“Recreational Mathematics”

New moonshine phenomenon

T.E.,H.Ooguri,Y.Tachikawa

T.E.,K.Hikami

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Monstrous moonshine

McKay-Thompson 1979

Expansion coefficients of Modular J function are given by the sum of dims. of irred. reps. of Monster group

◆ Modular group, modular form

$$q = e^{2\pi i\tau} \qquad \tau = \frac{\omega_2}{\omega_1} \qquad \omega_1, \omega_2$$; Periods of a torus

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$ or $$\tau' = \frac{a\tau + b}{c\tau + d}$$

$$SL(2;\mathbf{Z}) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in \mathbf{Z}, ab - cd = 1 \right\}$$

Generators

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

◆Modular J function

$$J(q^\gamma) = J(q), \quad q = e^{2\pi i\tau}, \quad q^\gamma = e^{2\pi i \cdot \frac{a\tau+b}{c\tau+d}}$$

J has an expansion

$$J(q) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + \ldots$$

$$= \sum_{n=-1} c(n)q^n$$

$$c(1) = 196884,$$
$$c(2) = 21493760,$$
$$c(3) = 864299970,$$
$$c(4) = 20245856256, \ldots$$

On the other hand dimensions of irred. reps. of Monster group are

$$d_0 = 1$$
$$d_1 = 196883$$
$$d_2 = 21296876$$
$$d_3 = 842609326, \ldots$$

Thus

$$c(1) = d_0 + d_1$$
$$c(2) = d_0 + d_1 + d_2$$
$$c(3) = 2d_0 + 2d_1 + d_2 + d_3$$
$$c(4) = 2d_0 + 3d_1 + 2d_2 + d_3 + d_5, \ldots$$

Expansion coefficients of J function are sum of dims. of irred. reps. of Monster group

Monster group: the largest sporadic discrete group

$$C_n, D_n, S_n, \text{Mathieu}, \text{Fischer}, \text{Suzuki}, \text{Janko}, \text{Conway}, \cdots, \text{Monster}$$

order of Monster

$$|\mathbf{M}| = 2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$$

Ambiguity of the decomposition is fixed by studying the McKay-Thompson series

$$J_g(q) = \sum Tr(g|_{V(n)})q^n, \quad J_{g=1}(q) = J(q)$$

There are 194 conjugacy classes in Monster

Study of monstrous moonshine has culminated in the work of Borchards (Fields medal)

- K3 surface (after Kummer, Kahler, Kodaira)

	$\dim_{\mathbf{C}}$	holonomy
torus	1	—
K3	2	SU(2)
Calabi-Yau	3	SU(3)

Standard Model of Black Hole:

Consider string theory compactified on K3 surface

String/K3+(Q1) D1 branes+(Q5) D5 brane wrapping K3

$Q_1 Q_5 = k \to \infty$ Macroscopic Black Hole geometry of tensor products of K3 surface

$Q_1 = Q_5 = k = 1$ Mini mini Black Hole geometry of K3

◆Mathieu moonshine

Quantum states of mini BH exhibits a phenomenon similar to monstrous moonshine

$$J(q) = \sum_{n=-1} c(n) q^n$$

$$\Downarrow$$

$$Z_{K3}(q,z) = 24 \cdot \chi_{\text{disc.rep}}(q,z) + \sum_{n=0} A(n) q^n \chi_{\text{conti.rep}}(q,z)$$

$$\chi_{\text{disc.rep}}(q,z), \chi_{\text{conti.rep}}(q,z) = 1 + \mathcal{O}(q) + \mathcal{O}(q^2) + \cdots$$

$Z_{K3}(q,z)$: Elliptic genus of K3 surface
Geometry of K3 surface is captured by its elliptic genus

n	1	2	3	4	5	6	7	8	$\cdots$
$A(n)$	2×45	2×231	2×770	2×2277	2×5796	2×13915	2×30843	2×65550	$\cdots$

On the other hand M24 has 26 irred. representations

$$\{1, 23, 252, 253, 1771, 3520, 45, \overline{45}, 990, \overline{990}, 1035, \overline{1035}, 1035', 231, \overline{231}, 770, \overline{770}, 483, 1265, 2024, 2277, 3312, 5313, 5796, 5544, 10395\}.$$

$$13915 = 3520 + 10395, \quad 30843 = 10395 + 5796 + 5544 + 5313 + 2024 + 1771, \cdots$$

We see a close analogue of monstrous moionshine.
We have constructed the analogues of McKay-Thompson series and have decomposed the coefficients A(n) into sum of dims. of M24 reps. up to n=600.

- Mathieu Group M24

Subgroup of S24, permutations of 24 objects.
Isomorphism group of binary Golay code.

◆Golay code

24 digit binary numbers: 2^{24} code words

select 2^{12} words so that any pair of words has a distance ≥ 8

then one can correct up to errors in three bits

	distance
$+++$ $+++$	0
$+++$ $++-$	1
$+++$ $-+-$	2

M24 maps 2^{12} words into each other.

Efficient error correcting code (used in various NASA project)

Efficient storage of information in mini BH ???