Exact solutions of relativistic conformal hydrodynamics

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arXiv: 1403.7693

arXiv: 1405.1984

Analytic approach to relativistic hydrodynamics

Hydrodynamics—a universal tool for long-distance phenomena

A set of nonlinear differential equations of many variables, need heavy numerics to solve.

With sufficient symmetry, one can find analytic solutions of ideal hydrodynamics (e.g., Hubble flow, Bjorken flow,…)

Difficult to analytically solve Navier-Stokes (first-order) equation (cf. Millennium prize problem)

Second-order hydro…hopeless? Conformal symmetry helps a lot.

Energy momentum tensor

projection operator

$\Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$

flow velocity $u^\mu u_\mu = -1$

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + (p + \Pi)\Delta^{\mu\nu} + \pi^{\mu\nu}$$

energy density

pressure

~~bulk pressure~~

shear-stress tensor
(vanishes in ideal hydro)

Hydrodynamic equation $\nabla_\mu T^{\mu\nu} = 0$

Shear-stress tensor in second-order hydro

Denicol, Niemi, Molnar, Rischke (2012)

Navier-Stokes

Israel-Stewart

$$
\begin{aligned}
\pi^{\mu\nu} \;=\; & -2\eta\sigma^{\mu\nu} - \tau_\pi \Delta^\mu_\alpha \Delta^\nu_\beta D\pi^{\alpha\beta} - \delta_{\pi\pi}\pi^{\mu\nu}\vartheta + \lambda_2 \pi^{\langle\mu}{}_\lambda \Omega^{\nu\rangle\lambda} \\
& + \lambda_1 \pi^{\langle\mu}{}_\lambda \pi^{\nu\rangle\lambda} + \lambda_3 \Omega^{\langle\mu}{}_\lambda \Omega^{\nu\rangle\lambda} + \lambda_{\pi\Pi}\Pi\sigma^{\mu\nu} + \varphi_6 \Pi\pi^{\mu\nu} - \tau_{\pi\pi}\sigma^{\langle\mu}{}_\lambda \pi^{\nu\rangle\lambda} \\
& + \eta_2 \sigma^{\mu\nu}\vartheta + \eta_3 \sigma^{\langle\mu}{}_\lambda \sigma^{\nu\rangle\lambda} + \eta_4 \sigma^{\langle\mu}{}_\lambda \Omega^{\nu\rangle\lambda} + \eta_6 \nabla^{\langle\mu} p \nabla^{\nu\rangle} p + \eta_9 \nabla^{\langle\mu}\nabla^{\nu\rangle} p
\end{aligned}
$$

shear tensor

$$\sigma^{\mu\nu} \equiv \nabla^{\langle\mu} u^{\nu\rangle} \equiv \left(\frac{1}{2}(\Delta^{\mu\alpha}\Delta^{\nu\beta} + \Delta^{\mu\beta}\Delta^{\nu\alpha}) - \frac{1}{3}\Delta^{\mu\nu}\Delta^{\alpha\beta} \right) \nabla_\alpha u_\beta$$

vorticity tensor

$$\Omega^{\mu\nu} \equiv \frac{1}{2}\Delta^{\mu\alpha}\Delta^{\nu\beta}(\nabla_\alpha u_\beta - \nabla_\beta u_\alpha)$$

Conformal hydrodynamics

- The equation of state is simple $\epsilon = 3p$
- Some transport coefficients are related

Baier,Romatschke,Son, Starinets, Stephanov (2008)
Bhattacharyya, Hubeny, Minwalla, Rangamani (2008)

- Utilize the Weyl transformation.
  A complicated flow in Minkowski space may look simpler in a different coordinate system.

$$d\hat{s}^2 = \hat{g}_{\mu\nu} d\hat{x}^{\mu} d\hat{x}^{\nu} = \frac{ds^2}{\Lambda^2}$$

Solutions of ideal hydrodynamics

Bjorken flow

QGP created in heavy-ion collisions is approximately boost-invariant.

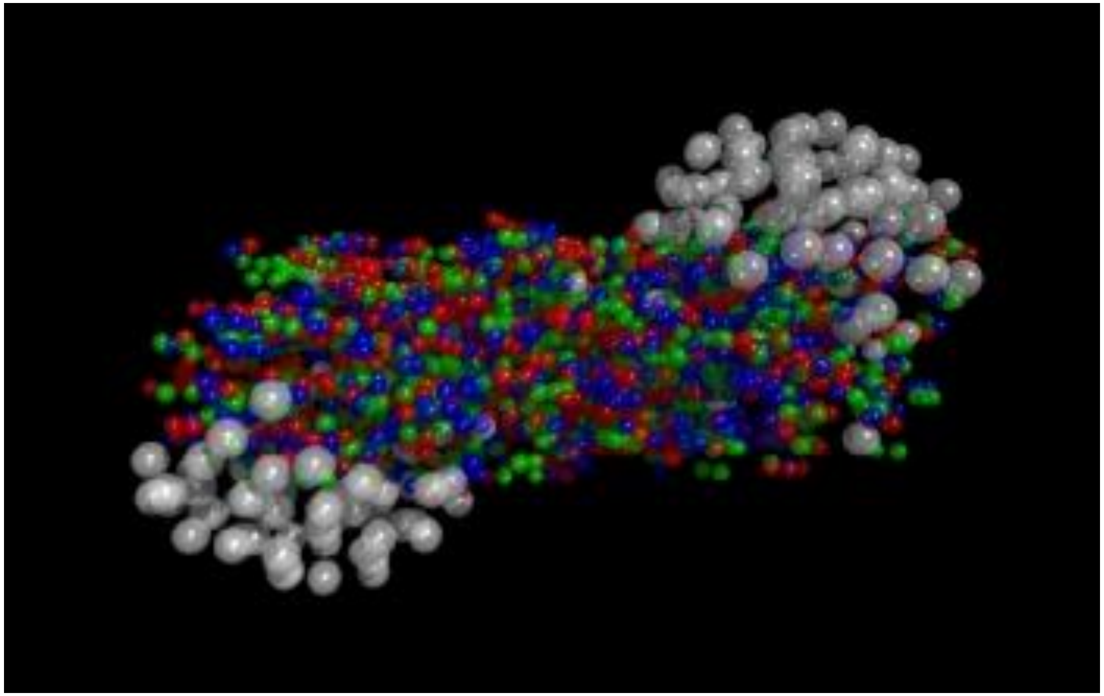
$$ds^2 = -d\tau^2 + \tau^2 d\eta^2 + dx^2 + dy^2$$

proper time $\tau \equiv \sqrt{t^2 - z^2}$ rapidity $\eta \equiv \tanh^{-1} \frac{z}{t}$

Comoving solution

$$u^\mu = \left(\frac{t}{\tau}, 0, 0, \frac{z}{\tau}\right)$$

$$\epsilon \propto \tau^{-4/3}$$



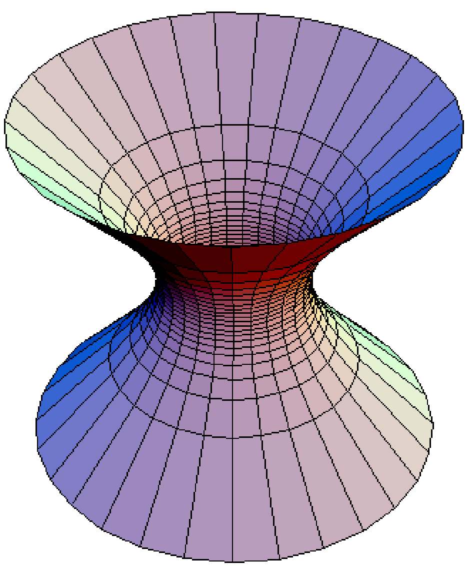
Gubser flow

Gubser (2010)

Minkowski space conformally equivalent to $dS_3 \times \mathbb{R}$

$$d\hat{s}^2 \equiv \frac{ds^2}{\tau^2} = \underbrace{\frac{-d\tau^2 + dx_\perp^2 + x_\perp^2 d\phi^2}{\tau^2}}_{\text{3D de Sitter space}} + d\eta^2$$

3D de Sitter space



$$v_\perp \equiv -\frac{u_\perp}{u_\tau} = \frac{2\tau x_\perp}{L^2 + \tau^2 + x_\perp^2}$$

$$\epsilon \propto \frac{1}{\tau^{4/3}} \frac{1}{(L^4 + 2(\tau^2 + x_\perp^2)L^2 + (\tau^2 - x_\perp^2)^2)^{4/3}}$$

Boost-invariant, cylindrically symmetric solution

Hubble flow in Milne universe

$$\begin{aligned} ds^2 &= -dt^2 + dr^2 + r^2 d\Omega^2 \\ &= -d\tau_r^2 + \tau_r^2 d\eta_r^2 + \tau_r^2 \sinh^2 \eta_r d\Omega^2 \\ &= -d\tau_r^2 + \tau_r^2 \left(\frac{dR^2}{1+R^2} + R^2 d\Omega^2 \right) \end{aligned}$$

Milne universe

3D proper time and rapidity

$$\tau_r = \sqrt{t^2 - r^2} \qquad \eta_r \equiv \tanh^{-1} \frac{r}{t}$$

$$u^\mu = \left(\frac{t}{\tau_r}, \frac{\vec{r}}{\tau_r} \right) \qquad \epsilon \propto \frac{1}{\tau_r^4}$$

Conformal soliton flow

Friess, Gubser, Michalogiorgakis, Pufu (2007)

$$t = \frac{L \sin \xi}{\cos \xi + \cos \sigma}, \qquad \vec{r} = \frac{L \sin \sigma}{\cos \xi + \cos \sigma} \vec{n}$$

$$ds^2 = \frac{L^2}{(\cos \xi + \cos \sigma)^2} \underbrace{\left(-d\xi^2 + d\sigma^2 + \sin^2 \sigma d\Omega^2\right)}_{\text{Einstein static universe}}$$

$$u_t = -\frac{L^2 + r^2 + t^2}{\sqrt{(L^2 + (r+t)^2)(L^2 + (r-t)^2)}},$$

$$\vec{u} = \frac{2t\vec{r}}{\sqrt{(L^2 + (r+t)^2)(L^2 + (r-t)^2)}}$$

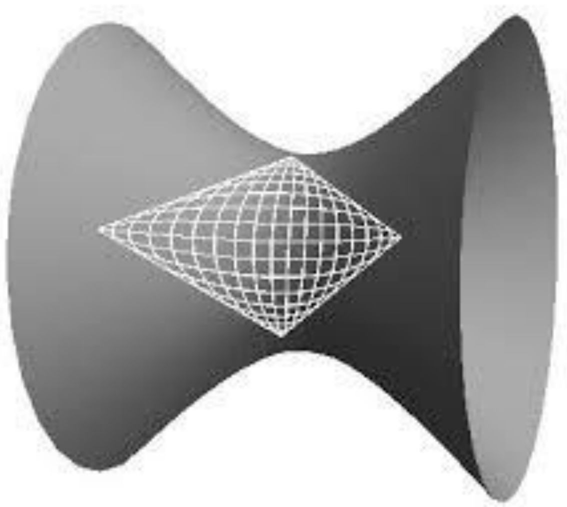
$$\epsilon \propto \frac{1}{(L^2 + (r+t)^2)^2 (L^2 + (r-t)^2)^2}$$

Alternative look

YH, Noronha, Xiao (2014)

$$d\hat{s}^2 \equiv \frac{ds^2}{x_\perp^2} = \frac{-dt^2 + dz^2 + dx_\perp^2}{x_\perp^2} + d\phi^2$$

$$= \underbrace{-\cosh^2\rho\, dT^2 + d\rho^2 + \sinh^2\rho\, d\bar{\Theta}^2}_{AdS_3} + \underbrace{d\phi^2}_{S^1}$$



3D anti-de Sitter space

$$X_0^2 - X_1^2 - X_2^2 + X_3^2 = L^2$$

Hydrostatic fluid in AdS → conformal soliton flow

Solutions of second-order hydro

Most general 2nd order equation with conformal symmetry

$$
\begin{aligned}
\pi^{\mu\nu} \; = \; & -2\eta\sigma^{\mu\nu} - \tau_\pi \left(\Delta^\mu_\alpha \Delta^\nu_\beta D\pi^{\alpha\beta} + \frac{4}{3}\vartheta\pi^{\mu\nu} \right) \\
& + \lambda_1 \pi^{\langle\mu}{}_\lambda \pi^{\nu\rangle\lambda} + \lambda_2 \pi^{\langle\mu}{}_\lambda \Omega^{\nu\rangle\lambda} + \lambda_3 \Omega^{\langle\mu}{}_\lambda \Omega^{\nu\rangle\lambda} \\
& - \tau_\sigma \left(\Delta^\mu_\alpha \Delta^\nu_\beta D\sigma^{\alpha\beta} + \frac{1}{3}\vartheta\sigma^{\mu\nu} \right) + \eta_3 \sigma^{\langle\mu}{}_\lambda \sigma^{\nu\rangle\lambda} + \eta_4 \sigma^{\langle\mu}{}_\lambda \Omega^{\nu\rangle\lambda} - \tau_{\pi\pi} \sigma^{\langle\mu}{}_\lambda \pi^{\nu\rangle\lambda}
\end{aligned}
$$

First consider spherical solutions

$$\sigma^{\mu\nu} = 0$$

$$\hat{\pi}^{\mu\nu} = -\frac{\tau_\pi}{\hat{\epsilon}^{1/4}} \hat{\Delta}^\mu_\alpha \hat{\Delta}^\nu_\beta \hat{D} \hat{\pi}^{\alpha\beta} + \frac{\lambda_1}{\hat{\epsilon}} \hat{\pi}^{\langle\mu}{}_\lambda \hat{\pi}^{\nu\rangle\lambda} + \frac{\lambda_2}{\hat{\epsilon}^{1/4}} \hat{\pi}^{\langle\mu}{}_\lambda \hat{}^{\,\nu\rangle\lambda} + \lambda_3 \hat{\epsilon}^{1/2}\, \hat{}^{\,\langle\mu}{}_\lambda \hat{}^{\,\nu\rangle\lambda}$$

2nd-order Hubble flow

$$\partial_{\tau_r}\epsilon + \frac{4}{\tau_r}\epsilon = 0\,,$$

$$4\epsilon\nabla_{\tau_r}u^{\mu} + \Delta^{\mu\alpha}\nabla_{\alpha}\epsilon + 3\Delta^{\mu}{}_{\nu}\nabla_{\alpha}\pi^{\alpha\nu} = 0\,,$$

$$\pi^{\mu\nu} = -\frac{\tau_\pi}{\epsilon^{1/4}}\left(\Delta^{\mu}_{\alpha}\Delta^{\nu}_{\beta}\nabla_{\tau_r}\pi^{\alpha\beta} + \frac{4}{\tau_r}\pi^{\mu\nu}\right) + \frac{\lambda_1}{\epsilon}\pi^{\langle\mu}{}_{\lambda}\pi^{\nu\rangle\lambda}$$

$$\epsilon = \frac{1}{\tau_r^4}\begin{cases}(\sinh\eta_r\sin\theta)^{\frac{9}{\lambda_1-3}}\,, \\ (\sinh\eta_r)^{\frac{9}{\lambda_1-3}}(\sin\theta)^{-\frac{9}{\lambda_1+6}}\,, \\ (\sinh\eta_r)^{-\frac{18}{\lambda_1+6}}\,,\end{cases}$$

$$\sinh\eta_r = \frac{r}{\tau_r}$$

spherical symmetry broken

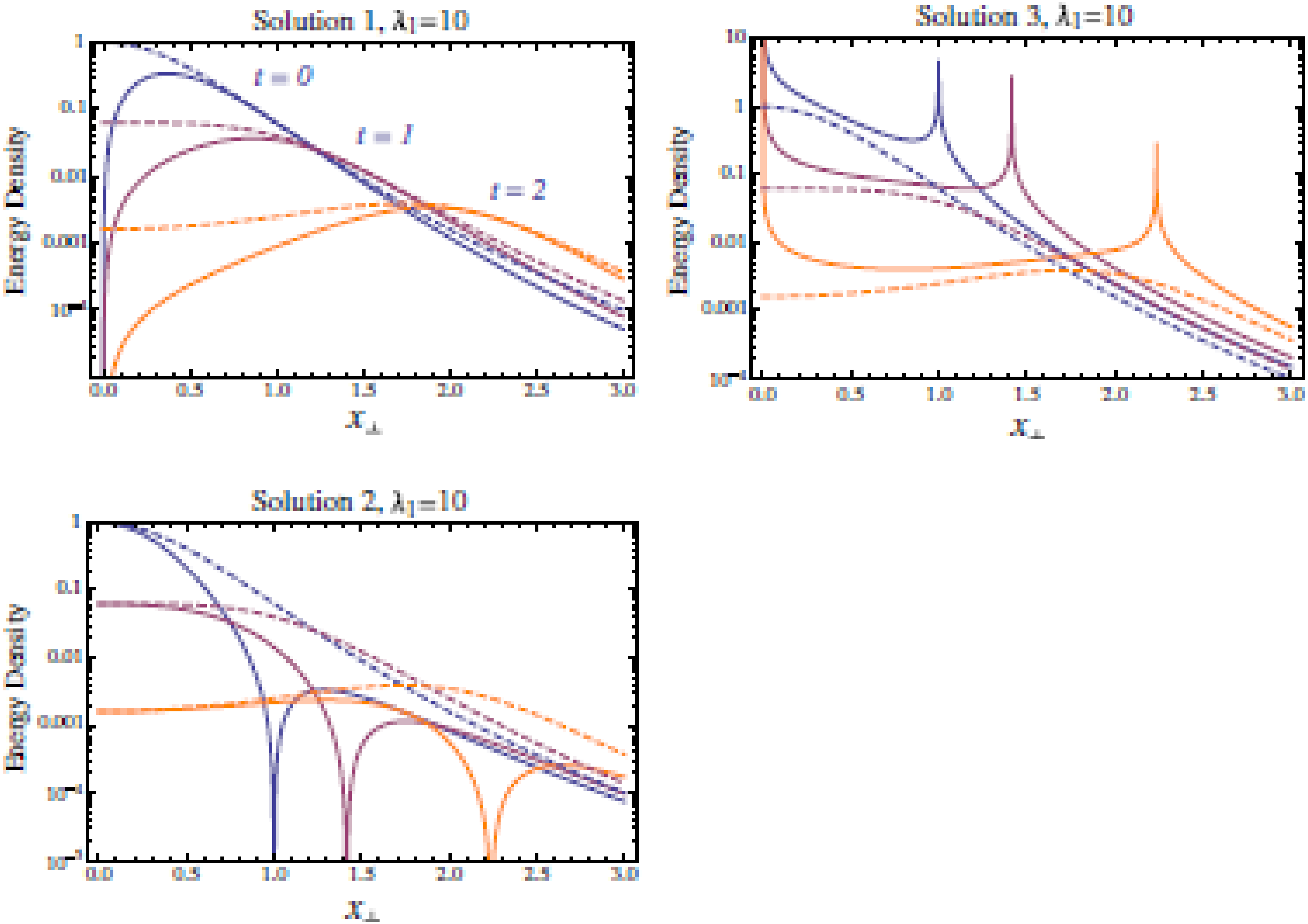
2^{nd}-order conformal soliton flow

Obtained via Einstein static universe

$$\epsilon \propto \begin{cases} \frac{1}{(L^2+(r+t)^2)^2(L^2+(r-t)^2)^2}\left(\frac{L^2 x_\perp^2}{(L^2+(r+t)^2)(L^2+(r-t)^2)}\right)^{\frac{9}{2(\lambda_1-3)}}, \\ \frac{1}{(L^2+(r+t)^2)^2(L^2+(r-t)^2)^2}\left(\frac{L^2 r^2}{(L^2+(r+t)^2)(L^2+(r-t)^2)}\right)^{\frac{9}{2(\lambda_1-3)}}\left(\frac{r^2}{x_\perp^2}\right)^{\frac{9}{2(\lambda_1+6)}} \\ \frac{1}{(L^2+(r+t)^2)^2(L^2+(r-t)^2)^2}\left(\frac{L^2 r^2}{(L^2+(r+t)^2)(L^2+(r-t)^2)}\right)^{-\frac{9}{\lambda_1+6}} \end{cases}$$

Obtained via $AdS_3 \times S^1$

$$\epsilon \propto \begin{cases} \frac{1}{(L^2+(t+r)^2)^2(L^2+(t-r)^2)^2}\left(\frac{4L^2 x_\perp^2}{(L^2+(t+r)^2)(L^2+(t-r)^2)}\right)^{\frac{9}{2(\lambda_1-3)}}, \\ \frac{1}{(L^2+(t+r)^2)^2(L^2+(t-r)^2)^2}\left(1-\frac{4L^2 x_\perp^2}{(L^2+(t+r)^2)(L^2+(t-r)^2)}\right)^{\frac{9}{2(\lambda_1-3)}}, \\ \frac{1}{(L^2+(t+r)^2)^2(L^2+(t-r)^2)^2}\left(\frac{4L^2 x_\perp^2\left((L^2+(t+r)^2)(L^2+(t-r)^2)-4L^2 x_\perp^2\right)}{(L^2+(t+r)^2)^2(L^2+(t-r)^2)^2}\right)^{-\frac{9}{2(\lambda_1+6)}} \end{cases}$$



Solution 1, λ1=10
t = 0
t = 1
t = 2
Energy Density
x⊥
1
0.1
0.01
0.001
0.0
0.5
1.0
1.5
2.0
2.5
3.0
Solution 3, λ1=10
10
Solution 2, λ1=10

2^{nd}-order Bjorken flow

Not spherical….more complicated! $\sigma^{\mu\nu} \neq 0$

The most general second-order equation

$$\partial_\tau \epsilon + \frac{4}{3\tau}\epsilon + \frac{1}{3\tau}(2\pi^\eta{}_\eta - \pi^x{}_x - \pi^y{}_y) = 0$$

where

$$\begin{aligned}\pi^{\mu\nu} &= -2\eta\epsilon^{3/4}\sigma^{\mu\nu} - \frac{\tau_\pi}{\epsilon^{1/4}}\left(\Delta^\mu_\alpha\Delta^\nu_\beta D\pi^{\alpha\beta} + \frac{4}{3}\vartheta\pi^{\mu\nu}\right) \\ &\quad + \frac{\lambda_1}{\epsilon}\pi^{\langle\mu}{}_\lambda\pi^{\nu\rangle\lambda} + \tau_\sigma\epsilon^{1/2}\left(\Delta^\mu_\alpha\Delta^\nu_\beta D\sigma^{\alpha\beta} + \frac{1}{3}\sigma^{\mu\nu}\vartheta\right) \\ &\quad - \tilde{\eta}_3\epsilon^{1/2}\sigma^{\langle\mu}{}_\lambda\sigma^{\nu\rangle\lambda} - \frac{\tau_{\pi\pi}}{\epsilon^{1/4}}\sigma^{\langle\mu}{}_\lambda\pi^{\nu\rangle\lambda}\end{aligned}$$

Exact solutions

$$\epsilon = \frac{C}{\tau^4} \qquad \pi^{\eta}{}_{\eta} = \frac{8}{3}\epsilon$$

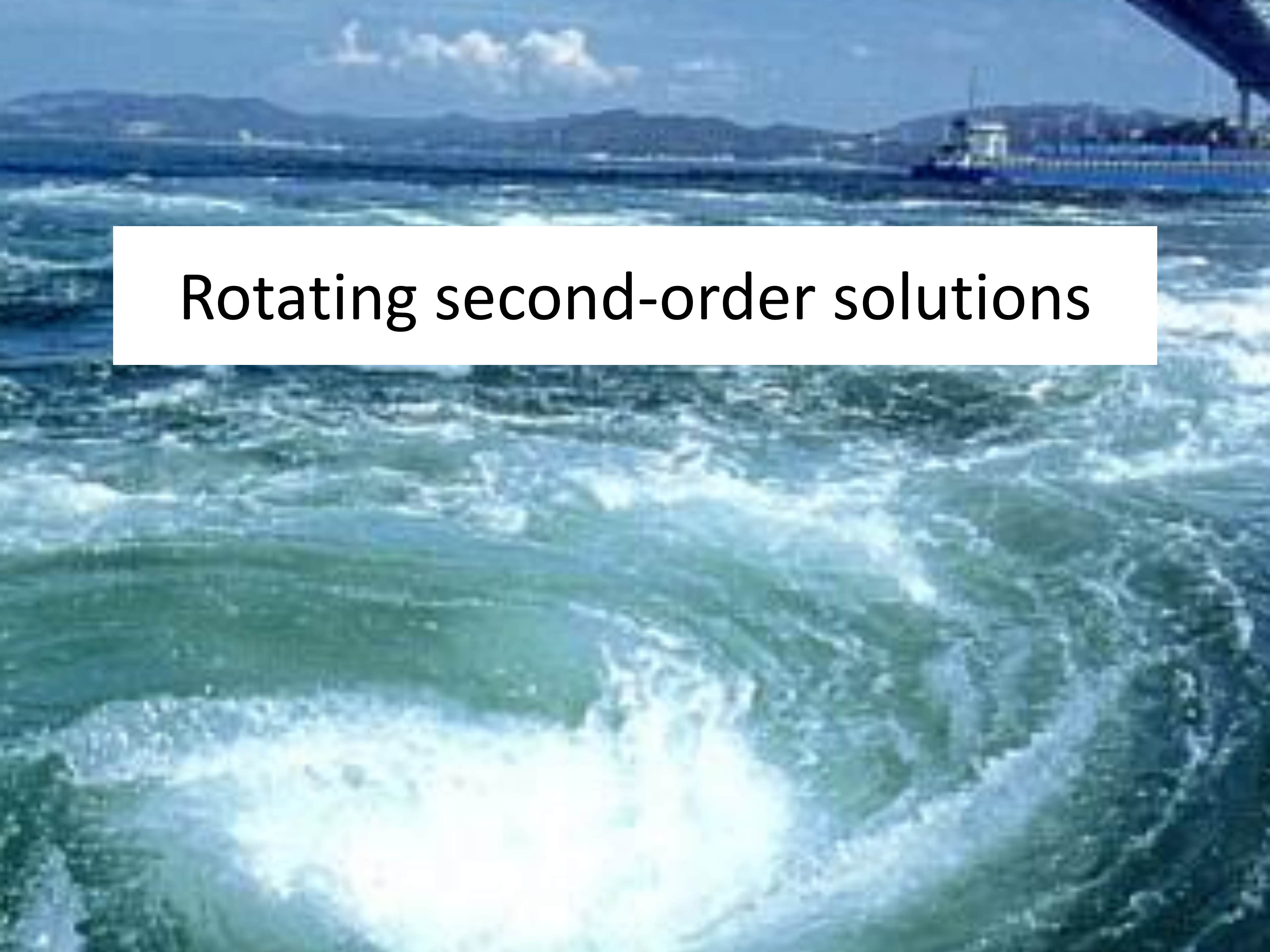
$$\begin{cases} \pi^{x}{}_{x} = \pi^{y}{}_{y} = -\frac{4}{3}\epsilon \\ C^{1/4} = \frac{3\eta - 16\tau_\pi + 2\tau_{\pi\pi} \pm \sqrt{(3\eta - 16\tau_\pi + 2\tau_{\pi\pi})^2 + 4(4\lambda_1 - 3)(2\tau_\sigma + \tilde{\eta}_3)}}{4(4\lambda_1 - 3)} \end{cases}$$

$$\begin{cases} \left(\frac{\pi^{x}{}_{x}}{\epsilon}, \frac{\pi^{y}{}_{y}}{\epsilon}\right) = -\frac{4}{3} \pm \sqrt{16 - \frac{1}{\lambda_1}\left(\frac{2(3\eta + 4\tau_{\pi\pi})}{3C^{1/4}} + \frac{2\tau_\sigma + \eta_3}{3C^{1/2}}\right)} \\ C^{1/4} = \frac{8\tau_\pi + \tau_{\pi\pi}}{8\lambda_1 + 3} \end{cases}$$

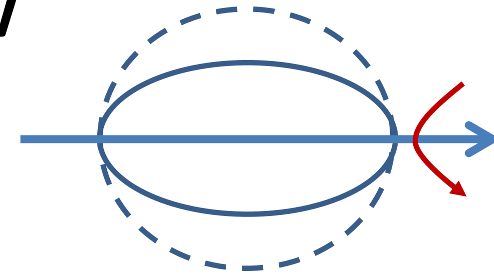
Explicitly depends on six transport coefficients!

$\pi^{\mu\nu} \approx -2\eta\sigma^{\mu\nu}$ does not hold.

Rotating second-order solutions



Rotating Hubble flow



$$\hat{\pi}^{\mu\nu} = -\frac{\tau_\pi}{\hat{\epsilon}^{1/4}}\hat{\Delta}^{\mu}_{\alpha}\hat{\Delta}^{\nu}_{\beta}\hat{D}\hat{\pi}^{\alpha\beta} + \frac{\lambda_1}{\hat{\epsilon}}\hat{\pi}^{\langle\mu}{}_{\lambda}\hat{\pi}^{\nu\rangle\lambda} + \frac{\lambda_2}{\hat{\epsilon}^{1/4}}\hat{\pi}^{\langle\mu}{}_{\lambda}\hat{\ }^{\nu\rangle\lambda} + \underline{\lambda_3\hat{\epsilon}^{1/2}\hat{\ }^{\langle\mu}{}_{\lambda}\hat{\ }^{\nu\rangle\lambda}}$$

ideal rotating solution

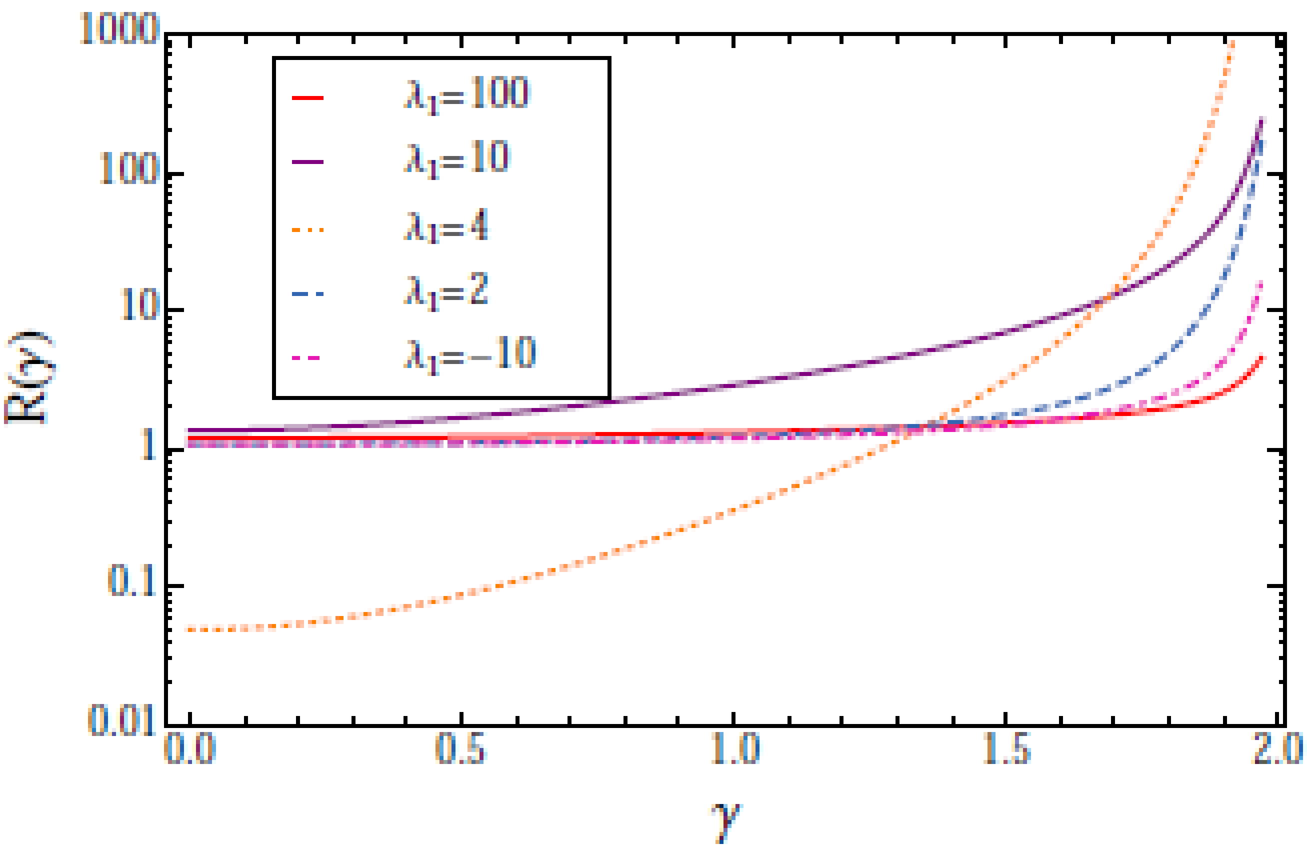
$$\hat{u}_\chi = -\frac{1}{\sqrt{1-\omega^2\gamma^2}}, \qquad \hat{u}_\phi = \frac{\omega\gamma^2}{\sqrt{1-\omega^2\gamma^2}} \qquad \gamma = \frac{x_\perp}{\sqrt{t^2-r^2}}$$

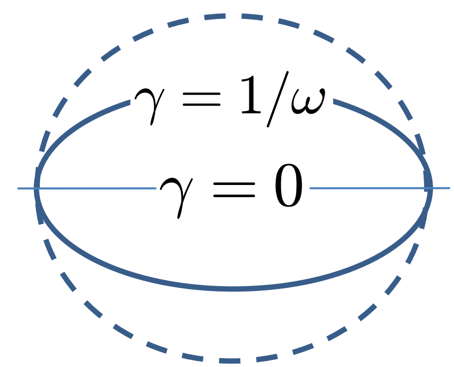
new term from vorticity

$$\hat{\ }^{\langle\mu}{}_{\lambda}\hat{\ }^{\nu\rangle\lambda} = \frac{(\hat{\ }_{\gamma\phi})^2}{3}\begin{pmatrix} \frac{\omega^2\gamma^4}{1+\gamma^2} & 0 & 0 & \frac{\omega\gamma^2}{1+\gamma^2} \\ 0 & \gamma^2(1-\omega^2\gamma^2) & 0 & 0 \\ 0 & 0 & -\frac{2\gamma^2(1-\omega^2\gamma^2)}{(1+\gamma^2)^2} & 0 \\ \frac{\omega\gamma^2}{1+\gamma^2} & 0 & 0 & \frac{1}{1+\gamma^2} \end{pmatrix}$$

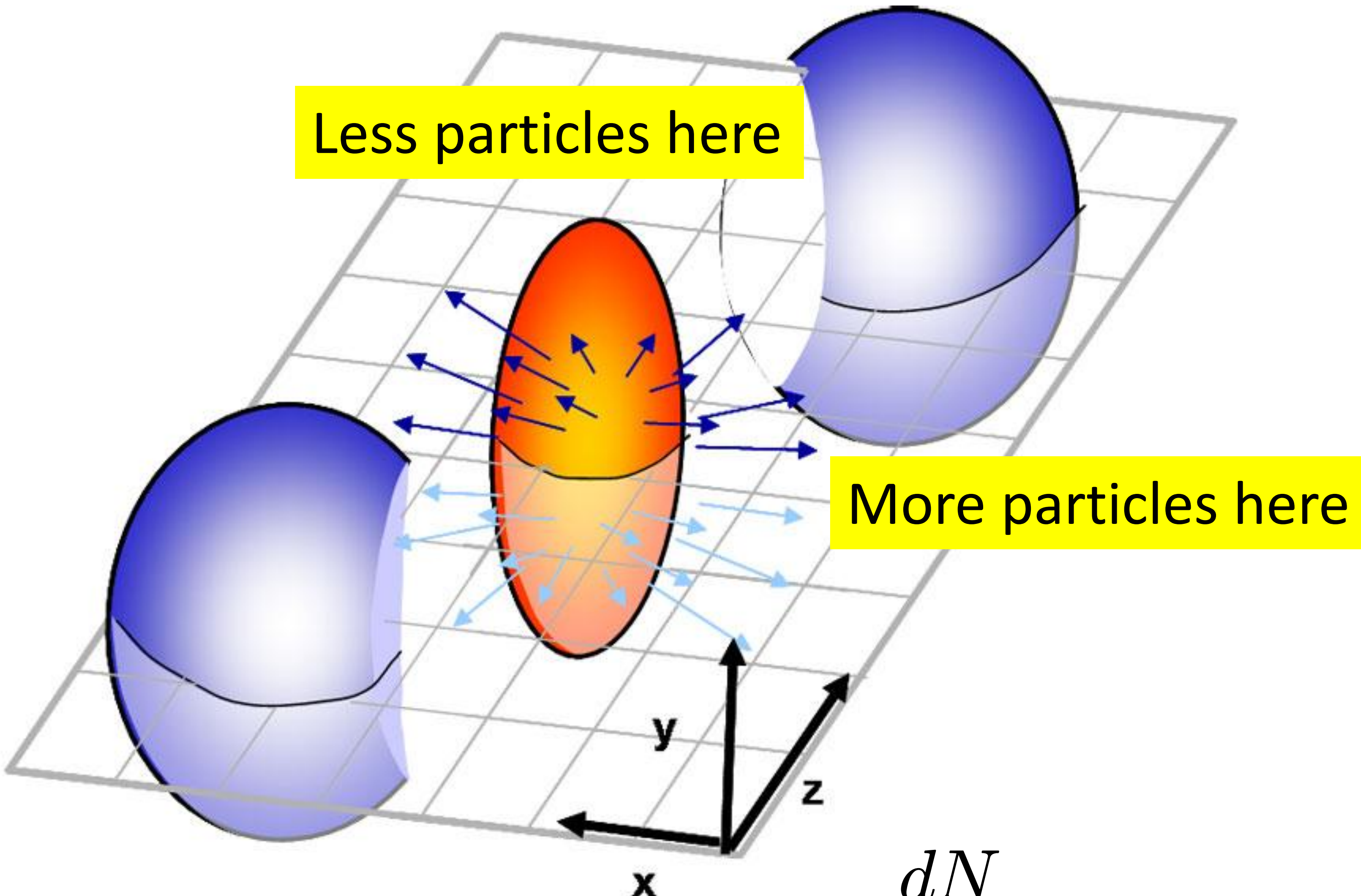
$$R(\gamma) = \frac{\epsilon_{2nd}(\gamma)}{\epsilon_{ideal}(\gamma)} = \left|1 + \frac{21}{4} b(\gamma)\right|^{2e_1(\lambda_1)} |1 + \lambda_1 b(\gamma)|^{2e_2(\lambda_1)}$$

$$e_1(\lambda_1) \equiv \frac{105 - 32\lambda_1}{7(4\lambda_1 - 21)}, \quad e_2(\lambda_1) \equiv \frac{9}{4\lambda_1 - 21}$$



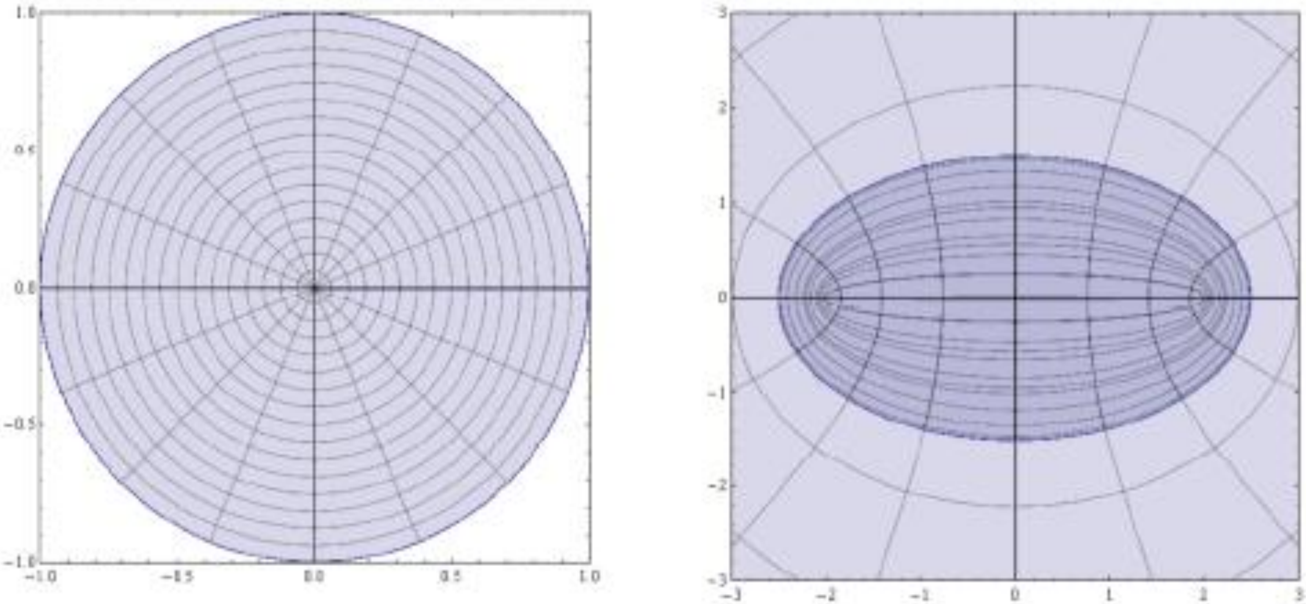


Elliptic flow in heavy-ion collisions



$$\frac{dN}{d\phi} \propto 1 + 2\underline{v_2}\cos 2\phi$$

Zhukovsky transform



A conformal map in 2D between a circle and an ellipse.

$$\hat{z} = z + \frac{a^2}{z}$$

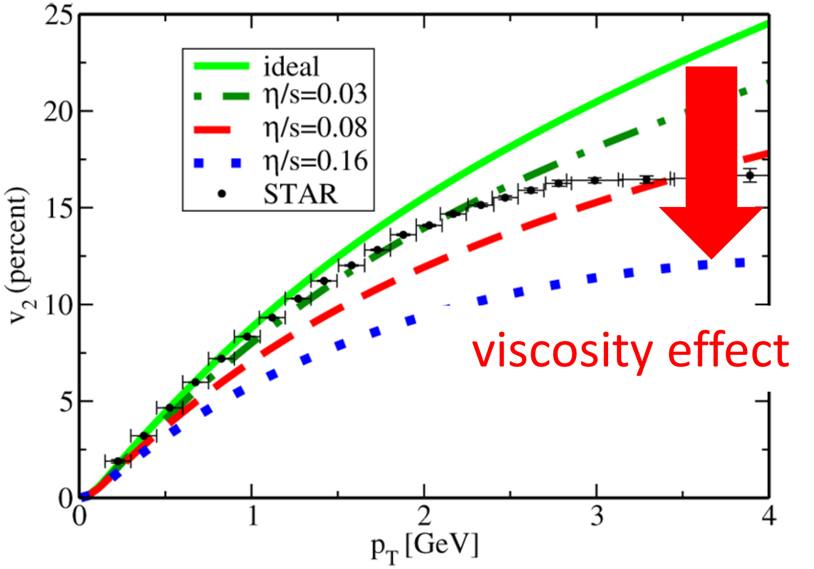
→ An approximate elliptic solution of
the Navier-Stokes from Gubser flow. YH, Xiao (2014)

Momentum anisotropy of the flow

YH, Xiao (2014)

$$\frac{dN}{d\phi} \propto 1 + 2\underline{v_2}\cos 2\phi$$

$$v_2 \propto \epsilon_p(\tau) \equiv \frac{\int dxdy(T_{xx} - T_{yy})}{\int dxdy(T_{xx} + T_{yy})}$$



$$\epsilon_p(\tau) = \frac{20a^2\tau^2}{3L^4}\left[\frac{6}{7} - \frac{3\eta_0}{2C}\left(\frac{L}{2\tau}\right)^{2/3} + \frac{513\eta_0^2}{70C^2}\left(\frac{L}{2\tau}\right)^{4/3}\right]$$