The Problem of Time in Quantum Gravity

A Semiclassical Approach

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Questions

Waiting for a theory of Quantum Gravity (Strings, loops, CDT, NCG, etc)

Open issue:
Can GR (or canonical GR) be derived from an underlying « theory of everything » ?

How can the framework of QFT in a given background spacetime be recovered?

Notion of classical spacetime should emerge in an appropriate limit from a theory of QG

(Clue) Schrodinger « looked for a wave equation which yields the Hamilton-Jacobi equation in some appropriate limit »

Optics ⟶ Mechanics

Hamilton – Jacobi equation: $$H\left(q_i, \frac{\partial S}{\partial q^i}\right) = \frac{1}{2m}(\nabla S)^2 + V = E$$

Ansatz: $\Psi \equiv \exp(iS/\hbar)$

- Second derivatives of Ψ with respect to q_i

Neglecting the second derivatives of *S* compared to its first derivatives (optics)

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\Psi = E\Psi$$

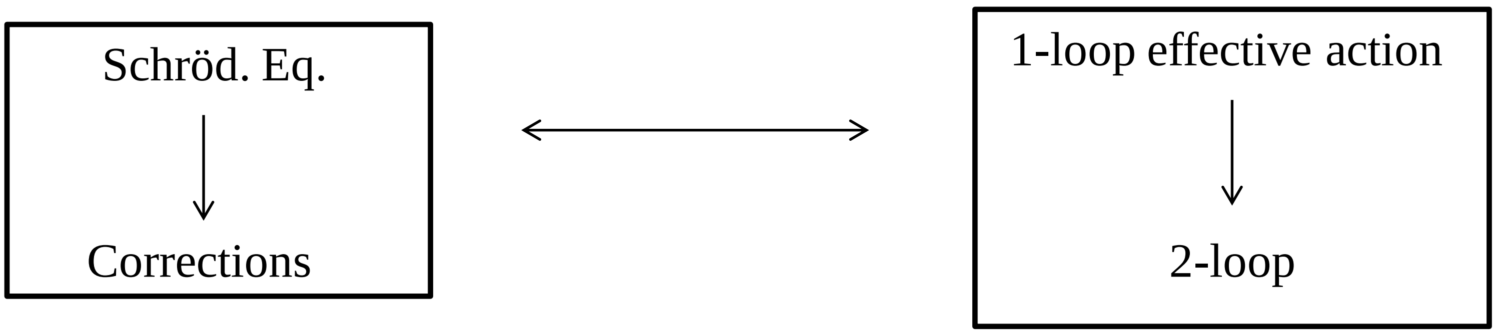
Waves in configuration space

Central equation of canonical QG is a constraint equation: $H\Psi = 0$

Reparametrization invariance (general covariance) of the theory :
« independent from the variables used to describe GR»

Semiclassical considerations may thus well be, at least to a large extent, **independent from the concrete form of the fundamental constraints**.

Schematically



Spacetime recovered with approximations

Background spacetime

Wheeler-DeWitt Equations

$$\begin{cases} H\,\Psi[h_{ab},\phi] = \left(-\dfrac{16\pi G\hbar^2}{c^2} G_{abcd} \dfrac{\delta^2}{\delta h_{ab}\delta h_{cd}} - \boxed{\dfrac{c^2}{16\pi G}\sqrt{h}\,{}^{(3)}R} + H^m \right)\Psi[h_{ab},\phi] = 0 \\ H_a\,\Psi[h_{ab},\phi] = \left(2ih_{ab}D_c \dfrac{\delta}{\delta h_{bc}} + H_a^m \right)\Psi[h_{ab},\phi] = 0 \end{cases}$$

We introduce the parameter $M_p^2 = \dfrac{c^2}{32\pi\hbar^2 G}$

$$\left(-\frac{1}{2M_P^2} G_{abcd} \frac{\delta^2}{\delta h_{ab}\delta h_{cd}} + \boxed{M_P^2 V} + H^m \right)\Psi = 0$$

Expansion with respect to M_P^2 if the mass scales of non-gravitational fields are much smaller than the Planck mass.

- **Without non-gravitational fields**: the M_P^2 expansion is equivalent to an $\hbar$ expansion.

- **With non-gravitational fields**: M_P^2 expansion equivalent to Born-Oppenheimer expansion

Born-Oppenheimer approximation

(e.g. molecular physics)

The full system is divided into two parts with very different scales

1. The « heavy » part: WKB
2. The « light » part: treated fully quantum and follows adiabatically the dynamics of the heavy part.

In QG:

1. The full gravitational field (large value of M)
2. All non-gravitational degrees of freedom

Time as an approximate concept from « timeless » QG

Relevant for observers within the Universe (intrinsic viewpoint)

Simple Quantum Mechanical problem

Total Syst. = « heavy » + « light »

Consider the following Hamiltonian for the full system:

$$H = \frac{P^2}{2M} + \frac{p^2}{2m} + V(Q) + h(q,Q) \approx \frac{P^2}{2M} + V(Q) + h(q,Q) \qquad M >> m$$

Approximate solution to the stationary Schrödinger equation:

$$H\Psi(q,Q) = E\Psi(q,Q)$$

In the Wheeler-DeWitt case: $E = 0$

$$H = -\frac{\hbar^2}{2M}\frac{\partial^2}{\partial Q^2} + V(Q) + h(q,Q)$$

We look for an expansion of the form

$$(a) \quad \Psi(q,Q) = \sum_n \chi_n(Q)\varphi_n(q,Q) \qquad "\Psi_{mol}(q,Q) = \varphi_{elec}(q,Q) \times \chi_{nucl}(Q)"$$

with assumption: $\langle \varphi_n | \varphi_m \rangle = \delta_{nm}$ for each value of Q

Inserting (a) into $H\Psi(q,Q) = E\Psi(q,Q)$, projecting on $\varphi_m(q,Q)$

and introducing:

$$\begin{cases} \varepsilon_{mn}(Q) \equiv \langle \varphi_m | h | \varphi_n \rangle \\ A_{mn}(Q) \equiv i\hbar \left\langle \varphi_m \middle| \frac{\partial \varphi_n}{\partial Q} \right\rangle \\ P_{mn} \equiv -i\hbar\delta_{mn}\frac{\partial}{\partial Q} - A_{mn} \end{cases}$$

One obtain first

$$(b)\quad \sum_n \left(\frac{P_{mn}^2}{2M} + \varepsilon_{mn}(Q) \right) \chi_n(Q) + V(Q)\chi_m(Q) = E\chi_n(Q)$$

Equation for « heavy part » / Spacetime

Without projection, one obtain

$$(c)\quad \sum_n \chi_n(Q) \left[h(q,Q) - \left(E - V(Q) + \frac{\hbar^2}{2M\chi_n} \frac{\partial^2 \chi_n}{\partial Q^2} \right) - \boxed{\frac{\hbar^2}{2M} \frac{\partial^2}{\partial Q^2}} - \boxed{\frac{\hbar^2}{M\chi_n} \frac{\partial \chi_n}{\partial Q} \frac{\partial}{\partial Q}} \right] \varphi_n(q,Q) = 0$$

Equation for « light part » / Non-gravitational d.o.f.

$$i\hbar \frac{\partial}{\partial t_n}$$

Neglected because of the slow variation wrt to Q

Approximations:

1. « heavy part » insensitive to changes in the « light » part - neglect off-diag elements
2. Standard WKB approximation for the « heavy part »

$$\chi_n(Q) = C_n(Q) e^{iMS_n(Q)/\hbar}$$

Taking into account approximations 1. and 2., and assuming M large, one obtain for the Q-derivatives:

$$\frac{\partial^2 \chi_n}{\partial Q^2} \approx -\left(\frac{M}{\hbar}\right)^2 \left(\frac{\partial S_n}{\partial Q}\right)^2 \chi_n \qquad \Rightarrow P_n \chi_n = M \frac{\partial S_n}{\partial Q} \approx -i\hbar \frac{\partial \chi_n}{\partial Q}$$

Equation (b) can then be written: $H_{cl}\chi_n = E\chi_n \quad \text{with} \quad H_{cl} \equiv \frac{P_n^2}{2M} + V(Q) + E_n(Q) = E$

And via the Hamilton equations, one can introduce a « **time coordinate** »:

$$\begin{cases} \frac{d}{dt_n} P_n = -\frac{\partial}{\partial Q} H_{cl} = -\frac{\partial}{\partial Q}\left(V(Q) + E_n(Q)\right) \\ \frac{d}{dt_n} Q = -\frac{\partial}{\partial P_n} H_{cl} = \frac{P_n}{M} \qquad \Rightarrow -\frac{\hbar^2}{M\chi_n}\frac{\partial \chi_n}{\partial Q} = -i\hbar \frac{P_n}{M} = -i\hbar \frac{dQ}{dt_n} \end{cases}$$

Then

$$-\frac{\hbar^2}{M\chi_n}\frac{\partial \chi_n}{\partial Q}\frac{\partial \varphi_n}{\partial Q} \approx -i\hbar \frac{\partial S_n}{\partial Q}\frac{\partial \varphi_n}{\partial Q} \equiv -i\hbar \frac{\partial \varphi_n}{\partial t_n}$$

Equation (c) becomes:

$$\sum_n \chi_n \left[h(q,t_n) - E_n(t_n) - i\hbar \frac{\partial}{\partial t_n} \right] \varphi_n(q,t_n) = 0$$

In QG, for the Wheeler – DeWitt equations

$$\begin{cases} -\frac{1}{2M}\frac{\partial^2}{\partial Q^2} \leftrightarrow & -\frac{1}{2M_P^2} G_{abcd} \frac{\delta^2}{\delta h_{ab}\delta h_{cd}} \\ V(Q) \leftrightarrow & -2M_P^2 \sqrt{h}\, {}^{(3)}R \\ h(q,Q) \leftrightarrow & H_m \\ \Psi(q,Q) \leftrightarrow & \left|\Psi[h_{ab}]\right\rangle \end{cases}$$

The wavefunction is written in the form

$$\Psi(q,Q) = \sum_n \chi_n(Q)\varphi_n(q,Q) \quad \leftrightarrow \quad \left|\Psi[h_{ab}]\right\rangle = C[h_{ab}] e^{iM_P^2 S[h_{ab}]} \left|\varphi[h_{ab}]\right\rangle$$

As for the previous case, we obtain an equation for $\chi[h_{ab}]$ or equivalently, an Hamilton-Jacobi equations for $S[h_{ab}]$

$$\begin{cases} \frac{M_P^2}{2} G_{abcd} \frac{\delta S}{\delta h_{ab}} \frac{\delta S}{\delta h_{cd}} - 2M_P^2 \sqrt{h}\, {}^{(3)}R + \langle \varphi | H^m | \varphi \rangle = 0 \\ -2M_P^2 h_{ab} D_c \frac{\delta S}{\delta h_{bc}} + \langle \varphi | H_a^m | \varphi \rangle = 0 \end{cases}$$

Einstein equations (Peres, 1962)

and equations for $\varphi[h_{ab}]$, similar to equation (c)

$$\begin{cases} \left(H^m - \langle \varphi | H^m | \varphi \rangle - iG_{abcd} \dfrac{\delta S}{\delta h_{ab}} \dfrac{\delta}{\delta h_{cd}} \right) | \varphi[h_{ab}] \rangle = 0 \\ \left(H_a^m - \langle \varphi | H_a{}^m | \varphi \rangle + 2ih_{ab} D_c \dfrac{\delta}{\delta h_{bc}} \right) | \varphi[h_{ab}] \rangle = 0 \end{cases}$$

From the Hamilton-Jacobi equations, one has a relation between the 3-metric, its momentum, the extrinsic curvature of 3-space and the « velocities » $\dot{h}_{ab}$

$$\dot{h}_{ab} = NG_{abcd} \frac{\delta S}{\delta h_{cd}} + 2D_{(a}N_{b)}$$

To get an evolutionary equation for the quantum state, one defines

$$\frac{\partial}{\partial t} | \varphi(t) \rangle = \int d^3x \, \dot{h}_{ab}(x,t) \frac{\delta}{\delta h_{ab}(x)} | \varphi[h_{ab}] \rangle$$

Functional Schrödinger equation for quantized matter fields in the chosen external gravitational field

$$i\frac{\partial}{\partial t}\left|\varphi(t)\right\rangle = \mathrm{H}^{m}\left|\varphi(t)\right\rangle$$

$$\mathrm{H}^{\mathrm{m}} \equiv \int d^{3}x\left\{N(x)H^{m}(x)+N^{a}(x)H_{a}^{m}(x)\right\}$$

The matter-field hamiltonian in the Schrödinger picture parametrically depends on (generally non-static) metric coefficients of the curved spacetime background.

Conclusion and perspectives

The standard concept of time in quantum theory thus emerges only in a semiclassical approximation, the Wheeler-DeWitt equation itself is « timeless ».

This approach should be independent of the expression of the Wheeler-DeWitt constraints.

- Can, in principle, be applied using the Ashtekar variables

- Tensor Model approach to QG: the constraints satisfies a Wheeler-DeWitt algebra

Can we apply this semiclassical approach to the Tensor Model?

Einstein Equations for Spacetime

Schrödinger Equation for Quantized Matter in a gravitational background