Quantum Entanglement of Local Operators

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4. arXiv.15xx.xxxxx[hep-th]
Introduction

Recently, (Renyi) entanglement entropy ((R)EE) has a center of wide interest in a broad array of theoretical physics.

• It is useful to study the distinctive features of various quantum state in condensed matter physics. (*Quantum Order Parameter*)

• (Renyi) entanglement entropy is expected to be an important quantity which may shed light on the mechanism behind the AdS/CFT correspondence. (*Gravity ↔ Entanglement*)
Introduction

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• In the lattice gauge theory, it is expected that entanglement entropy is a new order parameter which helps us study QCD more.

- But entanglement entropy in the gauge theory is ill-defined.
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It is important to study the properties of (Renyi) entanglement entropy.
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- In the lattice gauge theory, it is expected that entanglement entropy is a new order parameter which helps us study QCD more.

- But entanglement entropy in the gauge theory is ill-defined.

In this work, we investigate the time dependent property of (Renyi) entanglement entropy.
The Definition of (Renyi) Entanglement Entropy

- **Definition of Entanglement Entropy**

We divide the total Hilbert space into $A$ and $B$: $H_{tot} = H_A \otimes H_B$. The reduced density matrix $\rho_A$ is defined by

$$\rho_A \equiv Tr_B \rho_{tot}$$

This means the DOF in $B$ are traced out. The entanglement entropy is defined by von Neumann entropy $S_A$.

**(Renyi) Entanglement Entropy (REE)**

$$S_A^{(n)} = \frac{\log tr[\rho_A^n]}{1 - n}$$

$n \to 1$

Entanglement Entropy (EE)

$$S_A = -tr_A \rho_A \log \rho_A$$

on a certain time slice
Motivation

Previously, we studied the property of EE for the subsystem whose size ($l$) is very small in $d$ CFT.

\[ l \ll (\text{The Excess of Energy Density})^{-d} \]

\[ \Delta E_A = T_{\text{ent}} \Delta S_A \]

This temperature is universal.
We study the property of (R)EE for 

1. The size of subsystem is *infinite*. 

A half of the total system:

\[ x^1 \geq 0 \]
**Setup**

We study the property of \( (R)EE \) for

2. A state is defined by acting a local operator on the ground state:

\[
|\Psi\rangle = \mathcal{N} \mathcal{O}(t = -t, x^1 = -l, x)|0\rangle
\]
Setup

We study the property of (R)EE for

1. The size of subsystem is \textit{infinite}.
   A half of the total system:
   \[ x^1 \geq 0 \]

2. A state is defined by acting a local operator on the ground state:
   \[ |\Psi\rangle = \mathcal{N} \mathcal{O}(t, x^i) |0\rangle \]
We would like to focus on the time evolution of the (R)EE.

We define $\Delta S_A^{(n)}$ the excess of the (R)EE:

$$
\Delta S_A^{(n)} = S_A^{(n)Ex} - S_A^{(n)G},
$$

$S_A^{(n)Ex}$: (R)EE for $\hat{\rho}_A$ (Reduced Density Matrix for $|\Psi\rangle = \mathcal{N}\mathcal{O}(t, x^i) |0\rangle$)

$S_A^{(n)G}$: (R)EE for the ground state
Field Theory

1. Free massless scalar field theory

2. $U(N)$ or $SU(N)$ free massless scalar field theory in Large N limit

3. Free massless fermionic field theory

4. Holographic field theory
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1. Free massless scalar field theory

2. $U(N)$ or $SU(N)$ free massless scalar field theory in Large N limit

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4. Holographic field theory
1. Free massless scalar field theory

2. \( \Delta S_A^{(n)} \)  

   At late time

   Some Constants

3. Free massless fermionic field theory

4. Holographic field theory
1. Free massless scalar field theory

2. \[ \Delta S_A^{(n)} \] Logarithmically grows At late time

3. 

4. Holographic field theory
## Results

\( \Delta S_A^{(n)} \) At Late Time (\( t \gg l \))

<table>
<thead>
<tr>
<th>F.T.</th>
<th>Operator</th>
<th>( n \geq 2 )</th>
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<tbody>
<tr>
<td>Free Massless Scalar</td>
<td>( : (\partial^m \phi)^k : )</td>
<td>( \frac{1}{1-n} \log \left( \frac{1}{2^{n}k} \sum_{j=0}^{k} (kC_j)^n \right) )</td>
<td>( k \cdot \log 2 - \frac{1}{2^k} \sum_{j=0}^{k} kC_j \log kC_j )</td>
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<td>U(N), SU(N) Free Massless Scalar</td>
<td>( Tr(\phi_1 + i\phi_2)^J )</td>
<td>( \frac{2n - 1}{n - 1} \cdot \log 2 )</td>
<td>( \log \left( 2\sqrt{2N} \right) )</td>
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<td>Free Massless Fermion</td>
<td>( \overline{\psi} \psi )</td>
<td>( \frac{1}{1-n} \log \left[ \frac{2 \cdot 12^n + 4 \cdot 9^n + 4}{2^{6n}} \right] )</td>
<td>( \frac{3}{4} \log \left( \frac{128}{9} \right) )</td>
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<td>AdS/CFT</td>
<td>Gauge invariant Operator ( \mathcal{O}_\Delta )</td>
<td>( \frac{4n\Delta}{d(n - 1)} \log t ) ( 1 \ll \Delta \ll c )</td>
<td>( \frac{c}{6} \log t )</td>
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2d CFT \( \Delta \sim c \)
Results

\[ \Delta S_A^{(n)} \] At Late Time (\( t \gg l \))

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Example

We consider \textit{free massless scalar} field theory in \textit{d+1 dim.} Especially, we focus on that in \textit{4 dim.}

We act a local operator $\phi(-t, -l, x)$ on the ground state: $|\Psi\rangle = \mathcal{N} \phi(-t, -l, x) |0\rangle$.

We measure the (Renyi) entanglement entropies at $t=0$.

$\Rightarrow \textit{Time evolution!!}$

\[\begin{align*}
\text{x}_1 &= -l \\
\phi(-t, -l, x)
\end{align*}\]
Time Evolution of $\Delta S_A^{(2)}$

\[ \Delta S_A^{(2)}(t) \]

\[ \begin{align*}
  & t < l \\
  & \Delta S_A^{(2)} = 0 \\
  & t \geq l \\
  & \Delta S_A^{(2)} = \log \left[ \frac{2t^2}{t^2 + l^2} \right]
\end{align*} \]
Time Evolution of $\Delta S_A^{(2)}$

\[ \Delta S_A^{(2)}(t) = \begin{cases} t < l & \Delta S_A^{(2)} = 0 \\ t \geq l & \Delta S_A^{(2)} = \log \left[ \frac{2t^2}{t^2 + l^2} \right] \end{cases} \]

An entangled pair appears. Each of pair is included in the region B.
Time Evolution of $\Delta S_{A}^{(2)}$

In this region, two quanta is included in A and B respectively.

$t \geq l$

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In this region, two quanta is included in A and B respectively. Entanglement between quanta can contribute to $\Delta S_{A}^{(2)}$. 

$\Delta S_{A}^{(2)} = \log \left[ \frac{2t^2}{t^2 + l^2} \right]$
Time Evolution of $\Delta S_A^{(2)}$

Subsystem $= \text{a half of the total space}$

$\Delta S_A^{(2)}$ approaches $\text{Constant!!}$
Time Evolution of $\Delta S_{A}^{(2)}$

$\Delta S_{A}^{(2)}$ = Entanglement between Quasi-particles

Subsystem = a half of the total space
$\Delta S_{A}^{(2)}$ approaches Constant!!

$\Delta S_{A}^{(2)} = \text{Entanglement between Quasi-particles}!!$
Time Evolution of $\Delta S_{A}^{(2)}$:

$\Delta S_{A}^{(2)}$ = Entanglement between Quasi-particles at $t$!!

Subsystem

$\Delta S_{A}^{(2)}$ approaches = a half of the total space

(Renyi) entanglement entropies of (local) operators

Operator

$B$ $A$

$t < l$

$B$ $A$

$t > l$
### Results

\[ \Delta S_A^{(n)} \] At Late Time (t >> \l)

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Summary

• If non-perturbative effect for $c$ or $N$, $\Delta S_A^{(n)}$ can possibly approach constant.

• Free field theory (Infinite conserved charges) $\rightarrow$ Quasi-particle picture holds

  $\Rightarrow \quad \Delta S_A^{(n)} \rightarrow \text{Constant}$

Holographic CFT $\rightarrow$ Quasi-particle picture breaks down

  $\Rightarrow \quad \Delta S_A^{(n)} \rightarrow \text{Not constant}$
Future Problems

• In non-relativistic case, the time evolution of $\Delta S_A^{(n)}$

• In gauge field theory, $\Delta S_A^{(n)}$

• The (Renyi) entanglement entropies of operators in the interacting field theory. (also massive and charged Renyi.)

• Beyond large N, we investigate $\Delta S_A^{(n)}$.
  - approach constant?
  - diverge?