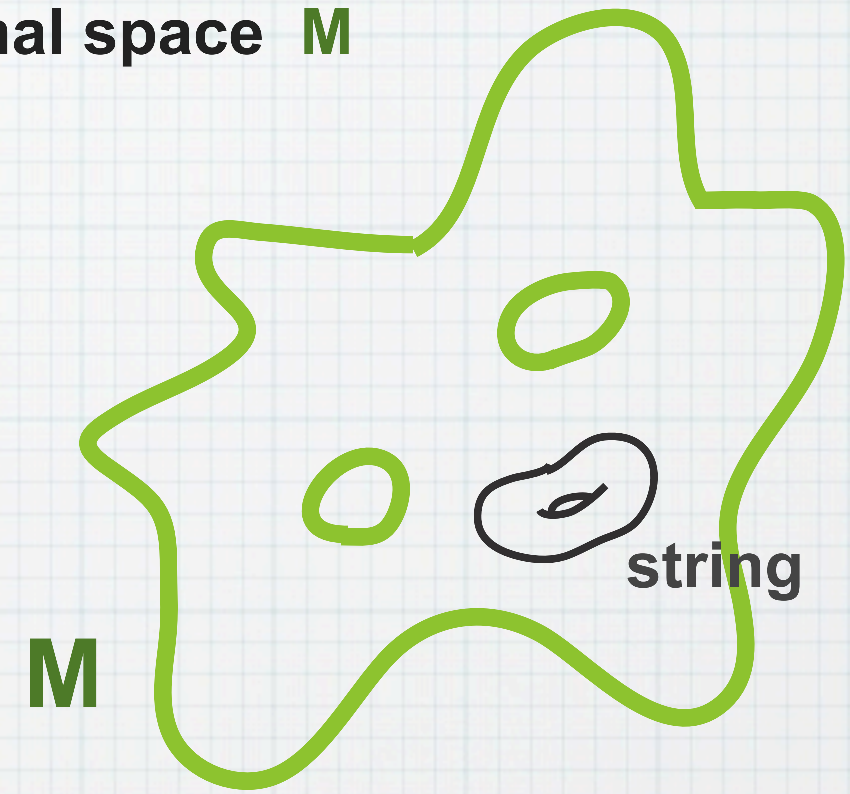
$$S \sim \int d\,\mathrm{Area}$$

< topological strings >

superstring : compactification on Minkowski X 6-dim. internal space M

↓

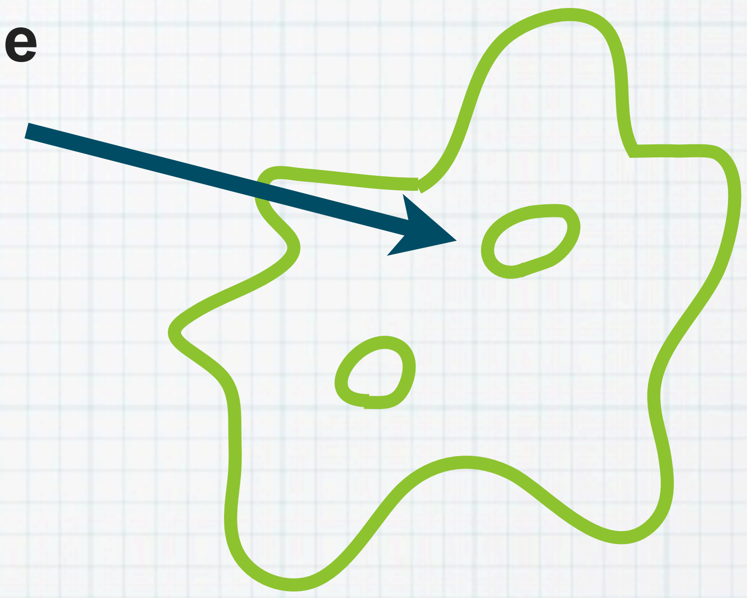
topological string : propagates only in the internal space M



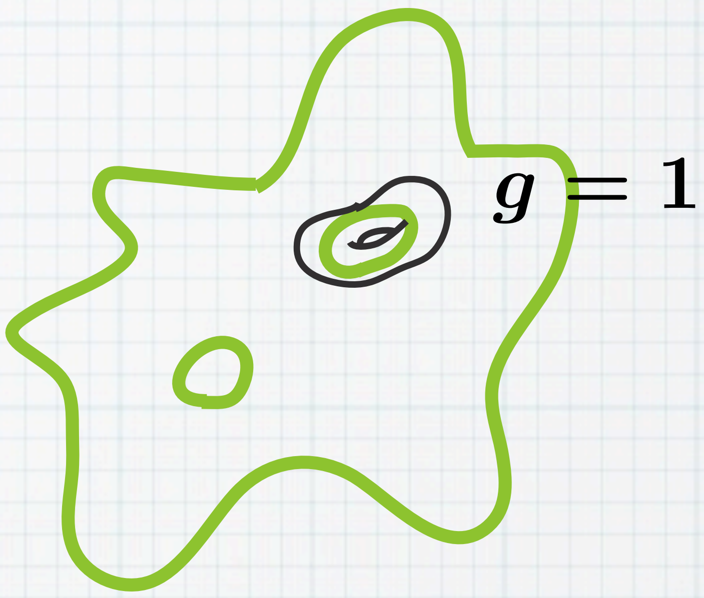
By using this model, we can omit detailed dynamics of the original superstring theory.

However, this model captures essential dynamics of the compactified superstring theory effectiviely.

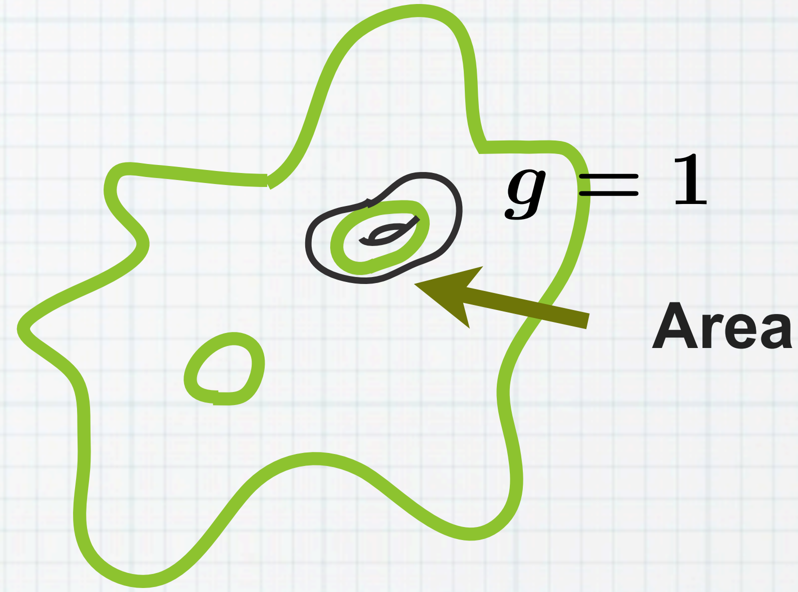
Dominant configuration which minimizes the energy (=area) is the string wraps minimal "holes" inside the M



Dominant configuration which minimizes the energy (=area) is the string wraps minimal "holes" inside the M

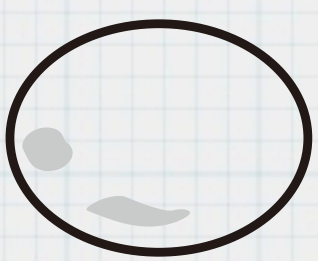
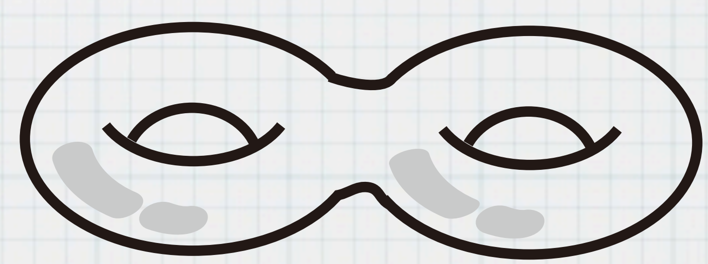


Dominant configuration which minimizes the energy (=area) is the string wraps minimal "holes" inside the M



free energy of topological strings

$$F = \text{“}\sum_{\text{minimal surface}}\text{”}\ \hbar^{2g-2}\, e^{-\text{Area}}$$

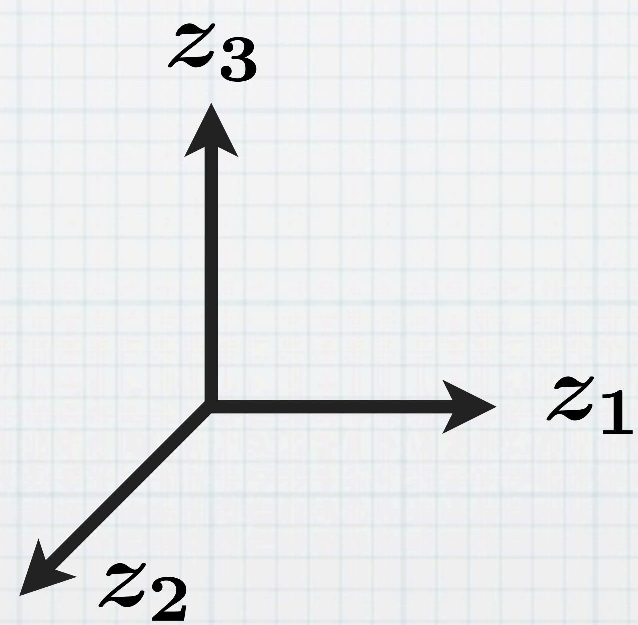
$= \hbar^{-2}$  $+$  $+\hbar^{2}$  $+\cdots$

< simplest example >

topological strings on $\mathbb{R}^{6} = \mathbb{C}^{3}$

No "holes" inside $\mathbb{R}^{6}$ (i.e. No neck on which string can wrap)

→ configuration vanishes to a point : **Area=0**

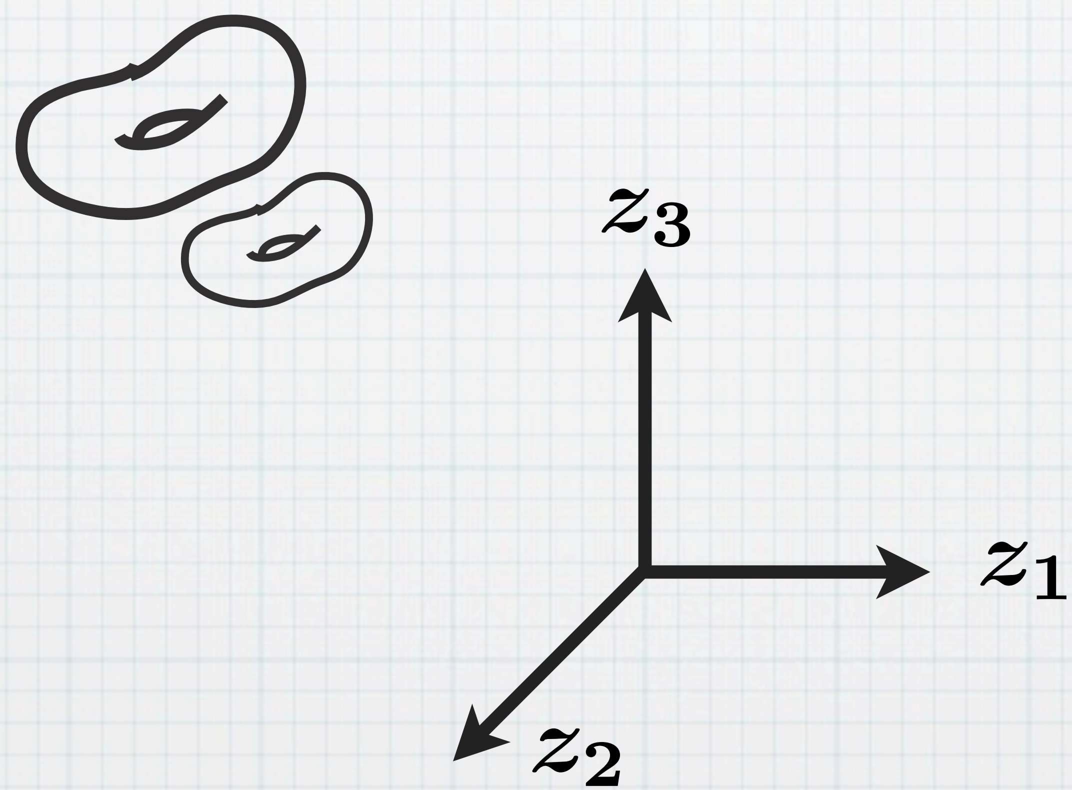


< simplest example >

topological strings on $\mathbb{R}^6 = \mathbb{C}^3$

No “holes” inside $\mathbb{R}^6$ (i.e. No neck on which string can wrap)

→ configuration vanishes to a point : Area=0

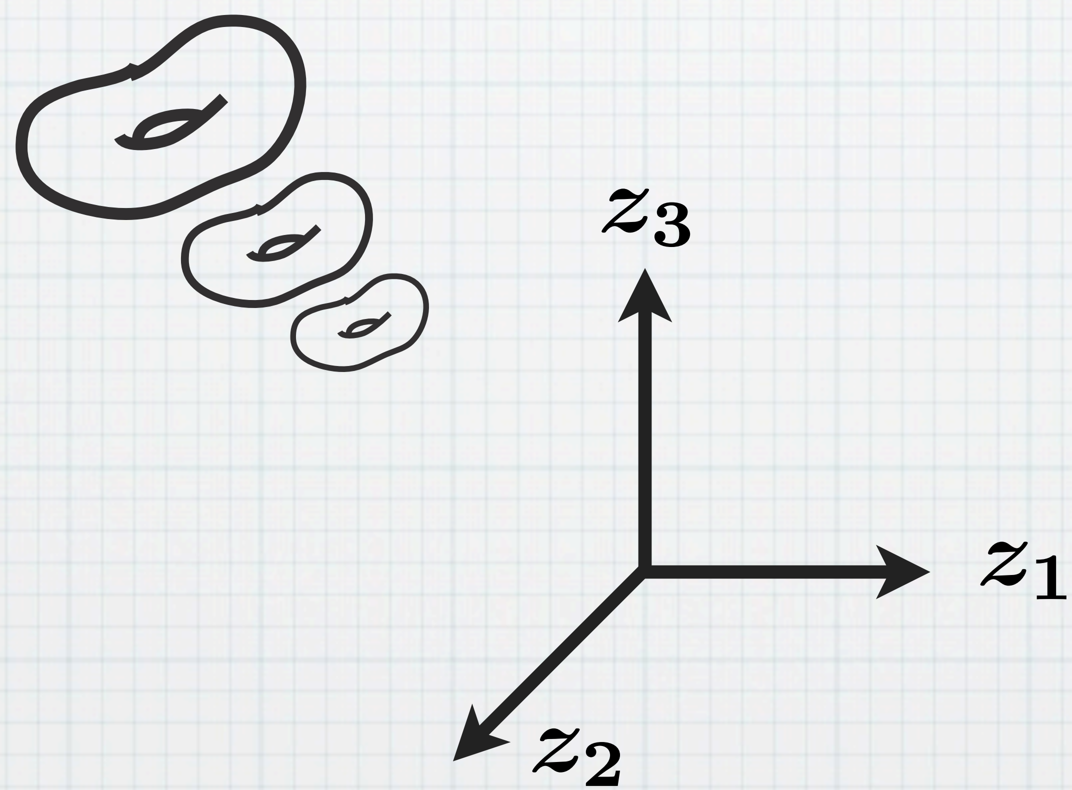


< simplest example >

topological strings on $\mathbb{R}^6 = \mathbb{C}^3$

No "holes" inside $\mathbb{R}^6$ (i.e. No neck on which string can wrap)

→ configuration vanishes to a point : Area=0

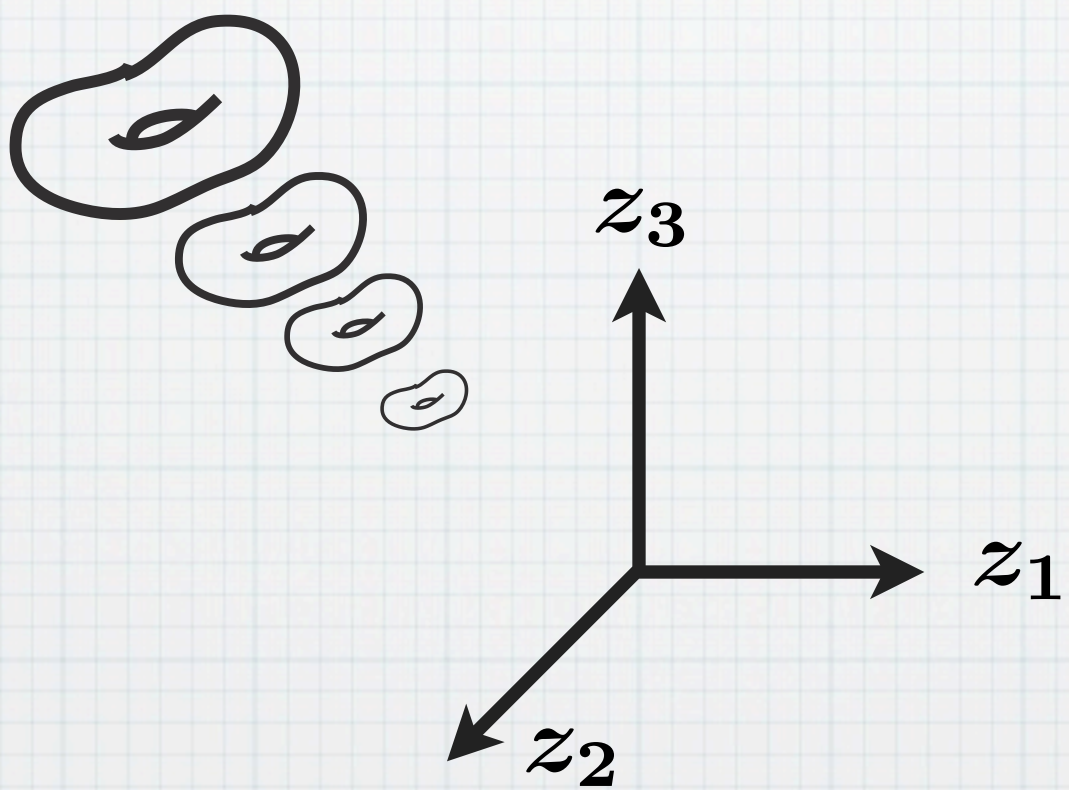


< simplest example >

topological strings on $\mathbb{R}^6 = \mathbb{C}^3$

No "holes" inside $\mathbb{R}^6$ (i.e. No neck on which string can wrap)

→ configuration vanishes to a point : Area=0

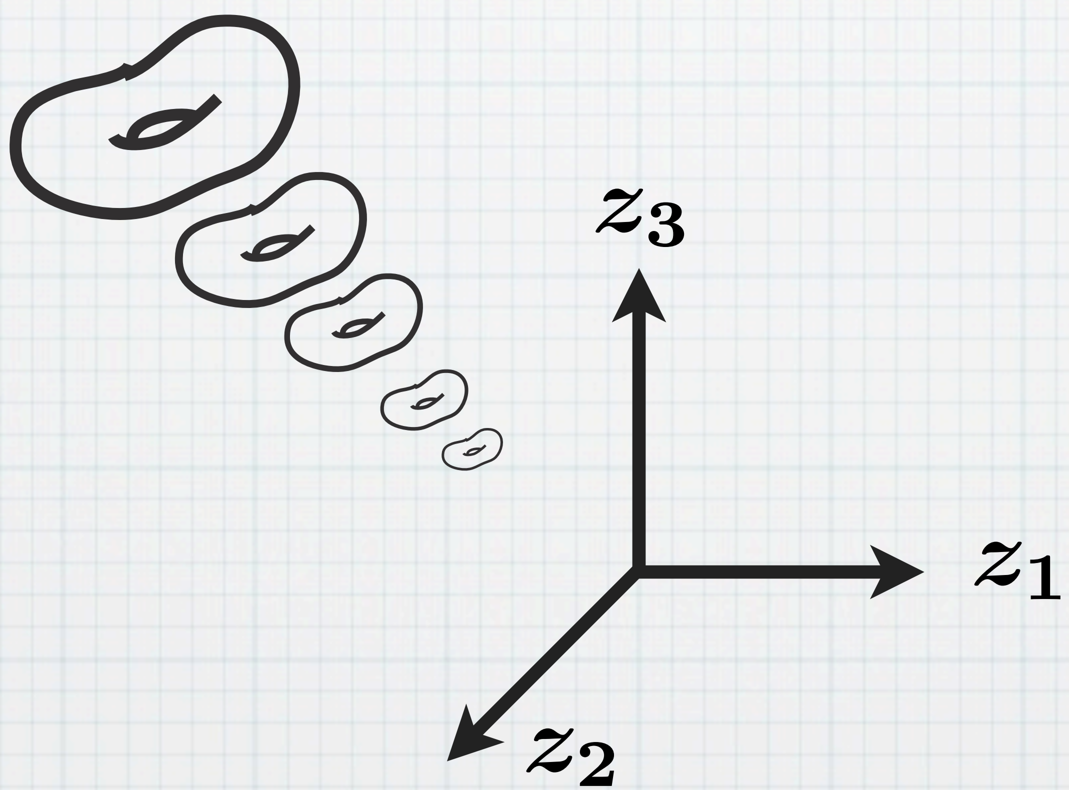


< simplest example >

topological strings on $\mathbb{R}^6 = \mathbb{C}^3$

No "holes" inside $\mathbb{R}^6$ (i.e. No neck on which string can wrap)

→ configuration vanishes to a point : Area=0

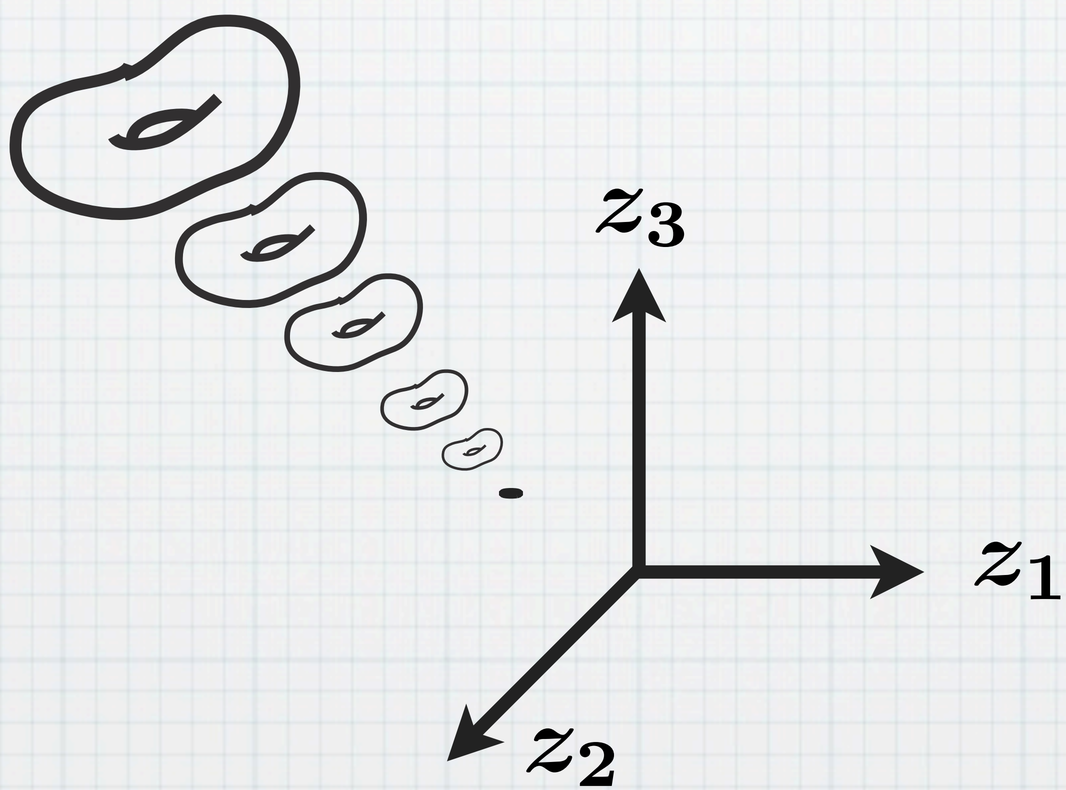


< simplest example >

topological strings on $\mathbb{R}^6 = \mathbb{C}^3$

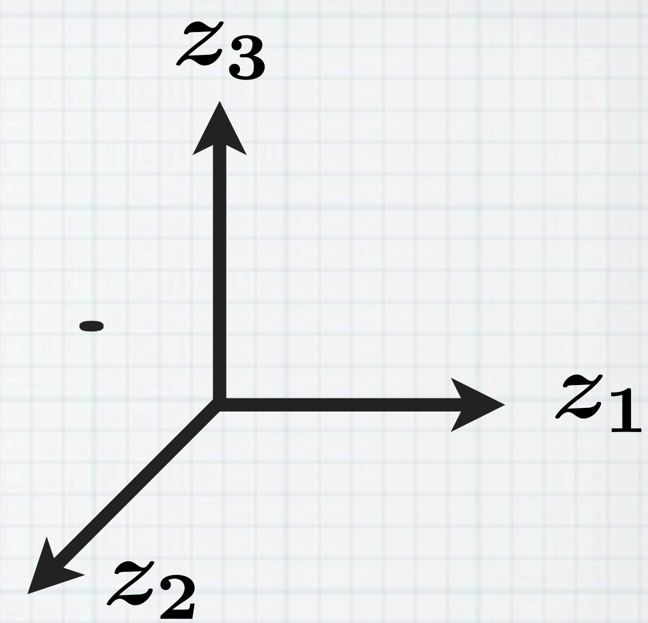
No "holes" inside $\mathbb{R}^6$ (i.e. No neck on which string can wrap)

→ configuration vanishes to a point : Area=0



< simplest example >

topological strings on $\mathbb{R}^6 = \mathbb{C}^3$



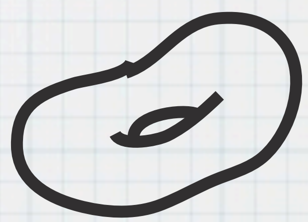
No “holes” inside $\mathbb{R}^6$ (i.e. No neck on which string can wrap)

→ configuration vanishes to a point : Area=0

Only d.o.f is the topological information of the small(vanishing) surface

$$F = \sum_{g=0}^{\infty} \hbar^{2g-2} \frac{(-1)^g B_g}{2g(2g-2)}$$

certain Euler number of genus g surface

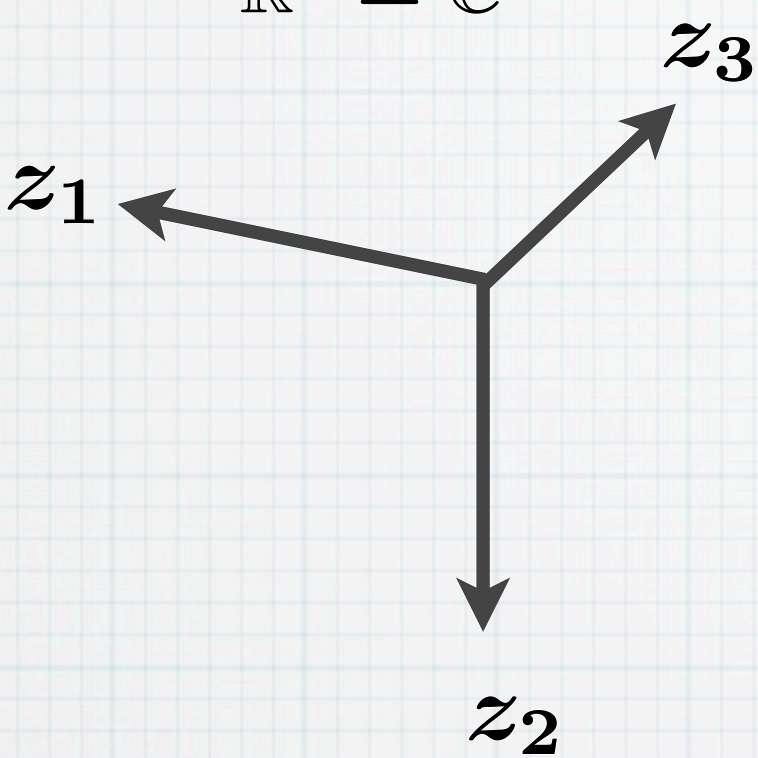


$g = 1$

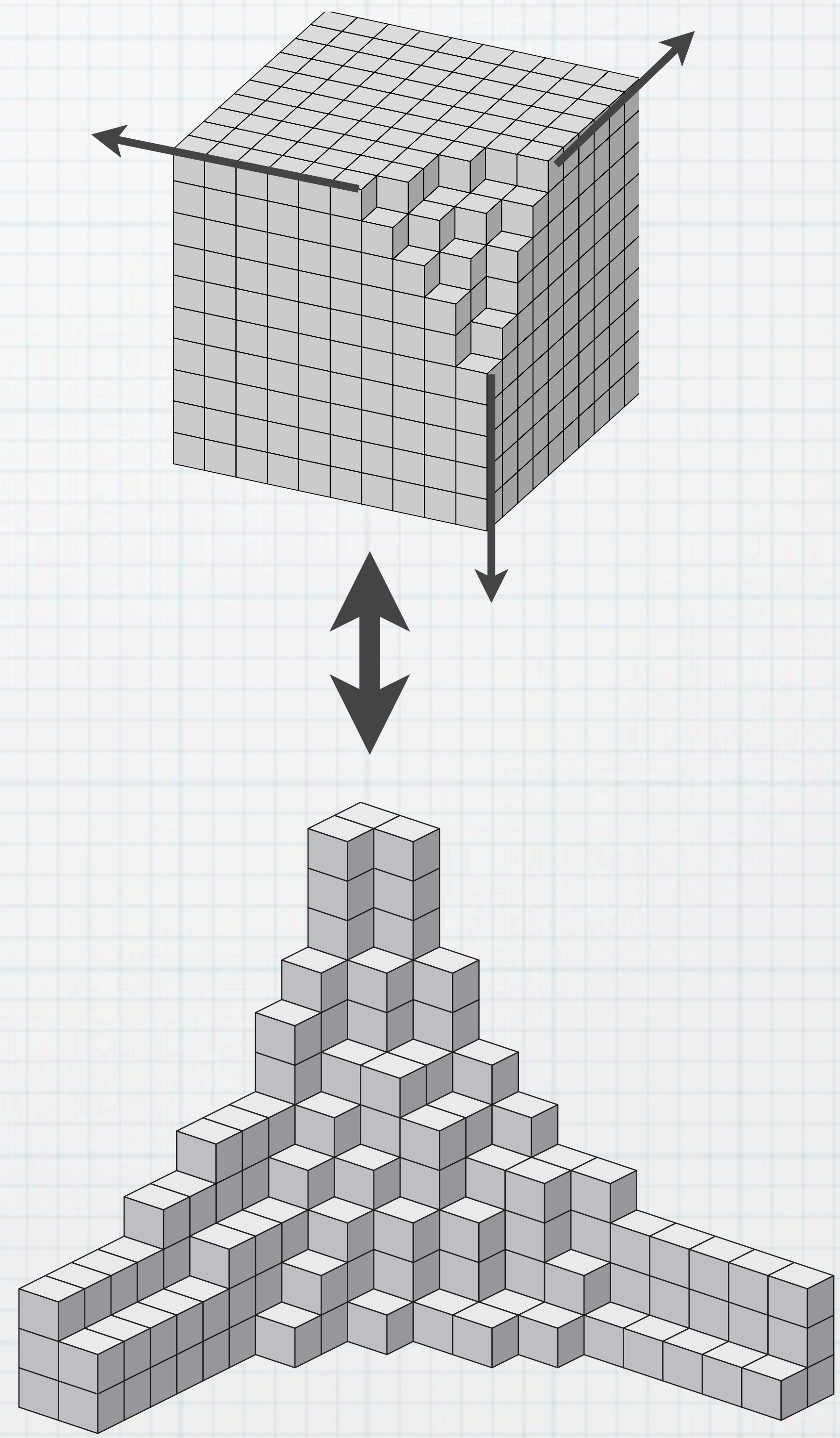
$$= -\sum_{n=1}^{\infty} n \log(1 - q^n) \qquad q = e^{-\hbar}$$

< duality and melting crystal corner > [Leshetekin-Okounkov-Vafa, `07]

$\mathbb{R}^6 = \mathbb{C}^3$

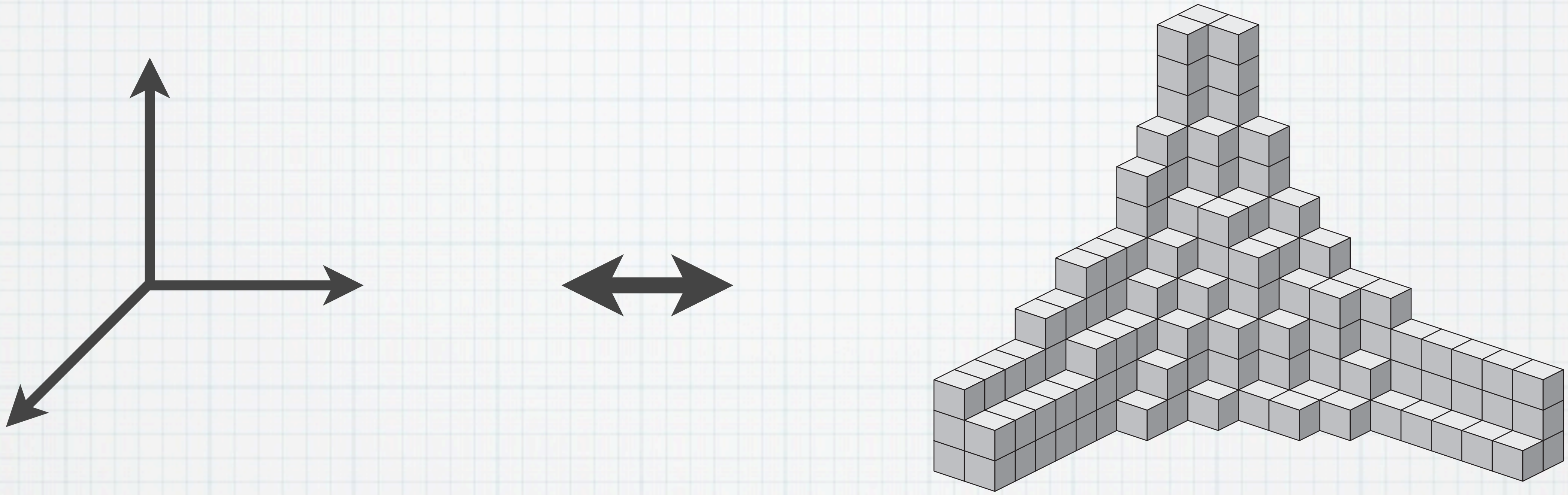


melting crystal corner



molten atoms of crystal corner

molten atoms of crystal corner

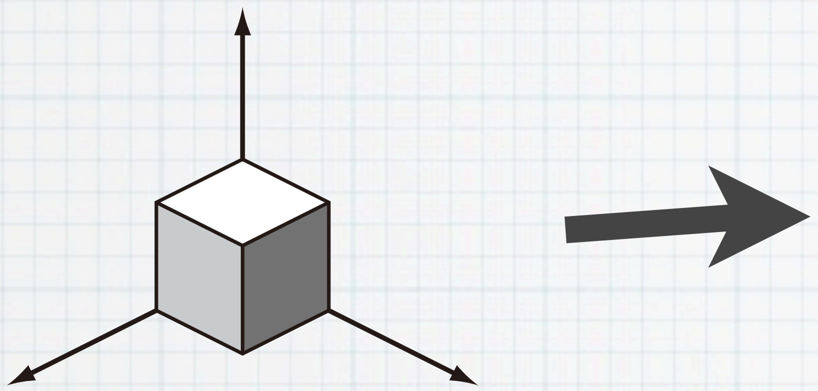


$$Z = \sum_{\text{crystals}} e^{-N\hbar}$$

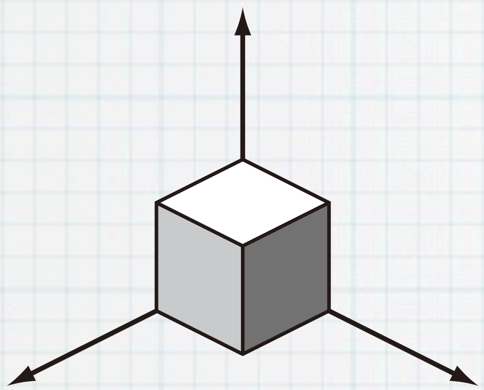
$$N = \#(\text{atoms})$$

$$= \sum q^N$$

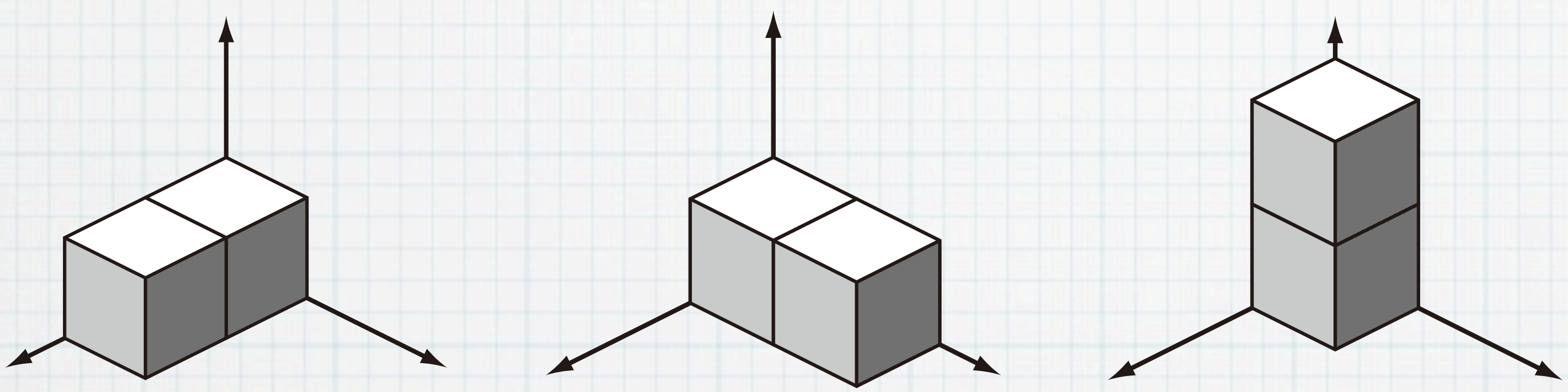
$$q = e^{-\hbar}$$



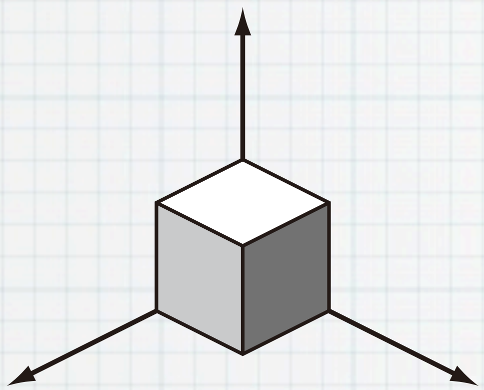
q^1



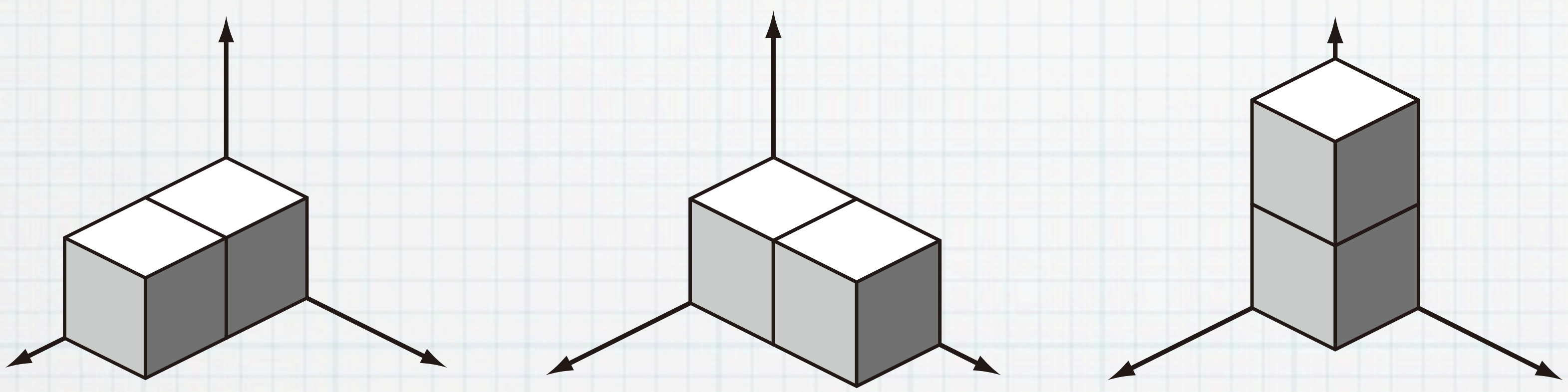
q^1



$3\,q^2$

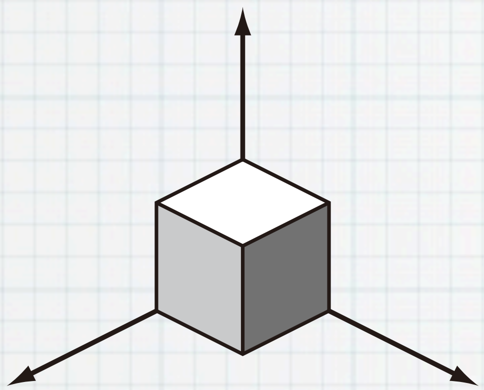


$$\rightarrow \quad q^1$$

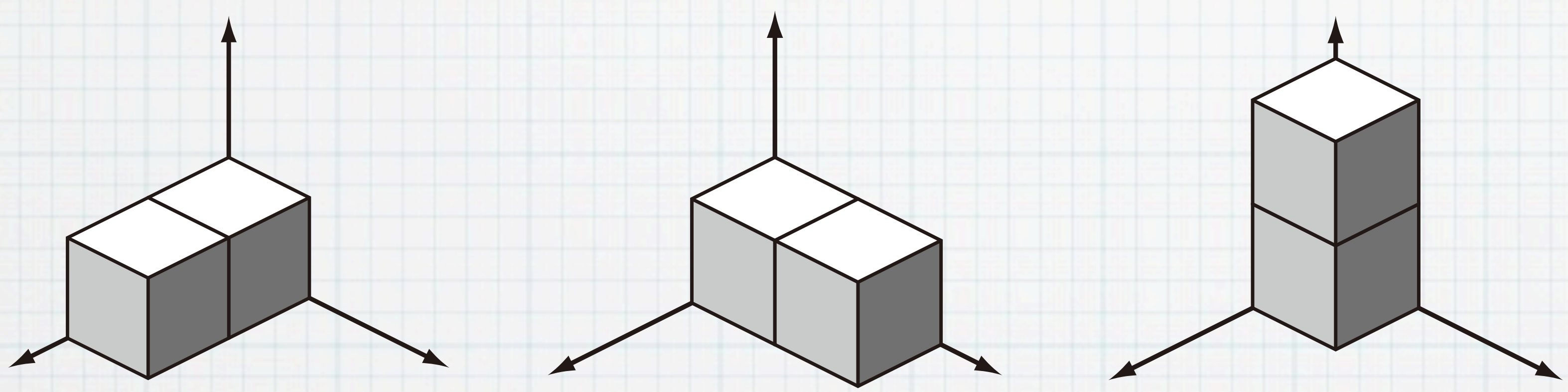


$$\rightarrow \quad 3\,q^2$$

$$Z = 1 + q + 3q^2 + \cdots$$



→ q^1



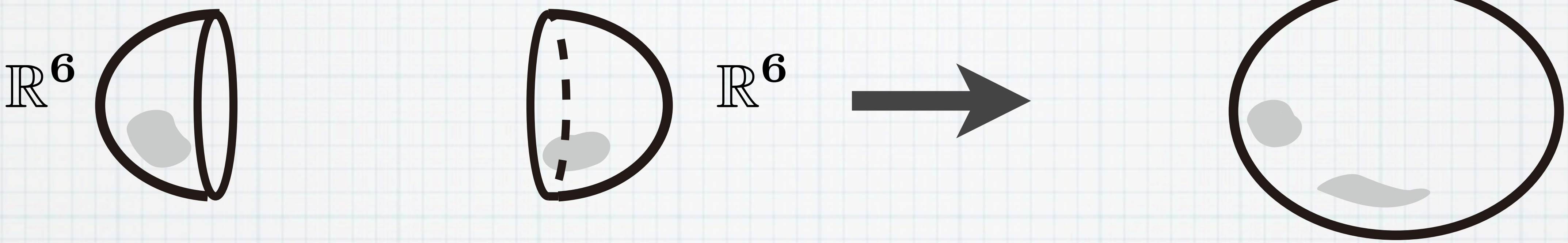
→ $3\,q^2$

$$Z = 1 + q + 3q^2 + \cdots$$

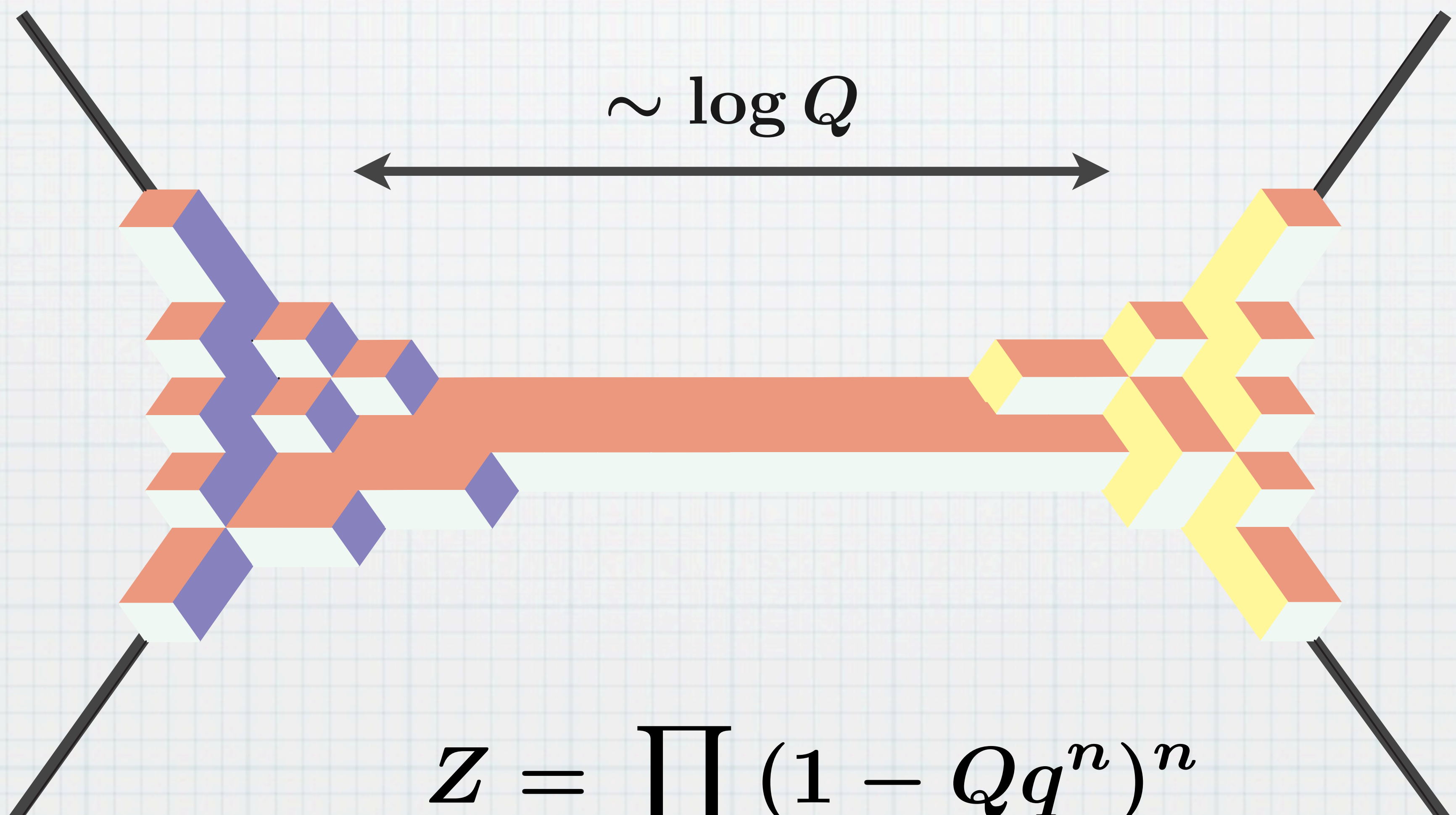
$$= \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n} \quad !!$$

[Percy MacMahon, 1915]

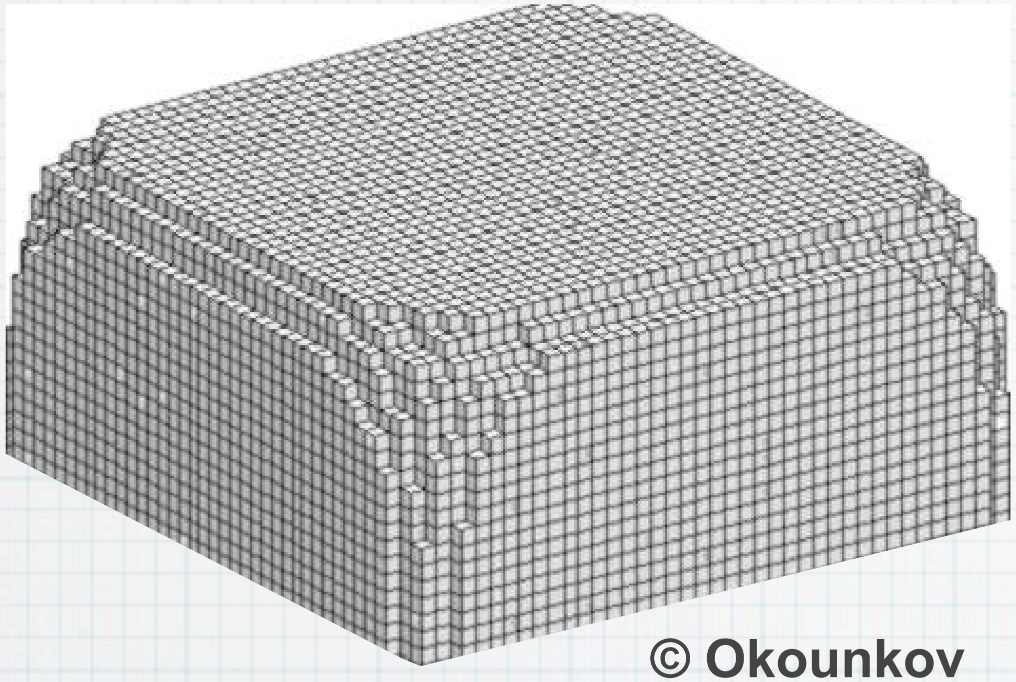
Any 6-dim. space is constructed by gluing local patches of $\mathbb{R}^6$



So, the corresponding crystal is a combination of these corners



$$Z = \prod_{n=1} (1 - Qq^n)^n$$



© Okounkov

(N=2) SU(2) SUSY
Yang-Mills theory

[Iqubal-Kashanipoor, `02]
[Eguchi-Kanno, `03]
[Iqubal-Kozcaz-Vafa, `07]
[M.T, `07]
. . .

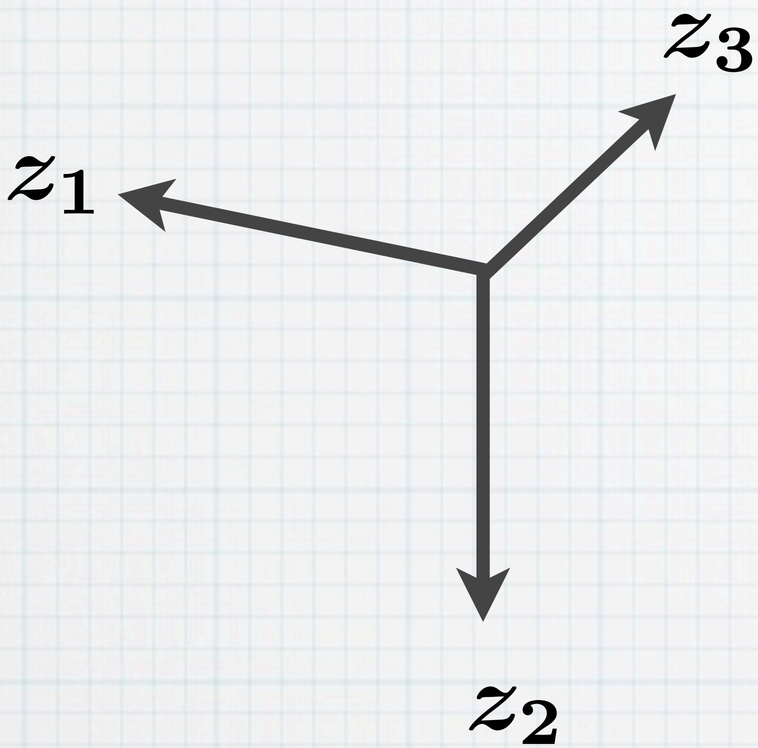
. . . . .

Lessons:

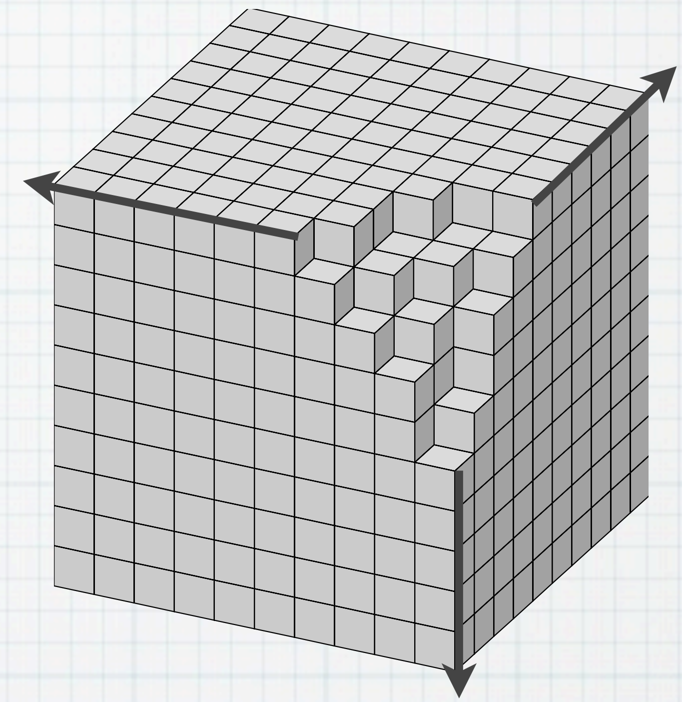
We can compute complicated stringy partition functions by using statistical models !

The resulting partition function captures the dynamics of gauge theory.

< crystal as a quantum gravity > [Iqubal-Nekrasov-Okounkov-Vafa, `07]



$\sum$
melting



classical geometry $\mathbb{R}^6 = \mathbb{C}^3$

It gives a toy model of quantum gravity on $\mathbb{R}^6$

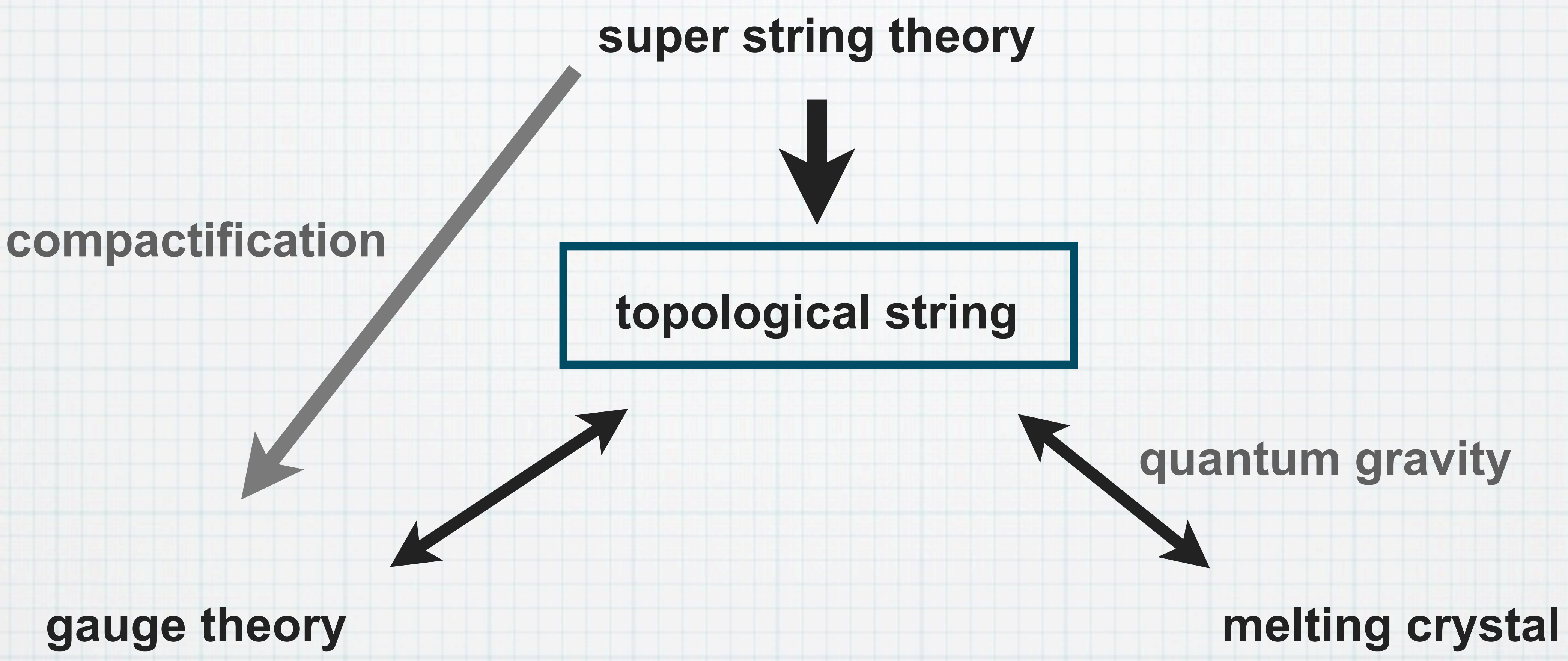
melting crystal = foamed geometry

stringy Kahler gravity

$$S_{\text{gravity}} = \frac{1}{3!\,\hbar^2}\int_M \omega \wedge \omega \wedge \omega$$

$$\omega = dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 + dz_3 \wedge d\bar{z}_3$$

2. Summary : Woods of Stringy Dualities



2. Summary : Woods of Stringy Dualities