MYERS-PERRY BLACK HOLES: BEYOND THE SINGLE ROTATION CASE

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Contents

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- For a fixed mass, solutions may exist which have arbitrary large angular momentum. c.f. Kerr $M > a$.
- $\lfloor \frac{D-1}{2} \rfloor$ rotation parameters are required to describe the rotation, where D is the total spacetime dimension.

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otherwise if D is even the x_{D-1} coordinate will be unpaired.
Therefore there are $\lfloor \frac{D-1}{2} \rfloor$ such pairs.
Writing each pair (x_i, x_{i+1}) in polar coordinates (r_i, ϕ_i) we see there is a rotation associated with each of the ∂_{ϕ_i} vectors.

The MP metric(s)

Even D; $$ds^2 = -dt^2 + r^2 d\alpha^2 + \sum_{i=1}^{d} \left(r^2 + a_i^2\right) \left(d\mu_i^2 + \mu_i^2 d\phi_i^2\right)$$
$$+ \frac{2Mr}{\Pi F} \left(dt + \sum_{i=1}^{d} a_i \mu_i^2 d\phi_i\right)^2 + \frac{\Pi F}{\Pi - 2Mr} dr^2,$$

Odd D; $$ds^2 = -dt^2 + \sum_{i=1}^{d} \left(r^2 + a_i^2\right) \left(d\mu_i^2 + \mu_i^2 d\phi_i^2\right)$$
$$+ \frac{2Mr^2}{\Pi F} \left(dt + \sum_{i=1}^{d} a_i \mu_i^2 d\phi_i\right)^2 + \frac{\Pi F}{\Pi - 2Mr^2} dr^2,$$

$$F = 1 - \sum_{i=1}^{d} \frac{a_i^2 \mu_i^2}{r^2 + a_i^2}, \quad \Pi = \prod_{i=1}^{d} (r^2 + a_i^2), \quad d = \begin{cases} \frac{D-2}{2}, & D \text{ even} \\ \frac{D-1}{2}, & D \text{ odd.} \end{cases}$$

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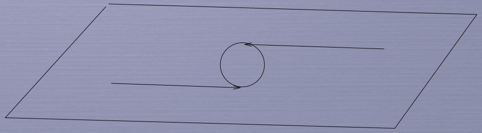
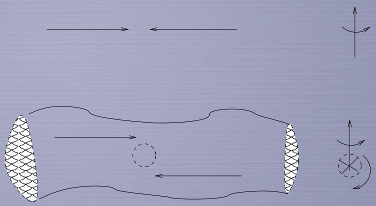
This can be loosely justified in brane world scenario’s, where colliding particle momentum will be confined to a brane and therefore any black hole created on this brane would only spin in this direction.



However, if the brane is thick then there is also the potential for angular momentum to occur in other directions.



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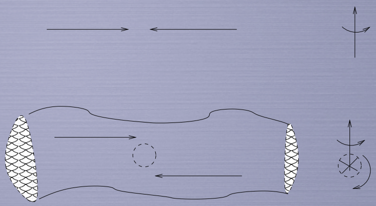

Furthermore, one could imagine planck-scale processes not confined to the brane (i.e., gravitons/strings etc) creating black holes in the bulk spacetime with arbitrary angular momentum. We are thus motivated to explore the solutions with more than one angular momentum parameter.

Angular momentum constraints

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This occurs when:

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Why is $D = 5$ special?
To "explain" this we calculate the angular momentum constraints in higher dimensions in general with all angular momentum parameters.

In D-even dimensions the condition for the location of the horizon is:

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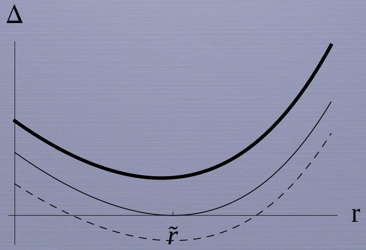
There are only three possibilities for the number of horizons:



Let $\tilde{r}$ be the unique minima of Δ for positive r.

Then the black hole will be free of naked singularities (that is to say a horizon exists) if:

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The inequality is saturated when $\Delta(\tilde{r}) = 0$, equality occurs iff $\tilde{r}$ is also an r-intercept. Therefore,

$$M \geq \frac{\Pi(\tilde{r}_h)}{2\tilde{r}_h},$$

where $\tilde{r}_h$ is a solution to the simultaneous set of equations

$$\Delta(\tilde{r}_h) = 0, \quad \partial_r \Delta(\tilde{r}_h) = 0.$$

Or:

$$0 = \tilde{r}_h \partial_r \Pi(\tilde{r}_h) - \Pi(\tilde{r}_h)$$

Using the product expansion:

$$\prod_{i=1}^{j}(r^2 + a_i^2) = \sum_{i=0}^{j} r^{2i} A_j^{j-i},$$

with the coefficients conveniently defined as

$$A_n^k = \sum_{\nu_1<\nu_2<\ldots\nu_k} a_{\nu_1}^2 a_{\nu_2}^2 \ldots a_{\nu_k}^2,$$

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$$P_2(\tilde{r}_h) = \sum_{i=0}^{d} (2i-1) \tilde{r}_h^{2i} A_d^{d-i} = 0.$$

$$d \le 4 \rightarrow \frac{D-2}{2} \le 4 \rightarrow D \le 10$$

In the D-odd case the location of the horizon is found by the equation:

$$\Delta = \Pi(l) - 2Ml = 0, \quad \Pi = \prod_{i=1}^{d}(l + a_i^2).$$

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$$\tilde{l}_h \partial_l \Pi(\tilde{l}_h) - \Pi(\tilde{l}_h) = 0, \quad M \geq \frac{\Pi(\tilde{l}_h)}{2\tilde{l}_h}$$

and the polynomial for the odd case is:

$$P_1(\tilde{l}_h) = \sum_{i=0}^{d}(i-1)\tilde{l}_h^i A_d^{d-i} = 0.$$

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$$d \leq 4 \to \frac{D-1}{2} \leq 4 \to D \leq 9$$

D=7 an example:

The polynomial is:

$$2\tilde{l}_h^3 + \tilde{l}_h^2(a_1^2 + a_2^2 + a_3^2) - a_1^2 a_2^2 a_3^2 = 0$$

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and the solution can be written in closed form:

$$\begin{aligned}
\tilde{l}_h &= \frac{1}{6}\left(-(a_1^2 + a_2^2 + a_3^2) + \frac{1}{p}(a_1^2 + a_2^2 + a_3^2)^2 + p\right), \\
p^3 &= -(a_1^2 + a_2^2 + a_3^2)^3 + 54 a_1^2 a_2^2 a_3^2 + 6\sqrt{3} \\
&\times \sqrt{-(a_1^2 + a_2^2 + a_3^2)^3 a_1^2 a_2^2 a_3^2 + 27(a_1^4 a_2^4 a_3^4)},
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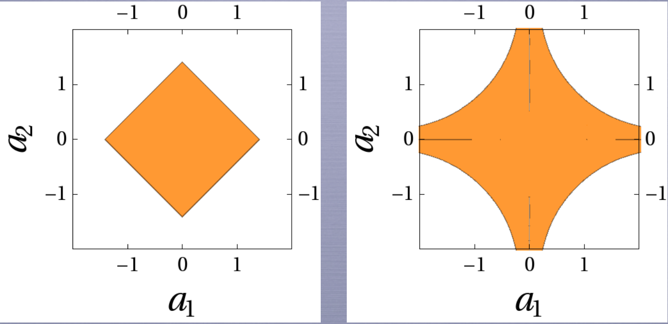
and the constraint reads:

$$M \geq \frac{(\tilde{l}_h + a_1^2)(\tilde{l}_h + a_2^2)(\tilde{l}_h + a_3^2)}{2\tilde{l}_h}.$$

Results. $M = 1$.



(left) $D = 5$. (right) $D = 6$.

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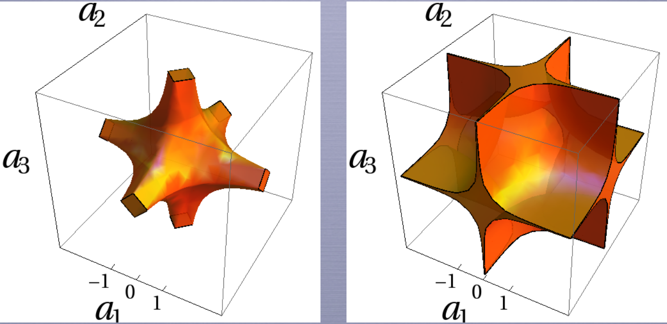

(left) $D = 7$. (right) $D = 8$.

D	All a_i non-zero	$\mathbf{a_1 = 0}$
4	$M \geq a_1$	$M \geq 0$
5	$2M \geq a_1^2 + a_2^2 + \lvert a_1 a_2 \rvert$	$M > \frac{a_2^2}{2}$
6	$M \geq \frac{(\tilde{r}_h^2 + a_1^2)(\tilde{r}_h^2 + a_2^2)}{2\tilde{r}_h};$ $3\tilde{r}_h^4 + (a_1^2 + a_2^2)\tilde{r}_h^2 - a_1^2 a_2^2 = 0$	$M \geq 0$
7	$M \geq \frac{(\tilde{l}_h + a_1^2)(\tilde{l}_h + a_2^2)(\tilde{l}_h + a_3^2)}{2\tilde{l}_h};$ $2\tilde{l}_h^3 + \tilde{l}_h^2(a_1^2 + a_2^2 + a_3^2) - a_1^2 a_2^2 a_3^2 = 0$	$M > \frac{a_1^2 a_2^2}{2}$
8	$M \geq \frac{(\tilde{r}_h^2 + a_1^2)(\tilde{r}_h^2 + a_2^2)(\tilde{r}_h^2 + a_3^2)}{2\tilde{r}_h};$ $5\tilde{r}_h^6 + 3(a_1^2 + a_2^2 + a_3^2)\tilde{r}_h^4$ $(a_1^2 a_2^2 + a_1^2 a_3^2 + a_2^2 a_3^2)\tilde{r}_h^2 - a_1^2 a_2^2 a_3^2 = 0$	$M \geq 0$
$\vdots$	$\vdots$	$\vdots$

Conclusion

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- In particular, it was shown that exact closed form expressions could be obtained for dimensions less than or equal to ten.
- These results can be used to show that none of the MP black holes can be spun into naked singularities using the Wald type gedanken experiment generalised into D dimensions.
- It is worth mentioning that even if a black hole satisfies the constraints presented it may not be stable; classical instabilities of the Gregory-Laflamme type are known to arise in the ultra spinning regimes.

If you are interested in these results please see my paper:

arXiv:1009.6118

Questions