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AdS/CFT correspondence and gluon scattering amplitudes

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Based on collaboration with
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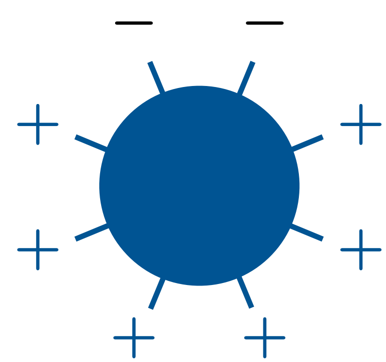
JHEP09(2010)064 (arXiv:1005.4487)
JHEP04(2011)100 (arXiv:1102.2477)

Perturbative computation of scattering amplitudes:
theoretically possible up to any desired order, but ...

Gluon scattering amplitudes in $\mathcal{N}=4$ U(N) super Yang-Mills ($N \to \infty$)

- the "simplest" gauge theory in four dimensions
- useful testing ground for perturbative QCD calculations

Significant progress has been made in recent years by making use of



- iterative structure in MHV amplitudes
  (Bern, Dixon, Kosower, Smirnov, ...)
- dual superconformal symmetry
  (Drummond, Henn, Korchemsky, Smirnov, Sokatchev, ...)

- BDS conjecture for MHV amplitudes (Bern-Dixon-Smirnov '05)

$$\mathcal{M}_n \equiv 1 + \sum_{l=1} a^l M_n^{(l)} \qquad \left(M_n^{(L)}(\epsilon) = A_n^{(L)} / A_n^{(0)} \right)$$

$$= \exp \left[\sum_{l=1} a^l \left(f^{(l)}(\epsilon) M_n^{(1)}(l\epsilon) + C^{(l)} + E_n^{(l)}(\epsilon) \right) \right]$$

$n = 4, 5$: confirmed (assuming the dual conformal symmetry)

$n \geq 6$: there appears deviation called "remainder function"

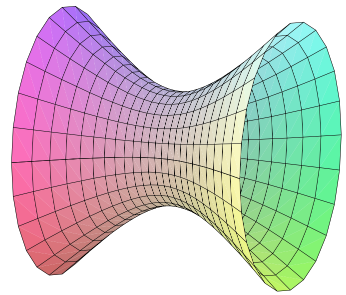
Goal: to determine the remainder function
as a function of gluon momenta and the coupling

- It is useful to know the strong coupling behavior of gluon scattering amplitudes by using the AdS/CFT correspondence

AdS/CFT Correspondence (Maldacena '97)

10d IIB Superstrings on

4d $\mathcal{N} = 4$ U(N) super Yang-Mills $\Longleftrightarrow$



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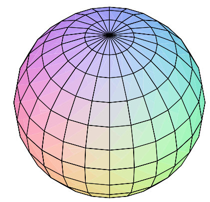


AdS_5 S^5

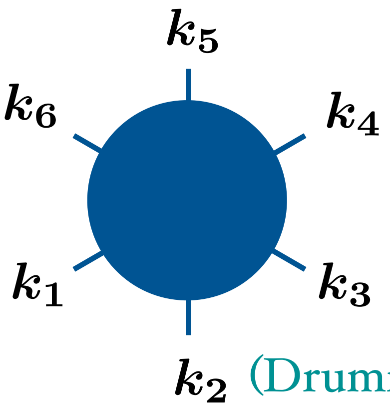
- Strong-weak correspondence

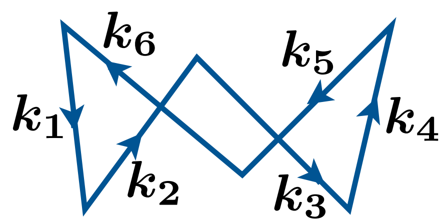
$$S = \frac{1}{\lambda} \int d^4x [(F_{\mu\nu})^2 + \cdots] \qquad S_\sigma = \frac{1}{4\pi\alpha'} \int d^2\sigma [G_{\mu\nu} \partial_a X^\mu \partial^a X^\nu + \cdots]$$

coupling: $\lambda = g^2_{\rm YM} N$ coupling: α'

$$4\pi\lambda = \frac{R^4}{\alpha'^2}$$

gluon scattering amplitude ⟺ Wilson loop





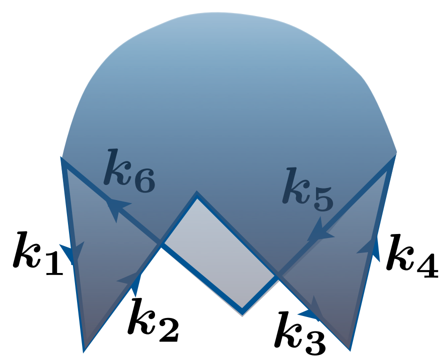
(Drummond, Henn, Korchemsky, Sokatchev '07)
(Brandhuber-Heslop-Travaglini '07)

AdS/CFT ⇕

(Rey-Yee '98)
(Maldacena '98)

Alday-Maldacena proposal

(Alday-Maldacena '07)



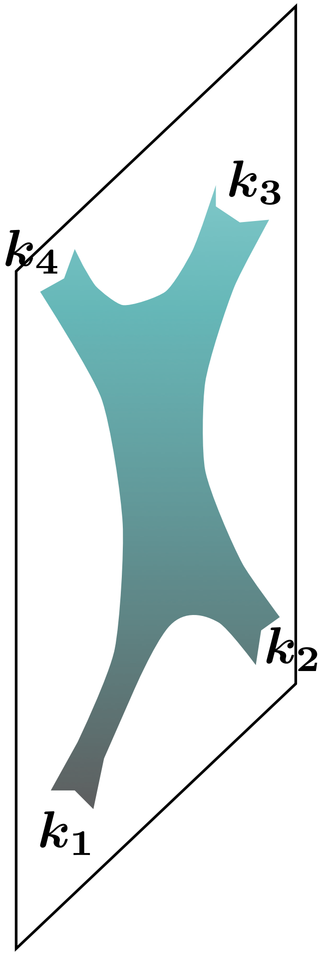
- At strong coupling,
the amplitude is given by

$$\mathcal{A}_n \sim e^{-\frac{\sqrt{\lambda}}{2\pi}A(k_1,\dots,k_n)}$$

the area of minimal surface in AdS_5
with a null polygonal boundary

String theory description of the gluon scattering (Alday-Maldacena '07)

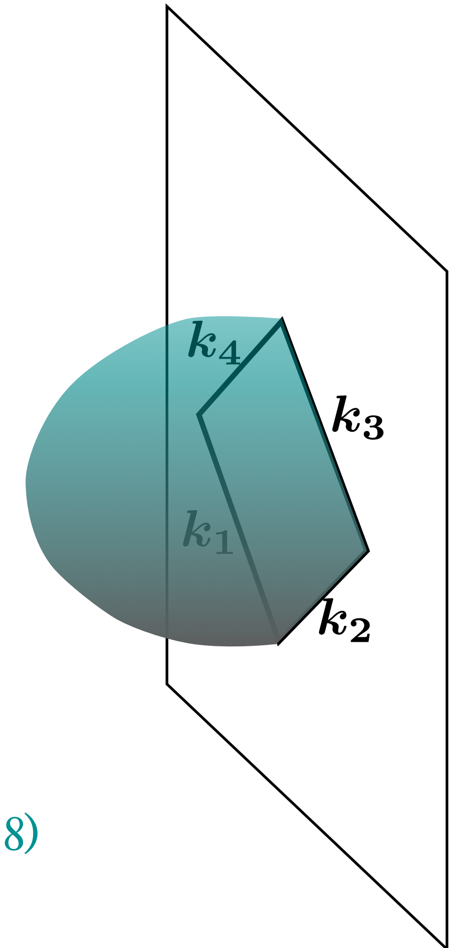
- Scattering of gluons is realized by that of open strings



$\Longleftrightarrow$

T-duality
(fermionic)

(Berkovits-Maldacena '08)
(Beisert-Ricci-Tseytlin-Wolf '08)



AdS$_5$ in the global coordinates

$$\vec{X}\cdot\vec{X} \equiv$$
$$-(X^{-1})^2-(X^0)^2+(X^1)^2+(X^2)^2+(X^3)^2+(X^4)^2=-1$$

Worldsheet coordinates $\quad z=\frac{1}{2}(s+it), \quad \bar{z}=\frac{1}{2}(s-it)$

Equations for classical strings in AdS$_5$ (minimal surfaces)

Equations of motion

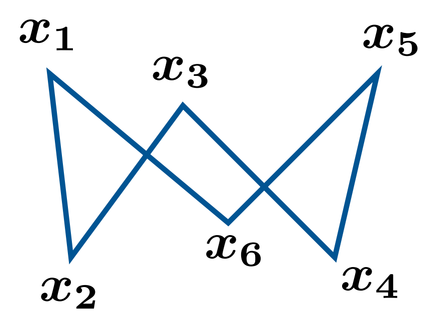
$$\partial\bar{\partial}\vec{X}-(\partial\vec{X}\cdot\bar{\partial}\vec{X})\vec{X}=0$$

Virasoro constraints

$$\partial\vec{X}\cdot\partial\vec{X}=\bar{\partial}\vec{X}\cdot\bar{\partial}\vec{X}=0$$

Poincaré coordinates

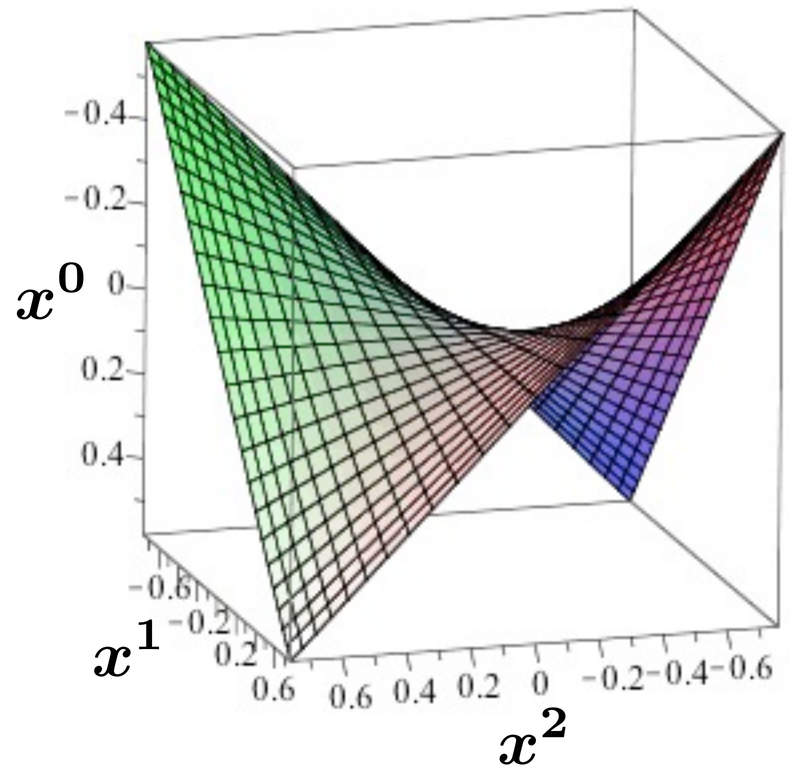
$$X^\mu=\frac{x^\mu}{r}, \quad \mu=0,1,2,3,$$
$$X^{-1}+X^4=\frac{1}{r} \qquad X^{-1}-X^4=\frac{r^2+x^\mu x_\mu}{r}$$



AdS boundary $r=0$ Boundary polygon $x_j^\mu-x_{j+1}^\mu=k_j^\mu$

The simplest 4-cusp solution

$$\begin{pmatrix} X^{-1} \\ X^{0} \\ X^{1} \\ X^{2} \\ X^{3} \\ X^{4} \end{pmatrix} = \begin{pmatrix} \cosh t \cosh s \\ \sinh t \sinh s \\ \cosh t \sinh s \\ \sinh t \cosh s \\ 0 \\ 0 \end{pmatrix}$$



$(x^i \equiv X^i / X^{-1})$

However, general $n(> 4)$-cusp solutions cannot be expressed in terms of any known special functions.

On the other hand, the equations are classically integrable.

EOM and Virasoro conditions of classical strings in AdS_5

$$\partial\bar{\partial}\vec{X} - (\partial\vec{X}\cdot\bar{\partial}\vec{X})\vec{X} = 0 \qquad \partial\vec{X}\cdot\partial\vec{X} = \bar{\partial}\vec{X}\cdot\bar{\partial}\vec{X} = 0$$

$\Longleftrightarrow$

Hitchin equations

$$D_z\Phi_{\bar{z}} = 0, \quad D_{\bar{z}}\Phi_z = 0$$
$$[D_z, D_{\bar{z}}] + [\Phi_z, \Phi_{\bar{z}}] = 0$$

$$D_z = \partial + [A_z, \ \ ], \ D_{\bar{z}} = \bar{\partial} + [A_{\bar{z}}, \ \ ]$$

$\Phi_z, \Phi_{\bar{z}}, A_z, A_{\bar{z}}$:
4 x 4 matrices
expressed
in terms of
$\vec{X}$ and derivatives
$\big(\mathfrak{so}(4,2) \cong \mathfrak{su}(2,2)\big)$

$\Longleftrightarrow$

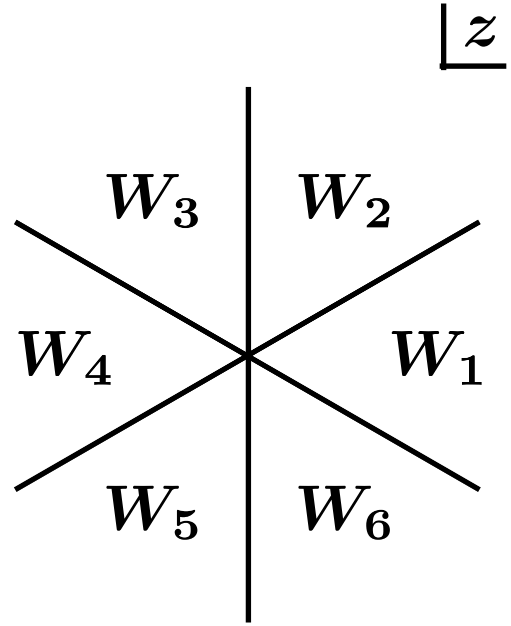
Auxiliary linear problem

$$\nabla_z^{\zeta} q(z,\bar{z};\zeta) = 0, \qquad \nabla_{\bar{z}}^{\zeta} q(z,\bar{z};\zeta) = 0$$

with

$$\nabla_z^{\zeta} = D_z + \zeta^{-1}\Phi_z, \qquad \nabla_{\bar{z}}^{\zeta} = D_{\bar{z}} + \zeta\Phi_{\bar{z}}$$

There are 4 linearly independent solutions to the auxiliary linear problem.



In each angular sector W_k, one can define the small solution $s_k(z, \bar{z}; \zeta)$ which shows the fastest decay for $|z| \to \infty$

Small solutions form an overcomplete basis of the space of solutions to the auxiliary linear problem

$$s_5(z, \bar{z}) = -s_1(z, \bar{z}) + a_2 s_2(z, \bar{z}) + b_3 s_3(z, \bar{z}) + c_4 s_4(z, \bar{z})$$

The coefficients $a_i(\zeta), b_i(\zeta), c_i(\zeta)$ characterize the minimal surface

Functional relations satisfied by the coefficients

$\Rightarrow$ Thermodynamic Bethe ansatz (TBA) equations !

TBA equations (AdS$_3$ case) $(\zeta \equiv e^{\theta})$

$$\epsilon_a(\theta) = m_a R \cosh\theta - \sum_{b=1}^{n/2-3} \int \frac{d\theta'}{2\pi} \frac{i I_{ab}}{\sinh(\theta - \theta' + \sigma_{ab} + \frac{\pi}{2} i)} \log(1 + e^{-\epsilon_b(\theta')})$$

(Alday-Maldacena-Sever-Vieira '10)

(Hatsuda-Ito-K.S.-Satoh '10)

The regularized area is essentially given by the free energy of the TBA system.

(Alday-Gaiotto-Maldacena '09)
(Alday-Maldacena-Sever-Vieira '10)

$$-F = \sum_{a=1}^{n/2-3} \int \frac{d\theta}{2\pi} |m_a| \cosh\theta \log(1 + e^{-\epsilon_a(\theta)})$$

The cross ratios of the gluon momenta are given by the pseudo energies $\epsilon_a(\theta)$ evaluated at special values

Underlying integrable model

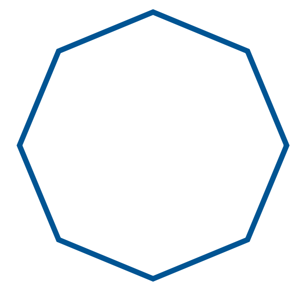
(Hatsuda-Ito-K.S.-Satoh '10)

TBA equations for the AdS_3 case is identified with that of the $SU(N)_2$ homogeneous sine-Gordon model.

(Fernandez-Pousa-Gallas-Hollowood-Miramontes '96, '97)

$$\left(N = \frac{n}{2} - 2\right)$$

CFT limit (high temperature limit)



boundary of the surface $\Longrightarrow$ regular polygon

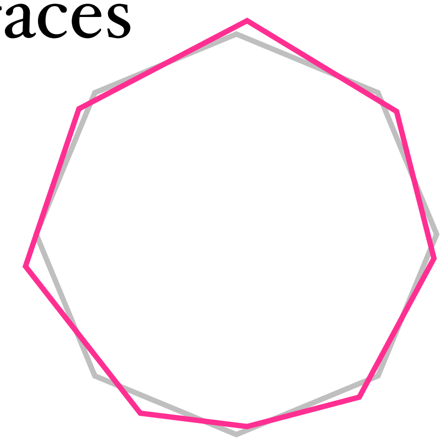
TBA equations $\Longrightarrow$ algebraic equations

free energy $\Longrightarrow$ central charge

- AdS_3 case:

homogeneous sine-Gordon model $\Longrightarrow$ generalized parafermion CFT $\dfrac{SU(N)_2}{U(1)^{N-1}}$

One can also compute the area of minimal surfaces with a general polygonal boundary by using conformal perturbation theory.



(Hatsuda-Ito-K.S.-Satoh '10)

- Conformal perturbation of the free energy

The free energy at temperature $1/L$

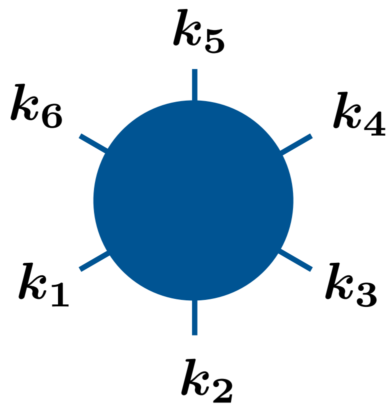
= Ground state energy of the system with circumference L

(Al. Zamolodchikov '90)

$$A_{\rm free} = \frac{\pi}{6}c_n + f_n^{\rm bulk} + (2\pi)^2 \sum_{k=1}^{\infty} \frac{(-\lambda)^k}{k!} \left(\frac{2\pi}{L}\right)^{2(\Delta-1)k}$$

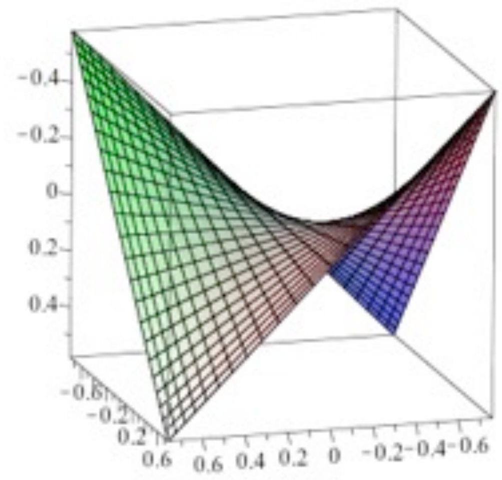
$$\times \int \left\langle \Phi_{\vec{\lambda},\vec{\lambda}^*}(z_1,\bar{z}_1)\cdots\Phi_{\vec{\lambda},\vec{\lambda}^*}(z_k,\bar{z}_k)\right\rangle_{\rm connected} \prod_{i=2}^{k}(z_i\bar{z}_i)^{\Delta-1} dz_2^2 \cdots dz_k^2$$

Outline



gluon scattering amplitudes
in 4d $\mathcal{N} = 4$ SYM

⇨

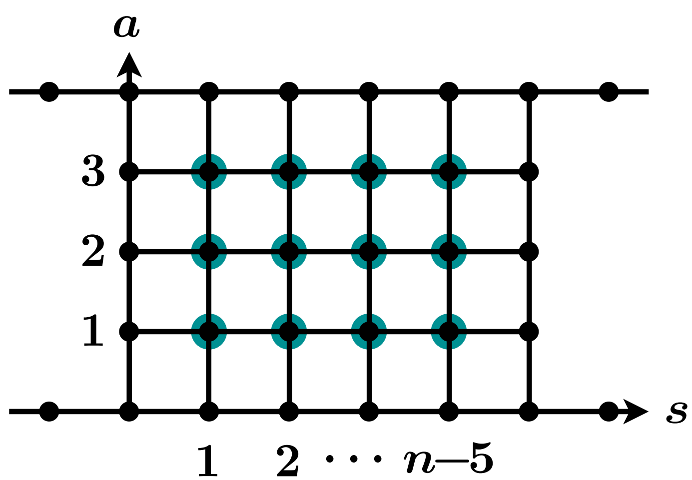


minimal surfaces
in AdS_5

⇩

2d Hitchin
system

⇦



TBA system

⇦

2d CFT

Comparison with two-loop results

- Rescaled remainder function (Brandhuber-Heslop-Khoze-Travaglini '09)

$$\bar{R}_{2n} := \frac{R_{2n} - R_{2n,\mathrm{reg}}}{R_{2n,\mathrm{reg}} - (n-2)R_{6,\mathrm{reg}}}$$

$$\bar{R}_{10} = \bar{C}_{8/5} \left| Y^{(2)}(\tilde{M}_1 e^{i\varphi_1}, \tilde{M}_2 e^{i\varphi_2}) \right|^2 \cdot l^{8/5} + (l^{12/5})$$

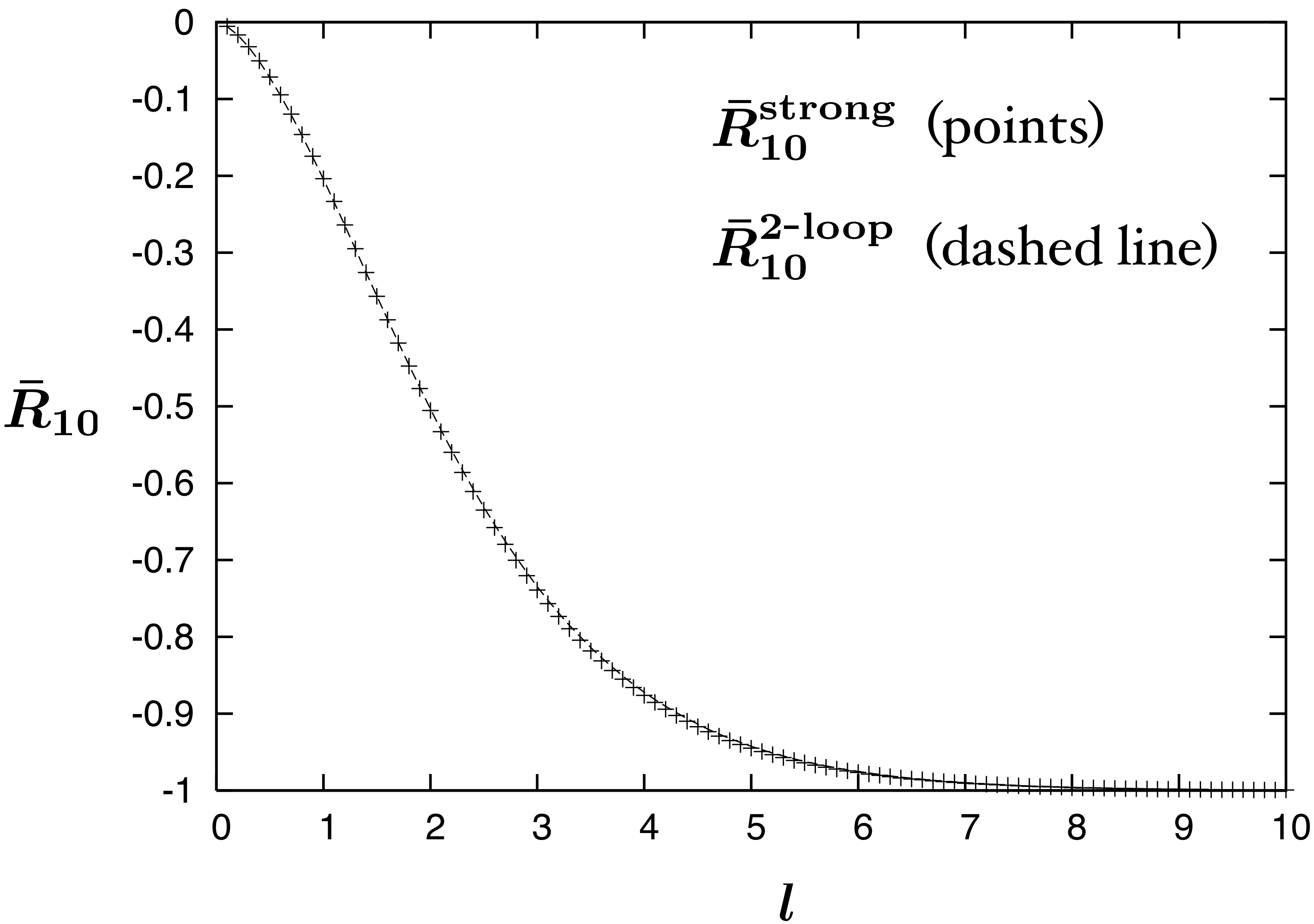
$$\left(Y^{(2)}(\tilde{M}_1, \tilde{M}_2) = \sum_{i,j=1}^{2} C_{ij} \tilde{M}_i^{2/5} \tilde{M}_j^{2/5} \right)$$

$$\bar{C}_{\frac{8}{5}}^{\mathrm{strong}} = \frac{-\frac{1}{5}\tan\frac{\pi}{5} + 20\cos^4(\frac{2\pi}{5}) - 4\sqrt{5}\cos^4(\frac{2\pi}{5})\log(2\cos\frac{\pi}{5})}{\frac{\pi}{5} - \frac{5}{2}\log^2(2\cos\frac{\pi}{5})} \approx -0.0441916$$

$$\bar{C}_{\frac{8}{5}}^{\text{2-loop}} = \frac{-3\sqrt{5} + 2^4\cos^2\frac{\pi}{5}\log(2\cos\frac{\pi}{5})}{2^8\sqrt{5}\cos^6\frac{\pi}{5}\log^2(2\cos\frac{\pi}{5})} \approx -0.0449039$$

Rescaled remainder functions at finite l

$(\tilde{M}_1 = \tilde{M}_2 = 1, \quad \varphi_1 = \varphi_2 = \pi/20)$



Summary

Gluon scattering amplitudes in $\mathcal{N} = 4$ U(N) SYM at strong coupling are described by TBA equations.

We identified the TBA equations in the AdS_3 case with those of the homogeneous sine-Gordon models.

We developed the conformal perturbation theory for the TBA system and obtained analytic perturbative expansions of the amplitudes around the CFT point.

The rescaled remainder functions at strong coupling and at two-loop show very similar structures.

Prospects

- $n(>10)$-point amplitudes

- Higher order corrections

- AdS_4 and AdS_5

  The full Y-system possesses an irregular structure.

- Quantum corrections

  The novel operator product expansion may be useful.

  (Alday-Gaiotto-Maldacena-Sever-Vieira '10)

- Form factors (Maldacena-Zhiboedov '10)