An operator method in entanglement entropy

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An operator method =A new computational method of entanglement entropy (EE)

Its application to Entanglement Entropy of Disjoint Regions in Locally Excited States

1.Introduction

Entanglement entropy (EE) is the quantity which measures the degree of quantum entanglement.

EE is a useful tool to study global properties of QFTs.

In the light of AdS/CFT correspondence, the geometries of gravitational spacetimes can be encoded in the quantum entanglement of dual CFTs.

The definition and basic properties of (Renyi) entanglement entropy

We decompose the total Hilbert space into subsystems A and B.

$$H_{tot} = H_A \otimes H_B$$

We trace out the degrees of freedom of B and consider the reduced density matrix of A.

$$\rho_A = Tr_B \rho_{tot}$$

EE is defined as von Neumann entropy.

$$S_A := -tr_A \rho_A \log \rho_A$$

The Renyi EE is the generalization of EE and defined as

$$S_A^{(n)} := \frac{1}{1-n} \log Tr({\rho_A}^n) \qquad S_A = \lim_{n \to 1} S_A^{(n)}$$

General properties

$$\rho_{AB} = \rho_A \otimes \rho_B$$

1. If a composite system $A \cup B$ is in a pure state, then

$S_{A\cup B} = 0$ and $S_A = S_B$

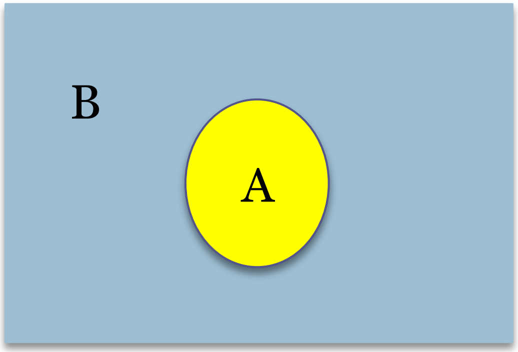
2. If $\rho_{A\cup B} = \rho_A \otimes \rho_B$, then $S_{A\cup B} = S_A + S_B$

EE in QFT (geometric entropy)

In (d+1) dimensional QFT, we define the subsystem geometrically.

Total system: d dimensional space-like manifold N

Subsystem: d dimensional domain $A \subset N$



Mutual (Renyi) information

$$I(A,B) := S_A + S_B - S_{A\cup B}$$

$$I^{(n)}(A,B) := S_A^{(n)} + S_B^{(n)} - S_{A\cup B}^{(n)}$$

2.Some computational methods of EE

Euclidean time path integral method:
When the subsystem is disjoint, this method is not useful.

Real time method:
This method is applicable only for the Gaussian density matrix

An operator method :
we can use the general properties of the operator to compute systematically the Renyi entropy for an arbitrary state and this method is useful when the subsystem is disjoint.

An operator method in EE

We consider the general scalar field in (d+1) dimensional spacetime and do not specify its Hamiltonian.

We consider n copies of the scalar fields and the j-th copy of the scalar field is denoted by $\{\phi^{(j)}\}$.
Thus the total Hilbert space, $H^{(n)}$, is the tensor product of the n copies of the Hilbert space,

$$H^{(n)} = H \otimes H \cdots \otimes H$$

where H is the Hilbert space of one scalar field.
We define the density matrix $\rho^{(n)}$ in $H^{(n)}$ as

$$\rho^{(n)} = \rho \otimes \rho \cdots \otimes \rho$$

where ρ is an arbitrary density matrix in H.
We can express $Tr\rho_{\Omega}{}^{n}$ as

$$Tr\rho_{\Omega}{}^{n} = Tr(\rho^{(n)} E_{\Omega})$$

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$$E_\Omega = \int \prod_{j=1}^{n} \prod_{x\in\Omega} DJ^{(j)}(x) DK^{(j)}(x) \exp\left[i\int d^d x \sum_{l=1}^{n} J^{(l+1)}(x)\phi^{(l)}(x)\right]$$
$$\times \exp\left[i\int d^d x \sum_{l=1}^{n} K^{(l)}(x)\pi^{(l)}(x)\right] \times \exp\left[-i\int d^d x \sum_{l=1}^{n} J^{(l)}\phi^{(l)}\right]$$

where $\pi(x)$ is a conjugate momenta of $\phi(x)$,
$[\phi(x), \pi(y)] = i\delta^d(x-y)$ and

$J^{(j)}(x)$ and $K^{(j)}(x)$ exist only in Ω and $J^{(n+1)} = J^{(1)}$

General properties of E_Ω

(1) Symmetry:

$$E_\Omega(\phi^{(1)},\ldots,\phi^{(n)},\pi^{(1)},\ldots,\pi^{(n)}) = E_\Omega(-\phi^{(1)},\ldots,-\phi^{(n)},-\pi^{(1)},\ldots,-\pi^{(n)}).$$

(2) Locality: when $\Omega = A \cup B$ and $A \cap B = 0$

$$E_{A\cup B} = E_A E_B$$

(3) For n arbitrary operators F_j (j = 1, 2, . . . , n) on H,

$$\mathrm{Tr}(F_1 \otimes F_2 \otimes \cdots \otimes F_n \cdot E_\Omega) = \mathrm{Tr}(F_{1\Omega} F_{2\Omega} \ldots F_{n\Omega}),$$

where $F_{j\Omega} = Tr_{\Omega^c} F_j$

(4) The cyclic property:

$$\mathrm{Tr}(F_1 \otimes F_2 \otimes \cdots \otimes F_n \cdot E_\Omega) = \mathrm{Tr}(F_2 \otimes F_3 \otimes \cdots \otimes F_n \otimes F_1 \cdot E_\Omega)$$

(5) The relation between E_Ω and E_{Ω^c} for pure states:

$$\langle\psi_1|\langle\psi_2|\ldots\langle\psi_n| E_\Omega |\phi_1\rangle|\phi_2\rangle\ldots|\phi_n\rangle = [\langle\phi_2|\langle\phi_3|\ldots\langle\phi_n|\langle\phi_1| E_{\Omega^c} |\psi_1\rangle|\psi_2\rangle\ldots|\psi_n\rangle]^*$$

where $|\phi_j\rangle$ and $|\psi_j\rangle$ are arbitrary pure states.

(5) The relation between E_{Ω} and E_{Ω^c} for pure states:

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where $|\phi_j\rangle$ and $|\psi_j\rangle$ are arbitrary pure states.

This is the generalization of $Tr\rho_{\Omega}{}^{n} = Tr\rho_{\Omega^c}{}^{n}$ for a pure state $\rho = |\psi\rangle\langle\psi|$

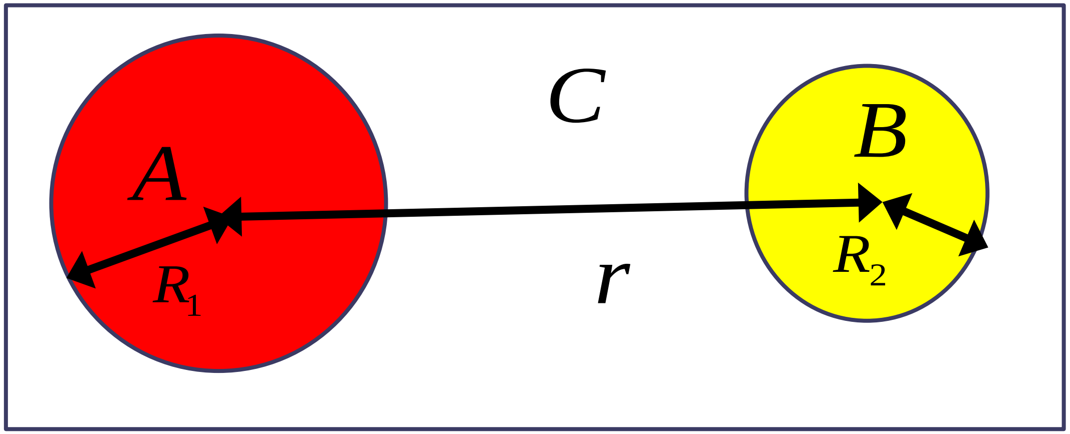
EE of Disjoint Regions in Locally Excited States

We consider the mutual Renyi information $I^{(n)}(A,B) := S_A^{(n)} + S_B^{(n)} - S_{A\cup B}^{(n)}$ of disjoint compact spatial regions A and B in the locally excited states.

$$|\Psi\rangle = N(O_{iA}O_{jB} + O_{i'A}O_{j'B})|0\rangle$$

where

$$\langle 0| O_{iA}^{\dagger} O_{i'A} |0\rangle = \langle 0| O_{jB}^{\dagger} O_{j'B} |0\rangle = 0.$$



$I^{(n)}(A,B)$ in the general QFT which has a mass gap

$$|\Psi\rangle = N(O_{iA}O_{jB} + O_{i'A}O_{j'B})|0\rangle$$

$$I^{(n)}(A,B) = \frac{2}{n-1}\ln\frac{(x+y)^n}{x^n+y^n}$$

$$x \equiv \langle 0|O_{iA}^{\dagger}O_{iA}|0\rangle\langle 0|O_{jB}^{\dagger}O_{jB}|0\rangle\,,$$

$$y \equiv \langle 0|O_{i'A}^{\dagger}O_{i'A}|0\rangle\langle 0|O_{j'B}^{\dagger}O_{j'B}|0\rangle$$

We can reproduce these results from the quantum mechanics.

$$|\Psi\rangle_{qm} = N(|i\rangle_A|j\rangle_B + |i'\rangle_A|j'\rangle_B),$$

$$I^{(n)}_{qm}(A,B) = \frac{2}{n-1}\ln\frac{(x_{qm}+y_{qm})^n}{x^n_{qm}+y^n_{qm}},$$

$$x_{qm} \equiv \langle i|i\rangle_A\langle j|j\rangle_B\,, \qquad y_{qm} \equiv \langle i'|i'\rangle_A\langle j'|j'\rangle_B$$

$$O_{i(i')A}|0\rangle \to |i(i')\rangle_A\,,$$

$$O_{j(j')B}|0\rangle \to |j(j')\rangle_B\,.$$

$I^{(n)}(A,B)$ in the free massless scalar field theory

$$|\Psi\rangle = N(O_{iA}O_{jB} + O_{i'A}O_{j'B})\,|0\rangle$$

we impose the condition that under the sign changing transformation $(\phi,\pi)\to(-\phi,-\pi)$
the operators O is transformed as

$$O \to (-1)^{|O|}O,$$

where |O|=0 or 1.

(i) The case $|O_{iA}|=|O_{i'A}|$

$$I^{(n)}(A,B) = I^{(n)}(A,B)|_{r\to\infty} + O(1/r^{2d-2}),$$

(ii) The case $|O_{iA}|\neq|O_{i'A}|$ and $|O_{jB}|\neq|O_{j'B}|$

$$I^{(n)}(A,B) = I^{(n)}(A,B)|_{r\to\infty} + O(1/r^{d-1}),$$

Conclusion

We developed the computational method of EE based on the idea that $Tr\rho_{\Omega}{}^{n}$ is written as the expectation value of the local operator at Ω.

$$Tr\rho_{\Omega}{}^{n} = Tr(\rho^{(n)} E_{\Omega})$$

The advantages of this methods are as follows:
(1) we can use ordinary technique in QFT such as OPE and the cluster decomposition property
(2) we can use the general properties of the operator to compute systematically the Renyi entropy for an arbitrary state.

We could apply the operator method to perturbative calculation in an interacting field theory.