YITP Lunch Seminar
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Buckling of a rotating elastic rod: Twist transport and geometric phases

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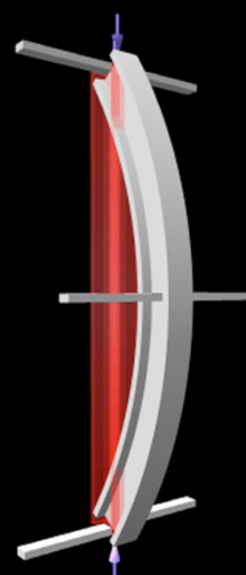
Buckling of a thin rod under stress

A ubiquitous phenomena in our daily life
A classical lesson on stability analysis



Euler buckling and Elastica
shape instability of a compressed rod

Leonhard Euler
(1707-1783)

"Euler and Lagrange are the earliest in the region of *elastic instability" -- from A.E.H. Love, "A Treatise on the Mathematical Theory of Elasticity"*

Gustav Kirchhoff
(1824-1887)

Progress in elastic theory of a thin rod

Kirchhoff rod equations

$$\begin{aligned} \partial_s \mathbf{F} - \mathbf{f} &= 0, \\ \partial_s \mathbf{M} + \partial_s \mathbf{r} \times \mathbf{F} - \mathbf{m} &= 0. \end{aligned}$$

Analogy to Lagrangian mechanics of a top "Kirchhoff's kinetic analogue"

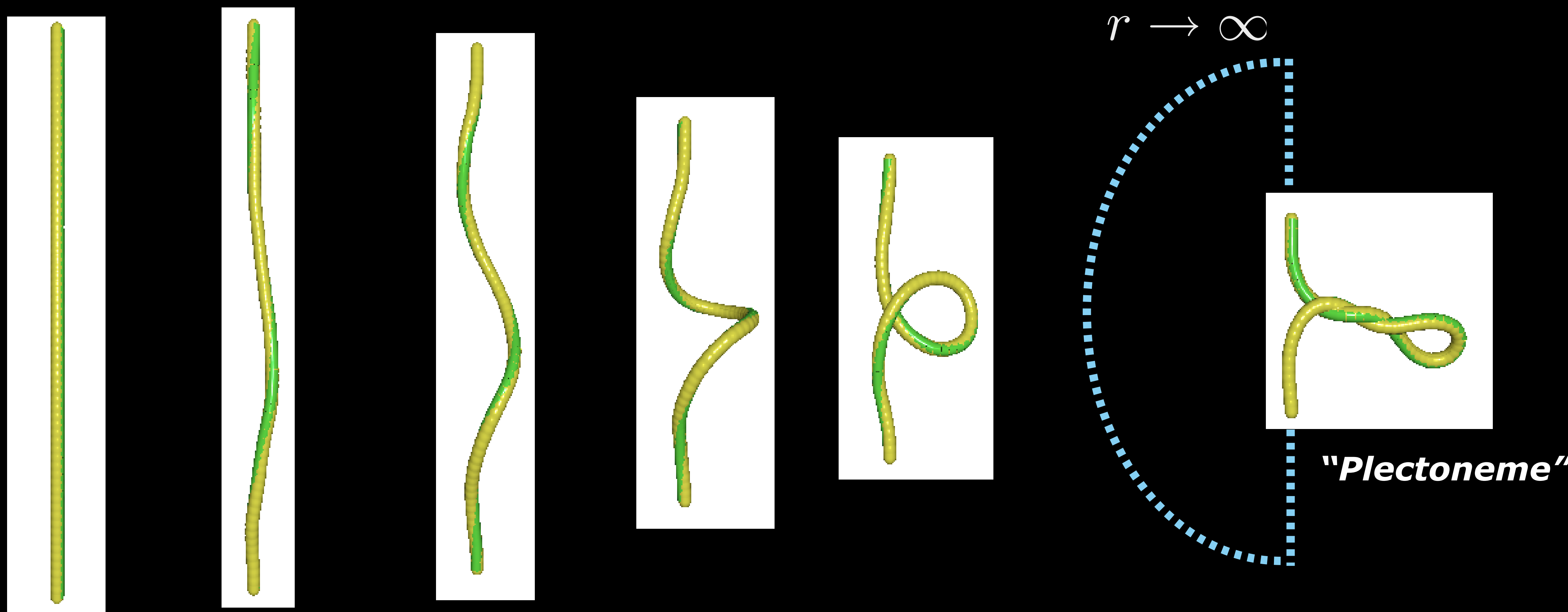
Twist and loop formation: Geometric aspects

Linking number is invariant:

$$\mathrm{Lk} = \mathrm{Wr} + \mathrm{Tw}$$

"writhe" "twist"

Chiral buckling of a twisted filament



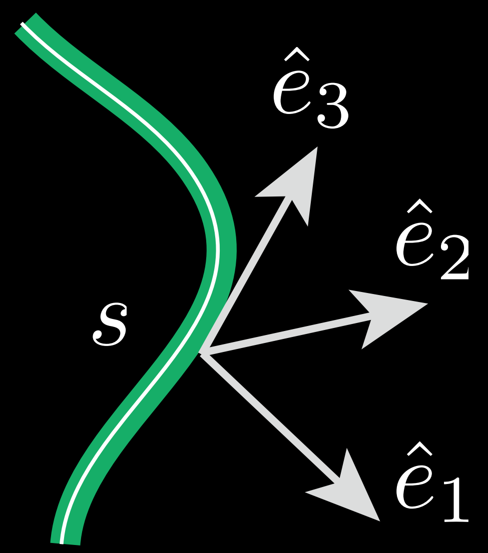
Axial twist is partly converted into centerline windings (writhe) through buckling

Self-crossing of the curve changes Wr by 2 discontinuously. (e.g., Dirac's belt-trick or Feyman's plate trick)

Violation of topological invariance Lk=Wr+Tw.

Theory for a thin elastic rod

--- Variational formulation



Kinematic equation

$$\partial_s \hat{\mathbf{e}}_i = \mathbf{\Omega} \times \hat{\mathbf{e}}_i$$

$\mathbf{\Omega}$ rotation rate vector

Elastic energy for an isotropic straight rod

$$E = \int_0^L ds \, \frac{A}{2} \left(\Omega_1^2 + \Omega_2^2\right) + \frac{C}{2} \, \Omega_3^2 - \int_0^L ds \, \Lambda(s),$$

Variational relations

change of line element

$$\delta(ds) = \hat{e}_3 \cdot \partial_s(\delta \mathbf{r}) ds$$

change of base vectors

$$\delta\hat{e}_1 = (\delta\chi)\hat{e}_2 - [\hat{e}_1 \cdot \partial_s(\delta\mathbf{r})]\hat{e}_3$$

$$\delta\hat{e}_2 = -(\delta\chi)\hat{e}_1 - [\hat{e}_2 \cdot \partial_s(\delta\mathbf{r})]\hat{e}_3$$

$$\delta\hat{e}_3 = \partial_s(\delta\mathbf{r}) - [\hat{e}_3 \cdot \partial_s(\delta\mathbf{r})]\hat{e}_3.$$

change of strain rates

$$\delta\Omega_1 = (\delta\chi)\Omega_2 - 2\Omega_1\hat{e}_3 \cdot \partial_s(\delta\mathbf{r}) - \hat{e}_2 \cdot \partial_s^2(\delta\mathbf{r}).$$

$$\delta\Omega_2 = -(\delta\chi)\Omega_1 - 2\Omega_2\hat{e}_3 \cdot \partial_s(\delta\mathbf{r}) + \hat{e}_1 \cdot \partial_s^2(\delta\mathbf{r})$$

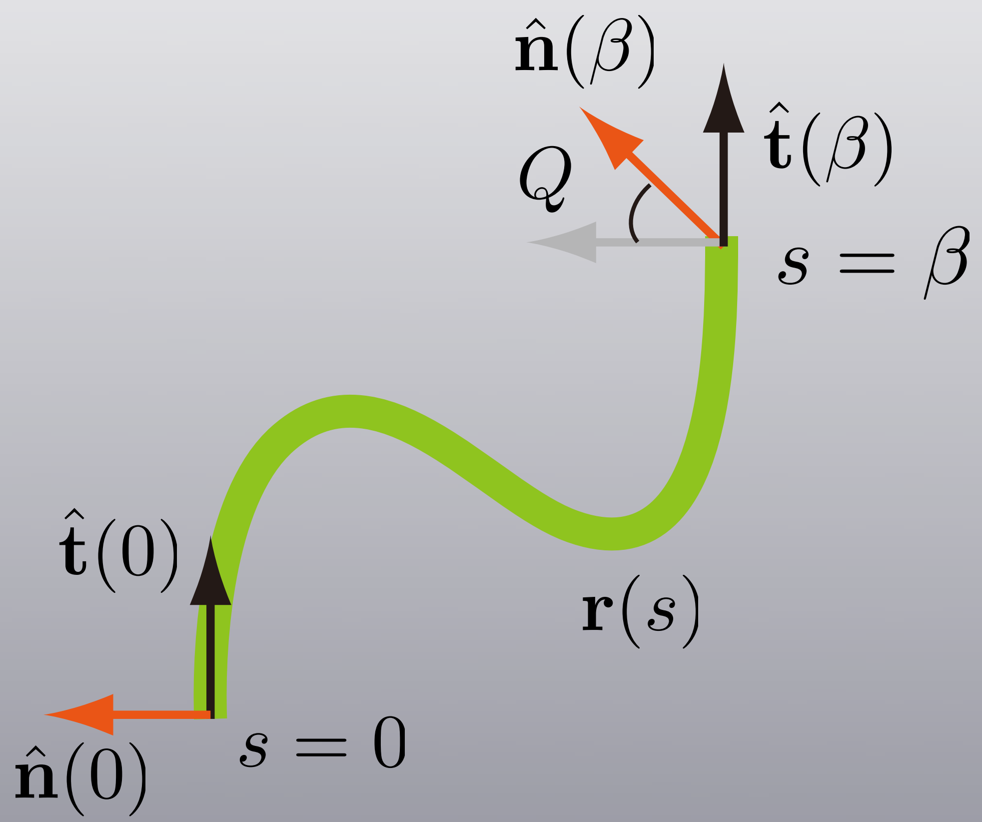
$$\delta\Omega_3 = \partial_s(\delta\chi) + (\Omega_1\hat{e}_1 + \Omega_2\hat{e}_2) \cdot \partial_s(\delta\mathbf{r}) - \Omega_3\hat{e}_3 \cdot \partial_s(\delta\mathbf{r})$$

axial rotation (spin) rate

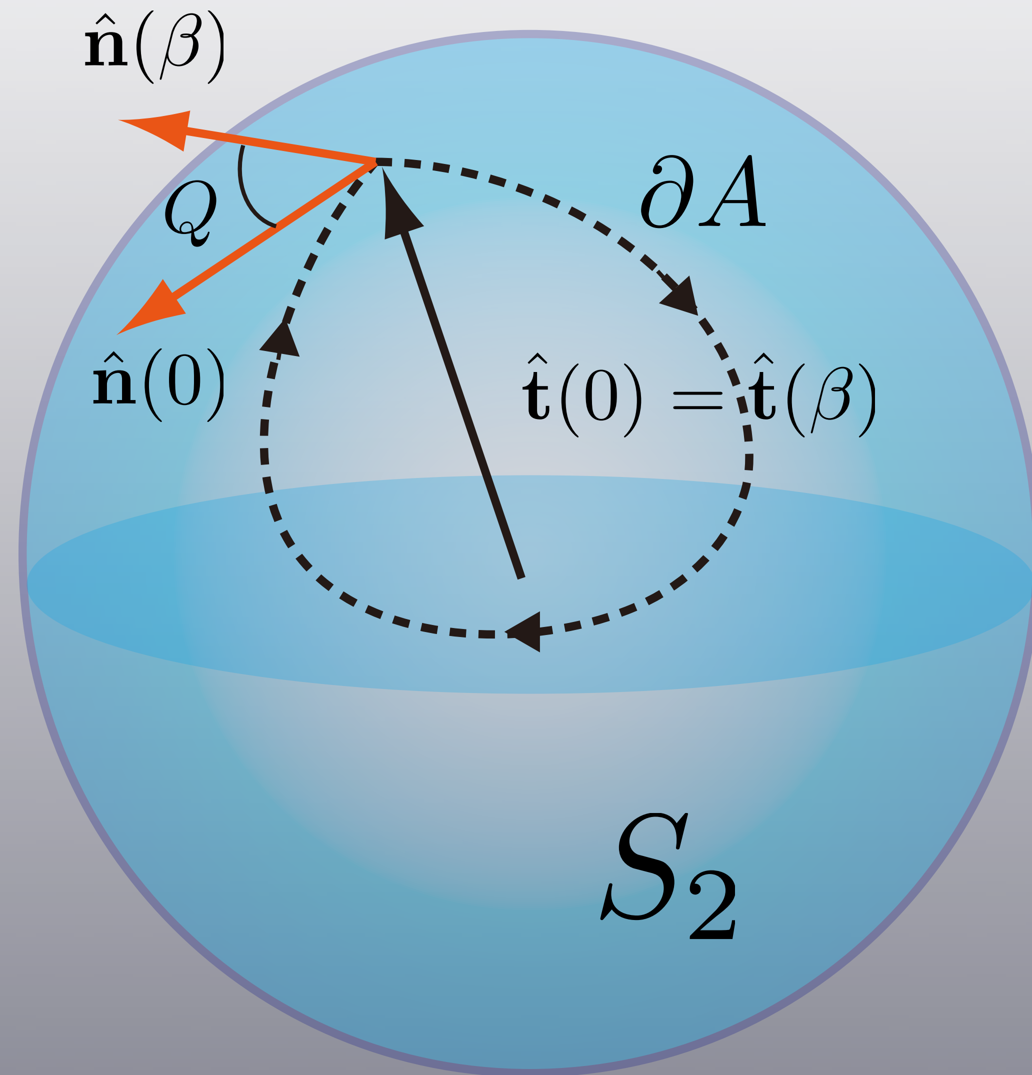
$$\delta\chi = \hat{e}_2 \cdot \partial_s\hat{e}_1$$

R. E. Goldstein, T. R. Powers and C H. Wiggins, PRL 80, 5232 (1998).

Simple but Not-so-precise argument on twist



Segment of a rod

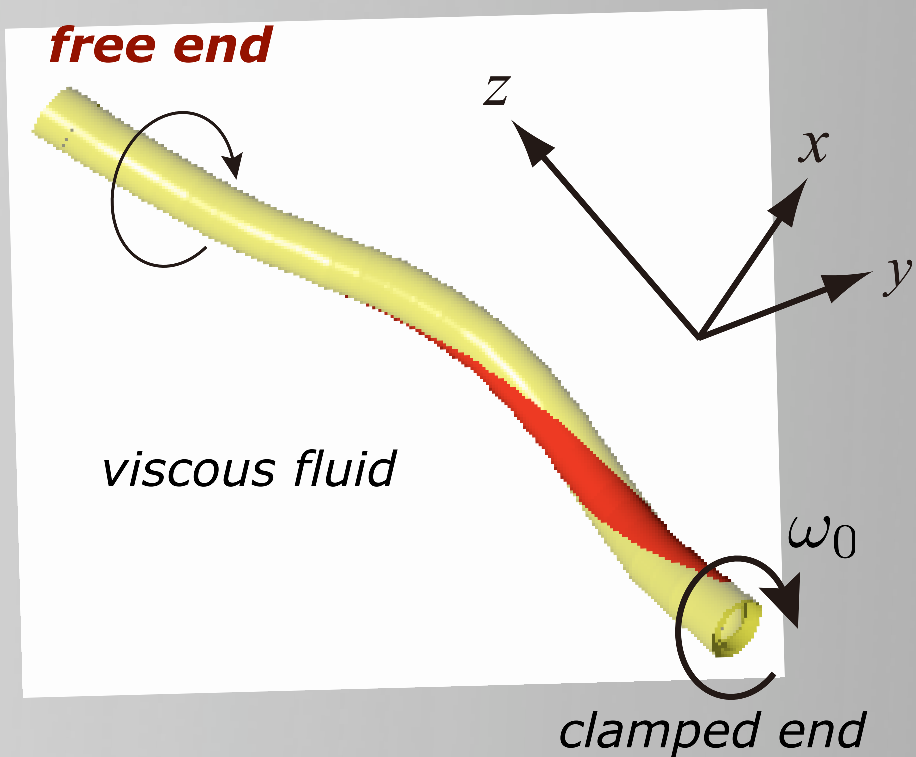


Poincare sphere

"dynamical phase" "geometric phase"

$\int_0^\beta ds\, \Omega_3(s)$ ~ (pure axial rotation) - (geometric Q)

(twist) (linking number) (writhe)



Rotating elastic rod in a viscous fluid: Dynamical analogue to a chiral buckling rod

Viscous equations of motion for bend and twist

$$\begin{aligned}\zeta\partial_t\mathbf{r} &= -A\partial_s^4\mathbf{r} + C\partial_s\left[\Omega(\partial_s\mathbf{r}\times\partial_s^2\mathbf{r})\right] - \partial_s(\Lambda\partial_s\mathbf{r}),\\ \zeta_r\partial_t\Omega &= C\partial_s^2\Omega + \zeta_r(\partial_s\mathbf{r}\times\partial_s^2\mathbf{r})\cdot\partial_s(\partial_t\mathbf{r}).\end{aligned}$$

Linear stability analysis for a straight rod

$$\Omega(s) = \frac{\zeta_r\omega_0}{C}(s-L)$$

$$\zeta\partial_t\xi = -A\partial_s^4\xi + i\zeta_r\omega_0\partial_s\left[(s-L)\partial_s^2\xi\right]$$

Critical frequency for buckling

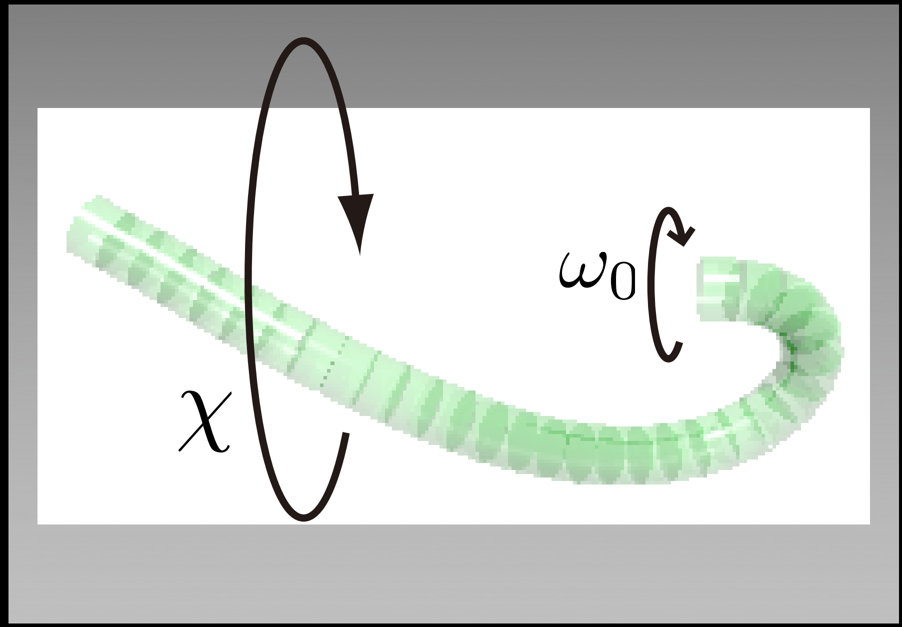
Wolgemuth et al. PRL (2000)

$$\omega_c = 8.9\frac{A}{\zeta_r L^2}$$

Post-buckled nonlinear dynamics

Wada et al EPL (2006)

Twist transport and nonlinear rotational response



Geometric relation for twist dynamics

$$\partial_t \Omega = \partial_s \omega - \Omega\, \hat{\mathbf{t}} \cdot \partial_s(\partial_t \mathbf{r}) + (\hat{\mathbf{t}} \times \partial_s \hat{\mathbf{t}}) \cdot \partial_s(\partial_t \mathbf{r})$$

twist change due to axial spinning | *twist change due to stretching* | *twist change due to out-of-plane bending ("writhing")*

Crankshafting motion of the rod centerline $\partial_t \mathbf{r} = \chi\, \hat{\mathbf{z}} \times \mathbf{r}(s)$

Geometry provides a local conservation law for twist $\partial_t \Omega + \partial_s j = 0$

"Effective" twist current $j(s) = -\omega(s) + \chi \cos\theta(s), \quad \cos\theta = \hat{\mathbf{z}} \cdot \hat{\mathbf{t}}$

"writhe current"

At steady state, current must be constant $j(s) = j(0) = -\omega_0 + \chi$

$$\omega_0 = \omega(s) + \chi(1 - \cos\theta(s))$$

Twist transport and nonlinear rotational response

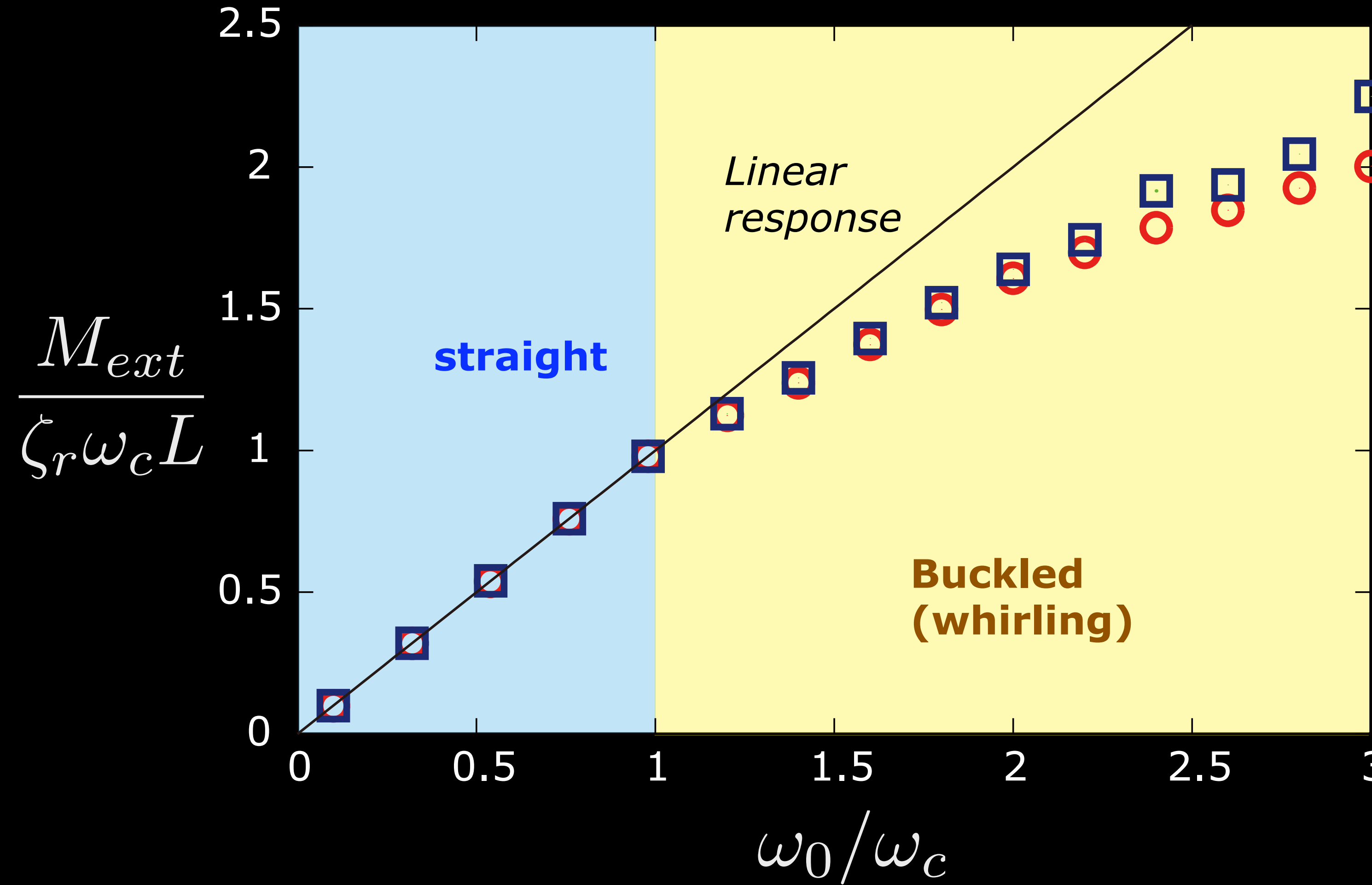
"Dynamical transition as a way to reduce the power dissipation."

purely geometric ("physicsless")

$$\omega_0 = \omega(s) + \chi(1 - \cos\theta(s))$$

+

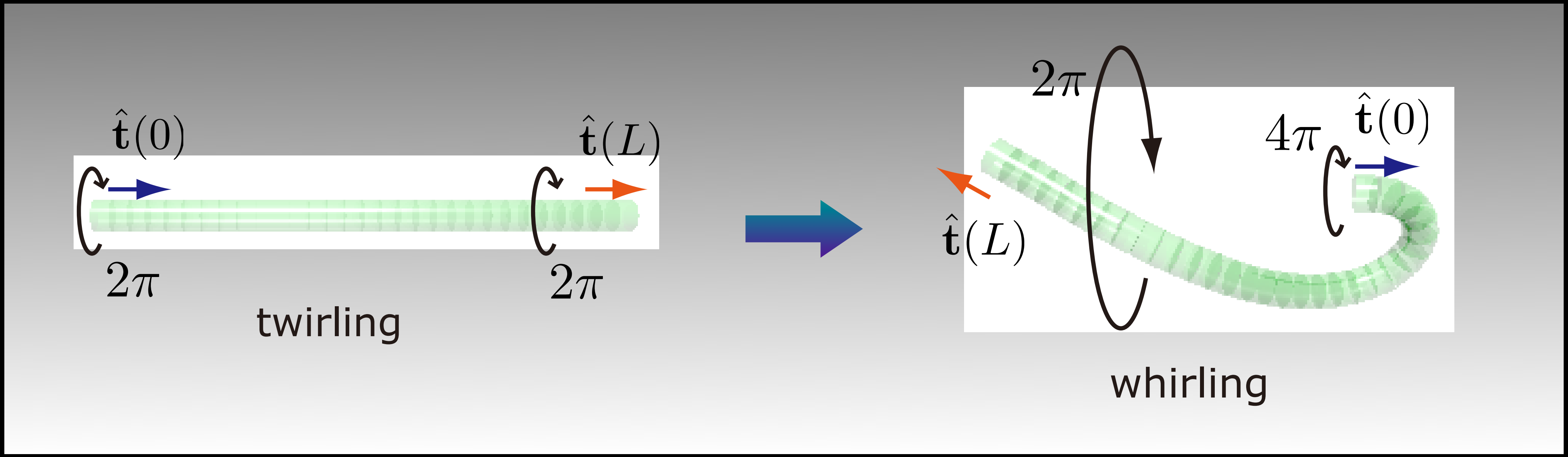
Viscous equations of motion
(Kirchhoff rod equations)



Nonlinear torque-frequency relationship

$$M_{ext}(\omega_0) = \zeta_r \omega_0 L \left[1 - \frac{\zeta_r L (1-\sigma)^2}{\int_0^L ds \left[\zeta_r (1 - \cos\theta)^2 + \zeta |\mathbf{r}_\perp|^2\right]} \right].$$

Underlying geometry and relation to geometric phases



Rationalization of the analogy to belt trick

$$\omega_0 = \omega(L) + \chi(1 - \cos\theta(L))$$

$$\omega(L) = 0,$$
$$\cos\theta(L) = \hat{\mathbf{t}}(0) \cdot \hat{\mathbf{t}}(L) = -1$$

$$\omega_0 = 2\chi$$

change rate

$$\oint \omega_0 dt = 2 \oint \chi dt = 4\pi$$

Global relation

Topologically equal to a “self-crossing” of a curve (jump of 2 in Wr): our example is its dynamical realization

Global topological quantities (Wr and Lk) are defined only to closed curves, but temporal changes in writhe and link are still meaningful in a form of “conservation law” (dynamical tradeoff) even for open filaments.