“Recreational Mathematics”
New moonshine phenomenon

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Monstrous moonshine

McKay-Thompson 1979

Expansion coefficients of Modular J function are given by the sum of dims. of irred. reps. of Monster group
Modular group, modular form

\[ q = e^{2\pi i \tau} \quad \tau = \frac{\omega_2}{\omega_1} \quad \omega_1, \omega_2 ; \text{Periods of a torus} \]

\[
\begin{pmatrix}
\omega'_1 \\
\omega'_2
\end{pmatrix}
= \begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\begin{pmatrix}
\omega_1 \\
\omega_2
\end{pmatrix}
\text{ or } \quad \tau' = \frac{a\tau + b}{c\tau + d}
\]

\[ SL(2; \mathbb{Z}) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}, ab - cd = 1 \right\} \]

Generators

\[ S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \]
Modular J function

\[ J(q^\gamma) = J(q), \quad q = e^{2\pi i \tau}, \quad q^\gamma = e^{2\pi i \frac{a \tau + b}{c \tau + d}} \]

J has an expansion

\[ J(q) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + \ldots \]

\[ = \sum_{n=-1} c(n)q^n \]

\[ c(1) = 196884, \]
\[ c(2) = 21493760, \]
\[ c(3) = 864299970, \]
\[ c(4) = 20245856256, \ldots \]
On the other hand dimensions of irred. reps. of Monster group are

\[
\begin{align*}
  d_0 &= 1 \\
  d_1 &= 196883 \\
  d_2 &= 21296876 \\
  d_3 &= 842609326, \ldots
\end{align*}
\]

Thus

\[
\begin{align*}
  c(1) &= d_0 + d_1 \\
  c(2) &= d_0 + d_1 + d_2 \\
  c(3) &= 2d_0 + 2d_1 + d_2 + d_3 \\
  c(4) &= 2d_0 + 3d_1 + 2d_2 + d_3 + d_5, \ldots
\end{align*}
\]

Expansion coefficients of J function are sum of dims. of irred. reps. of Monster group
Monster group: the largest sporadic discrete group

\[ C_n, D_n, S_n, Mathieu, Fischer, Suzuki, Janko, Conway, \ldots, \text{Monster} \]

order of Monster

\[ |M| = 2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \]

Ambiguity of the decomposition is fixed by studying the McKay-Thompson series

\[ J_g(q) = \sum Tr(g|_V(n))q^n, \quad J_{g=1}(q) = J(q) \]

There are 194 conjugacy classes in Monster

Study of monstrous moonshine has culminated in the work of Borchards (Fields medal)
K3 surface (after Kummer, Kahler, Kodaira)

<table>
<thead>
<tr>
<th></th>
<th>dim${}_{\mathbb{C}}$</th>
<th>holonomy</th>
</tr>
</thead>
<tbody>
<tr>
<td>torus</td>
<td>1</td>
<td>—</td>
</tr>
<tr>
<td>K3</td>
<td>2</td>
<td>SU(2)</td>
</tr>
<tr>
<td>Calabi-Yau</td>
<td>3</td>
<td>SU(3)</td>
</tr>
</tbody>
</table>

Standard Model of Black Hole:
Consider string theory compactified on K3 surface
String/K3+(Q1) D1 branes+(Q5) D5 brane wrapping K3

\[ Q_{1}Q_{5} = k \rightarrow \infty \]

Macroscopic Black Hole
geometry of tensor products of K3 surface

\[ Q_{1} = Q_{5} = k = 1 \]

Mini mini Black Hole
geometry of K3
Mathieu moonshine

Quantum states of mini BH exhibits a phenomenon similar to monstrous moonshine

\[ J(q) = \sum_{n=-1}^{\infty} c(n)q^n \]
\[ \downarrow \]
\[ Z_{K3}(q, z) = 24 \cdot \chi_{\text{disc. rep}}(q, z) + \sum_{n=0} A(n)q^n \chi_{\text{conti. rep}}(q, z) \]
\[ \chi_{\text{disc. rep}}(q, z), \chi_{\text{conti. rep}}(q, z) = 1 + O(q) + O(q^2) + \cdots \]

\[ Z_{K3}(q, z) \quad : \quad \text{Elliptic genus of K3 surface} \]

Geometry of K3 surface is captured by its elliptic genus

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>( \cdots )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A(n) )</td>
<td>( 2 \times 45 )</td>
<td>( 2 \times 231 )</td>
<td>( 2 \times 770 )</td>
<td>( 2 \times 2277 )</td>
<td>( 2 \times 5796 )</td>
<td>( 2 \times 13915 )</td>
<td>( 2 \times 30843 )</td>
<td>( 2 \times 65550 )</td>
<td>( \cdots )</td>
</tr>
</tbody>
</table>
On the other hand M24 has 26 irred. representations

\{1, 23, 252, 253, 1771, 3520, 45, 45, 990, 990, 1035, 1035, 1035', 231, 231, \\
770, 770, 483, 1265, 2024, 2277, 3312, 5313, 5796, 5544, 10395\}.

13915 = 3520 + 10395, \quad 30843 = 10395 + 5796 + 5544 + 5313 + 2024 + 1771, \ldots

We see a close analogue of monstrous moonshine. We have constructed the analogues of McKay-Thompson series and have decomposed the coefficients A(n) into sum of dims. of M24 reps. up to \ n=600.

\[\mathbf{Mathieu\ Group\ M24}\]

Subgroup of S24, permutations of 24 objects. Isomorphism group of binary Golay code.
Golay code

24 digit binary numbers: \(2^{24}\) code words
select \(2^{12}\) words so that any pair of words has a distance \(\geq 8\)
then one can correct up to \(3\) errors in three bits

\[
\begin{array}{c|c}
\text{distance} & \text{M24 maps} \ 2^{12} \ \text{words into each} \\
+ + + + & 0 \\
+ + + + & \text{other.}
\end{array}
\]

\[
\begin{array}{c|c}
+ + + + & 1 \\
+ + + - & \text{Efficient error correcting code} \\
& \text{(used in various NASA project)}
\end{array}
\]

\[
\begin{array}{c|c}
+ + + + & 2 \\
- + - - & \text{Efficient storage of information in} \\
& \text{mini BH ???}
\end{array}
\]