Supersymmetry and Localization

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Message

Localization theorem is a powerful tool to study SUSY path integrals.

A Toy Example

Harish-Chandra-Itzykson-Zuber (HCIZ) Integral:

$$I(A,B) \;=\; \int \mathrm{d}U e^{-\mathrm{Tr}(AUBU^\dagger)}$$

U : unitary, A, B : Hermite. Without loss of generality,

$$A = \mathrm{diag}(a_1, \cdots, a_n),\;\; B = \mathrm{diag}(b_1, \cdots, b_n)$$

Problem:

1. Show that HCIZ integral is supersymmetric.
2. Use localization theorem to show

$$I(A,B) = \mathrm{const} \cdot \frac{\det(e^{-a_i b_j})}{\prod_{i<j}(a_i - a_j)(b_i - b_j)}$$

Integral over a Kahler manifold

$$I(A,B) \;=\; \int \mathrm{d}U e^{-H}, \quad H \equiv \mathrm{Tr}(AUBU^{\dagger})$$

* Restrict the integration domain to the coset, $X = U(n)/U(1)^n$,
where $U(1)^n$ = the group of U which commutes with B.

* X is **Kahler** (= a complex manifold with a closed 2-form J).

- **coordinates:** $(z^1, \cdots, z^N); \quad N \equiv n(n-1)/2.$
- **metric:** $\mathrm{d}s^2 = g_{\mu\bar{\nu}} \mathrm{d}z^{\mu} \mathrm{d}\bar{z}^{\nu}.$
- **Kahler form:** $J = \frac{i}{2} g_{\mu\bar{\nu}} \mathrm{d}z^{\mu} \wedge \mathrm{d}\bar{z}^{\nu}.$

$$\Rightarrow \quad I(A,B) = \int_X \frac{J^N}{N!} e^{-H(z,\bar{z})} = \int_X e^{-H(z,\bar{z})+J}.$$

Emergence of SUSY QFT

$$I(A,B)=\int_X e^{-H(z,\bar{z})+\frac{i}{2}g_{\mu\bar{\nu}}\mathrm{d}z^\mu\mathrm{d}\bar{z}^\nu}$$

* Introduce <u>anti-commuting</u> variables θ^μ, $\bar{\theta}^\mu$ and rewrite further.

0-dimensional "Quantum Field Theory"

$$I(A,B)=\int \mathrm{d}^{2N}z\mathrm{d}^{2N}\theta\cdot e^{-S},\quad S=H(z,\bar{z})-\frac{i}{2}g_{\mu\bar{\nu}}\theta^\mu\bar{\theta}^\nu$$

* The <u>action</u> S is invariant under a **SUSY**

$$Qz^\mu=\theta^\mu,\quad Q\theta^\mu=-2ig^{\mu\bar{\nu}}\frac{\partial H}{\partial\bar{z}^\nu},$$

$$Q\bar{z}^\mu=\bar{\theta}^\mu,\quad Q\bar{\theta}^\mu=+2ig^{\nu\bar{\mu}}\frac{\partial H}{\partial z^\nu}.$$

SUSY

A symmetry between **bosons** and **fermions**

- key idea to go beyond the **Standard Model**
- key feature that allows for **exact computations** of observables

Extended SUSY

SUSY changes the spin of particles by $\pm 1/2$.

$\mathcal{N}$-extended SUSY relates particles of spin-difference $\mathcal{N}/2$ or less.

Gauge theories with extended SUSY have massless scalars.

⇒ Infinitely many degenerate vacua parametrized by their condensates.

Partition function (0-point amplitude)

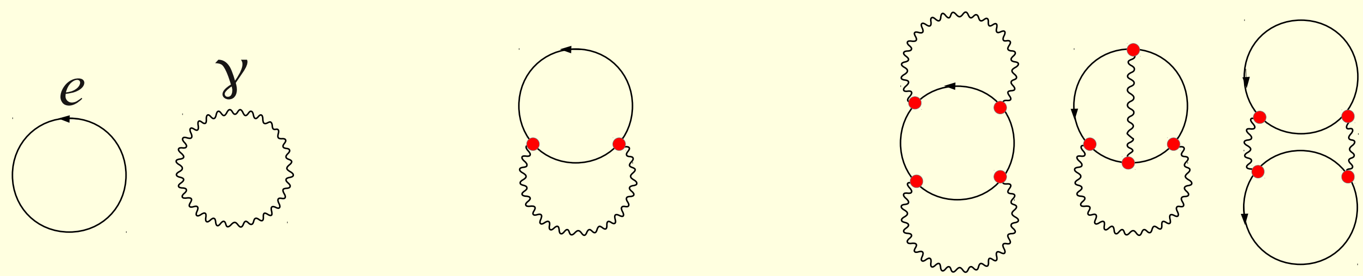
$$Z(\text{coupling}, \text{vacuum}) = \int D(\text{fields}) e^{-\text{Action}}$$

. . . contains lots of information on low-energy dynamics.

Perturbation Theory

$$\ln Z = \text{(sum over bubble diagrams)}$$

[Example: QED]



1-loop $\sim g^0$ 2-loop $\sim g^2$ 3-loop $\sim g^4$

$$= \sum_{\ell} F_\ell \, g^{2\ell-2} + \text{(non-perturbative)}$$

SUSY Localization Theorem

In systems with supersymmetry,
non-zero contributions to path integrals
arise only from **fixed points.**

(= points invariant under SUSY)

SUSY Localization Theorem

In theories with SUSY Q,

the VEVs of Q-exact operators are all zero.

$$\langle Q\mathcal{V}\rangle \equiv \int D(\text{fields})e^{-\text{Action}} \cdot Q\mathcal{V} \;=\; 0.$$

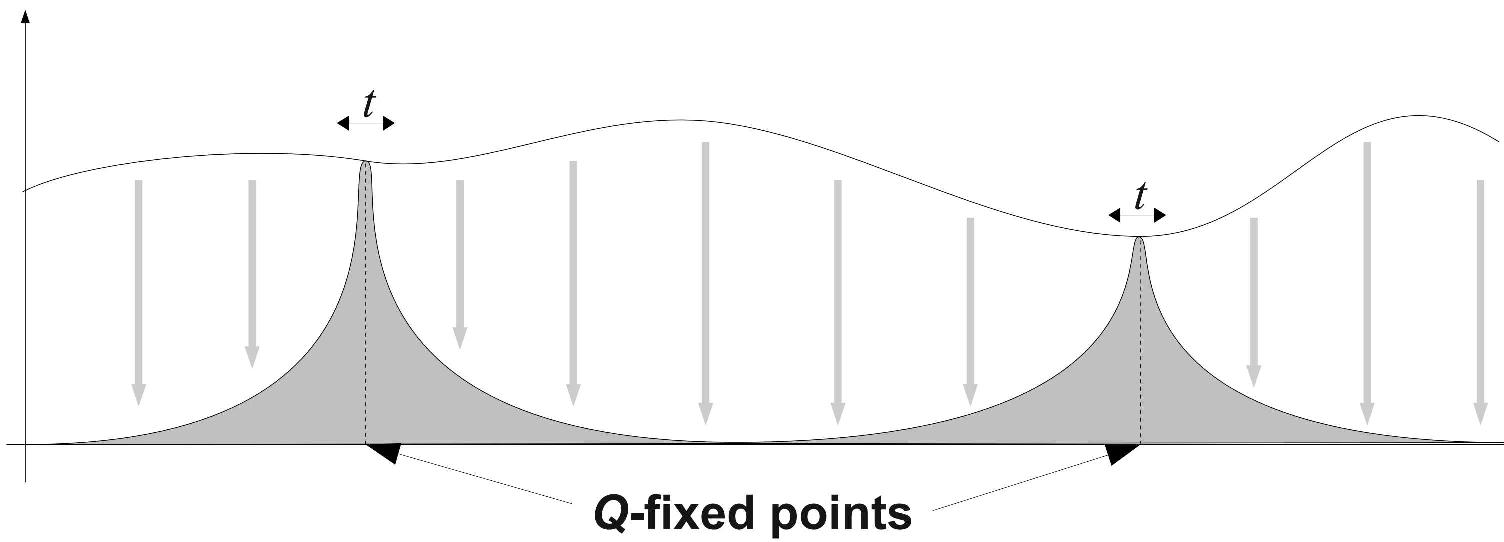
Then one can consider the deformation of path integral,

$$Z[t] = \int D(\text{fields}) \exp\left[-\text{Action} - \frac{1}{t^2}Q\mathcal{V}\right] \quad (Q^2\mathcal{V} = 0)$$

which is actually t-independent.

$$Z = \int D(\text{fields}) \exp\left[-\text{Action} - \frac{1}{t^2} Q\mathcal{V}\right]$$

For suitable V and small t, the integral localizes to Q-fixed points.



$t \to 0$

In the limit $t \to 0$,

the Gaussian approximation at each fixed point becomes exact.

Application to HCIZ integral

$$I(A,B) = \int [\mathrm{d}^{2N} z \mathrm{d}^{2N}\theta] e^{-S}, \quad S = H(z,\bar{z}) - \frac{i}{2} g_{\mu\bar{\nu}} \theta^\mu \bar{\theta}^\nu$$

$$Qz^\mu = \theta^\mu, \quad Q\theta^\mu = -2ig^{\mu\bar{\nu}} \partial_{\bar{z}^\nu} H,$$

$$Q\bar{z}^\mu = \bar{\theta}^\mu, \quad Q\bar{\theta}^\mu = +2ig^{\nu\bar{\mu}} \partial_{z^\nu} H.$$

* Q-fixed points are characterized by $\mathrm{d}H = 0.$

* We consider the deformation, $S \longrightarrow S + \frac{1}{t^2} Q\mathcal{V},$

where we choose

$$\mathcal{V} = i\theta^i \partial_{z^i} H - i\bar{\theta}^{\bar{i}} \partial_{\bar{z}^{\bar{i}}} H,$$

$$Q\mathcal{V} = 4g^{i\bar{j}} \partial_{z^i} H \partial_{\bar{z}^{\bar{j}}} H - 2i\theta^i \bar{\theta}^{\bar{j}} \partial_{z^i} \partial_{\bar{z}^{\bar{j}}} H$$

Fixed points : $\mathrm{d}H = 0$; $H = \mathrm{Tr}(AUBU^\dagger)$

Under $\delta U = ihU$ (h: arbitrary small hermite matrix),

$$\delta H = \mathrm{Tr}\Big[ih(UBU^\dagger A - AUBU^\dagger)\Big] \stackrel{!}{=} 0.$$

The solutions are labeled by a permutation p.

For $A = \mathrm{diag}(a_1, \cdots, a_n)$, $B = \mathrm{diag}(b_1, \cdots, b_n)$,

one finds $UBU^\dagger = \mathrm{diag}(b_{p(1)}, \cdots, b_{p(n)})$

at the fixed point labeled by p.

HCIZ integral localizes to $n!$ points.

* parametrize U near the fixed point $U = \mathrm{id}$ as

$$U = e^{ih}\,;\ \ h = (z_{ij}) : \text{Hermite}$$

* deform the HCIZ integral by

$$Q\mathcal{V} = 4\sum_{i<j}(a_i - a_j)^2(b_i - b_j)^2 z_{ij}\bar{z}_{ij}$$
$$-2i\sum_{i<j}(a_i - a_j)(b_i - b_j)\theta_{ij}\bar{\theta}_{ij} + (\text{cubic or higher})$$

$$I(A,B) \equiv \int [\mathrm{d}z][\mathrm{d}\theta]\exp\left[-S - \frac{1}{t^2}Q\mathcal{V}\right]$$

e^{-H} evaluated at $U = \mathrm{id}.$

$$= \text{const}\cdot\frac{e^{-\sum_i a_i b_i}}{\prod_{i<j}(a_i - a_j)(b_i - b_j)}$$

Gaussian integral

$$+\ (\text{contributions from other fixed points})$$

Summing over all the fixed points, one obtains

$$I(A,B) \equiv \int \mathrm{d}U\, e^{-\mathrm{Tr}(AUBU^\dagger)}$$

$$= \mathrm{const} \cdot \sum_p \frac{e^{-\sum_i a_i b_{p(i)}}}{\prod_{i<j}(a_i - a_j)(b_{p(i)} - b_{p(j)})}$$

$$= \mathrm{const} \cdot \frac{\det(e^{-a_i b_j})}{\prod_{i<j}(a_i - a_j)(b_i - b_j)}$$

Similar localization theorem was applied to 4D N=2 SUSY gauge theories, and partition functions were obtained.

* Omega background of R^4 (Nekrasov '02)
* 4-sphere (Pestun '07)
* 4D ellipsoid (Hama-Hosomichi '12)