The Partition Function of ABJ Theory

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Introduction

AdS_4/CFT_3

M-theory on
$AdS_4 \times S^7/\mathbb{Z}_k$

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Type IIA on
$AdS_4 \times \mathbb{CP}^3$

⟺

$d = 3, \mathcal{N} = 6$
$U(N)_k \times U(N)_{-k}$
Chern-Simons-matter
SCFT
(ABJM theory)

$$R \sim (kN)^{1/6} l_{11} \sim \left(\frac{N}{k}\right)^{1/4} l_s, \quad g_s \sim N^{1/2} k^{-5/2}$$

- Low E physics of N M2-branes on $\mathbb{C}^4/\mathbb{Z}_k$
- Can tell us about M2 dynamics in principle (cf. $F \sim N^{3/2}$)
- More non-trivial example of AdS/CFT

ABJ theory

- Generalization of ABJM [Hosomichi-Lee[3]-Park, ABJ]
- $U(N_1)_k \times U(N_2)_{-k}$ $\mathcal{N} = 6$ CS-matter SCFT
- Low E dynamics of N_1 M2 and $(N_2 - N_1)$ fractional M2
- Dual to M on $AdS_4 \times S^7/\mathbb{Z}_k$ with 3-form A_3

$H_3(S^7/\mathbb{Z}_k, \mathbb{Z}) = \mathbb{Z}_k$

Seiberg duality

- $U(N+l)_k \times U(N)_{-k} = U(N)_k \times U(N+k-l)_{-k}$

"ABJ triality" [Chang+Minwalla+Sharma+Yin]

- $N_2, k \to \infty$ with $\lambda = \frac{N_2}{k}, N_1$: fixed

⇨ dual to Vasiliev higher spin theory?

Localization & ABJM

- Powerful technique in susy field theory
  - Reduces field theory path integral to matrix model
  - Tool for precision holography ($1/N$ corrections)
- Application to ABJM [Kapustin+Willett+Yaakov]
  - Large N: reproduces $F \sim \frac{N^2}{\sqrt{\lambda}} \sim N^{3/2}$, $\lambda \equiv \frac{N}{k}$

    [Drukker+Marino+Putrov]
  - All pert. $\frac{1}{N}$ corrections [Fuji+Hirano+Moriyama][Marino+Putrov]
  - Wilson loops [Klemm+Marino+Schiereck+Soroush]

ABJ: not as well-studied as ABJM;
MM harder to handle

↓

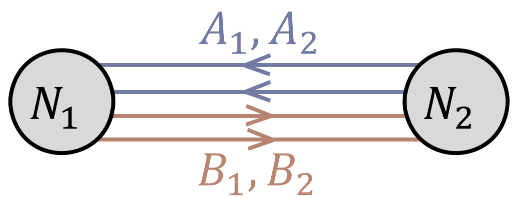
Goal: develop framework to study ABJ

- Rewrite ABJ MM
  in more tractable form
- Fun to see how it works
- Interesting physics emerges!

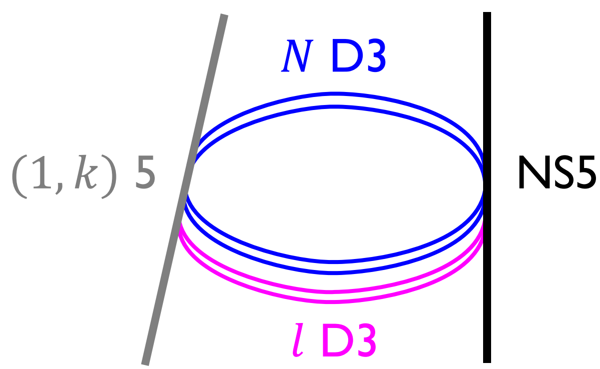
Outline & main results

ABJ theory

- $d = 3, \mathcal{N} = 6 \;\; U(N_1)_k \times U(N_2)_{-k}$ CSM theory



- Superconformal
- IIB picture



$U(N)_k \times U(N+l)_{-k}$

Lagrangian

$$\mathcal{L} = -\frac{\epsilon^{\mu\nu\rho}}{2k}\mathrm{tr}\left(A_\mu\partial_\nu A_\rho + \frac{2}{3}A_\mu A_\nu A_\rho - \tilde{A}_\mu\partial_\nu\tilde{A}_\rho - \frac{2}{3}\tilde{A}_\mu\tilde{A}_\nu\tilde{A}_\rho\right)$$
$$-\mathrm{tr}\left(D_\mu\bar{\Phi}^\alpha D^\mu\Phi_\alpha - i\bar{\Psi}_\alpha\gamma^\mu D_\mu\Psi^\alpha\right) - ik\epsilon^{\alpha\beta\gamma\delta}\mathrm{tr}\left(\Phi_\alpha\bar{\Psi}_\beta\Phi_\gamma\bar{\Psi}_\delta\right) + ik\epsilon_{\alpha\beta\gamma\delta}\mathrm{tr}\left(\bar{\Phi}^\alpha\Psi^\beta\bar{\Phi}^\gamma\Psi^d\right)$$
$$-ik\mathrm{tr}\left(\bar{\Phi}^\alpha\Phi_\alpha\bar{\Psi}_\beta\Psi^\beta - \Phi_\alpha\bar{\Phi}^\alpha\Psi^\beta\bar{\Psi}_\beta + 2\bar{\Phi}^\alpha\Psi^\beta\bar{\Psi}_\alpha\Phi_\beta - 2\Phi_\alpha\bar{\Psi}_\beta\Psi^\alpha\bar{\Phi}^\beta\right)$$
$$+\frac{1}{3}k^2\mathrm{tr}\left(\Phi_\alpha\bar{\Phi}^\alpha\Phi_\beta\bar{\Phi}^\beta\Phi_\gamma\bar{\Phi}^\gamma + \bar{\Phi}^\alpha\Phi_\alpha\bar{\Phi}^\beta\Phi_\beta\bar{\Phi}^\gamma\Phi_\gamma\right) + \frac{4}{3}k^2\mathrm{tr}\left(\Phi_\alpha\bar{\Phi}^\gamma\Phi_\beta\bar{\Phi}^\alpha\Phi_\gamma\bar{\Phi}^\beta\right)$$
$$-2k^2\mathrm{tr}\left(\Phi_\alpha\bar{\Phi}^\alpha\Phi_\beta\bar{\Phi}^\gamma\Phi_\gamma\bar{\Phi}^\beta\right) \; .$$

ABJ MM [Kapustin-Willett-Yaakov]

Partition function of ABJ theory on S^3:

vector multiplets

$$Z_{\mathrm{ABJ}}(N_1,N_2)_k = \mathcal{N}_{\mathrm{ABJ}} \int \prod_{i=1}^{N_1} \frac{d\mu_i}{2\pi} \prod_{a=1}^{N_2} \frac{d\nu_a}{2\pi} \frac{\Delta_{\mathrm{sh}}(\mu)^2 \Delta_{\mathrm{sh}}(\nu)^2}{\Delta_{\mathrm{ch}}(\mu,\nu)^2} e^{-\frac{1}{2g_s}\left(\sum_{i=1}^{N_1}\mu_i^2 - \sum_{a=1}^{N_2}\nu_a^2\right)}$$

matter multiplets

$$g_s \equiv \frac{2\pi i}{k}, \quad k \in \mathbb{Z}_{\neq 0}$$

$$\Delta_{\mathrm{sh}}(\mu) = \prod_{1\le i<j\le N_1}\left(2\sinh\left(\frac{\mu_i-\mu_j}{2}\right)\right) \qquad \Delta_{\mathrm{sh}}(\nu) = \prod_{1\le a<b\le N_2}\left(2\sinh\left(\frac{\nu_a-\nu_b}{2}\right)\right)$$

$$\Delta_{\mathrm{ch}}(\mu,\nu) = \prod_{i=1}^{N_1}\prod_{a=1}^{N_2}\left(2\cosh\left(\frac{\mu_i-\nu_a}{2}\right)\right) \qquad \mathcal{N}_{\mathrm{ABJ}} := \frac{i^{-\frac{\kappa}{2}(N_1^2-N_2^2)}}{N_1!N_2!}, \qquad \kappa := \operatorname{sign} k\ .$$

- Pert. treatment — easy
- Rigorous treatment — obstacle: $\Delta_{\mathrm{ch}}^{-2}$

Strategy

"Analytically continue" lens space ($S^3/\mathbb{Z}_2$) MM

$$Z_{\text{lens}}(N_1, N_2)_k = \mathcal{N}_{\text{lens}} \int \prod_{i=1}^{N_1} \frac{d\mu_i}{2\pi} \prod_{a=1}^{N_2} \frac{d\nu_a}{2\pi} \Delta_{\text{sh}}(\mu)^2 \Delta_{\text{sh}}(\nu)^2 \Delta_{\text{ch}}(\mu,\nu)^2 e^{-\frac{1}{2g_s}\left(\sum_{i=1}^{N_1}\mu_i^2 + \sum_{a=1}^{N_2}\nu_a^2\right)}$$

$$\mathcal{N}_{\text{lens}} = \frac{i^{-\frac{\kappa}{2}(N_1^2+N_2^2)}}{N_1!N_2!}$$

Perturbative relation: [Marino][Marino+Putrov]

$$Z_{\text{ABJ}}(N_1, N_2)_k \sim Z_{\text{lens}}(N_1, -N_2)_k$$

Simple Gaussian integral ⇨ Explicitly doable!

Expression for Z_{lens}

$$Z_{\text{lens}}(N_1, N_2)_k = i^{-\frac{\kappa}{2}(N_1^2+N_2^2)} \left(\frac{g_s}{2\pi}\right)^{\frac{N}{2}} q^{-\frac{1}{6}N(N^2-1)}(1-q)^{\frac{1}{2}N(N-1)} G_2(N+1;q)\, S(N_1, N_2)$$

$$S(N_1, N_2) = \sum_{(\mathcal{N}_1,\mathcal{N}_2)} \prod_{C_j<D_a} \frac{q^{C_j}+q^{D_a}}{q^{C_j}-q^{D_a}} \prod_{D_a<C_j} \frac{q^{D_a}+q^{C_j}}{q^{D_a}-q^{C_j}}$$

$$q \equiv e^{-g_s} = e^{-\frac{2\pi i}{k}}, \qquad N \equiv N_1 + N_2$$

- $G_2(N+1;q)$: q-Barnes G function

$$(1-q)^{\frac{1}{2}N(N-1)} G_2(N+1;q) = \prod_{j=1}^{N-1} (1-q^j)^{N-j}$$

- $(\mathcal{N}_1, \mathcal{N}_2)$: partition of $\{1, \dots, N\}$ into two sets

$$\mathcal{N}_1 = \{C_1, \dots, C_{N_1}\}, \qquad \mathcal{N}_2 = \{D_1, \dots, D_{N_2}\},$$
$$C_1 < \cdots < C_{N_1}, \qquad D_1 < \cdots < D_{N_2}$$

Now we analytically continue:

$$N_2 \to -N_2$$

It's like knowing discrete data

$$n!$$

and guessing

$$\Gamma(-z)$$

Analytically cont'd Z_{ABJ}

$$Z_{\mathrm{ABJ}}(N_1,N_2)_k = i^{-\frac{\kappa}{2}(N_1^2+N_2^2)}(-1)^{\frac{1}{2}N_1(N_1-1)}\,2^{-N_1}\left(\frac{g_s}{2\pi}\right)^{\frac{N_1+N_2}{2}}(1-q)^{\frac{M(M-1)}{2}}G_2(M+1;q)$$
$$\times\frac{1}{N_1!}\sum_{s_1,\ldots,s_{N_1}\geq 0}(-1)^{s_1+\cdots+s_{N_1}}\prod_{j=1}^{N_1}\frac{(q^{s_j+1})_M}{(-q^{s_j+1})_M}\prod_{j<k}^{N_1}\frac{(1-q^{s_k-s_j})^2}{(1+q^{s_k-s_j})^2}$$

$$M\equiv N_2-N_1\geq 0,\qquad q\equiv e^{-g_s}=e^{-\frac{2\pi i}{k}}$$

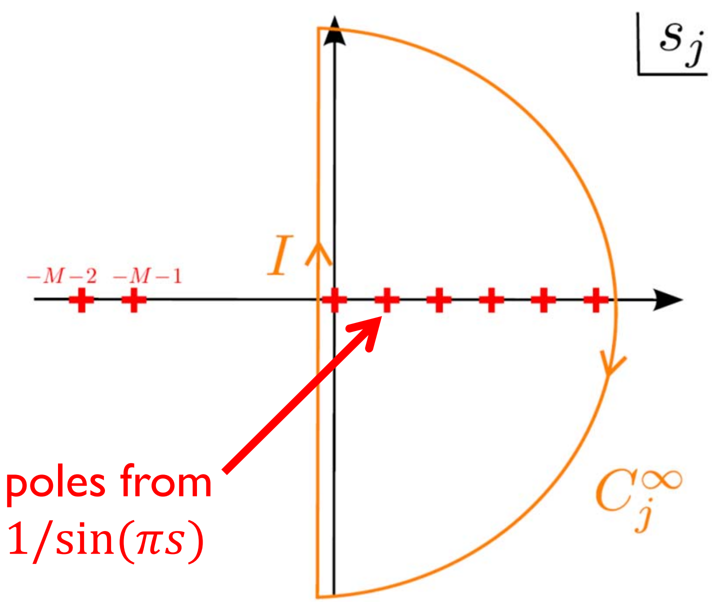
q-Pochhammer symbol:

$$(a)_n\equiv(1-a)(1-aq)\cdots(1-aq^{n-1})$$

- Analytic continuation $N_2\to -N_2$ has ambiguity
  - ⇨ criterion: reproduce pert. expansion
- Non-convergent series — need regularization

Z_{ABJ}: integral rep (1)

$$Z_{\mathrm{ABJ}}(N_1,N_2)_k = i^{-\frac{\kappa}{2}(N_1^2+N_2^2)}(-1)^{\frac{1}{2}N_1(N_1-1)}\,2^{-N_1}\left(\frac{g_s}{2\pi}\right)^{\frac{N_1+N_2}{2}}(1-q)^{\frac{M(M-1)}{2}}G_2(M+1;q)$$
$$\times\frac{1}{N_1!}\prod_{j=1}^{N_1}\left[\frac{-1}{2\pi i}\int_I\frac{\pi\,ds_j}{\sin(\pi s_j)}\right]\prod_{j=1}^{N_1}\frac{(q^{s_j+1})_M}{(-q^{s_j+1})_M}\prod_{1\le j<k\le N_1}\frac{(1-q^{s_k-s_j})^2}{(1+q^{s_k-s_j})^2}$$

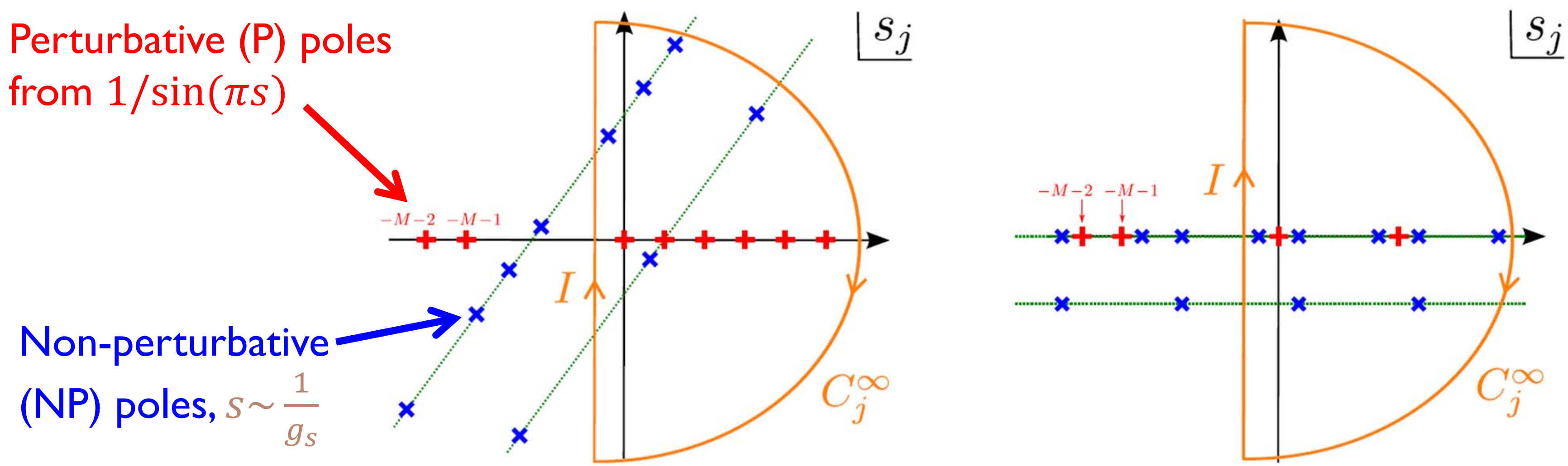


- Finite
- "Agrees" with the formal series
- Watson-Sommerfeld transf in Regge theory

Z_{ABJ}: integral rep (2)

$$Z_{\text{ABJ}}(N_1,N_2)_k = i^{-\frac{\kappa}{2}(N_1^2+N_2^2)}(-1)^{\frac{1}{2}N_1(N_1-1)}\,2^{-N_1}\left(\frac{g_s}{2\pi}\right)^{\frac{N_1+N_2}{2}}(1-q)^{\frac{M(M-1)}{2}}G_2(M+1;q)$$
$$\times\frac{1}{N_1!}\prod_{j=1}^{N_1}\left[\frac{-1}{2\pi i}\int_I\frac{\pi\,ds_j}{\sin(\pi s_j)}\right]\prod_{j=1}^{N_1}\frac{(q^{s_j+1})_M}{(-q^{s_j+1})_M}\prod_{1\le j<k\le N_1}\frac{(1-q^{s_k-s_j})^2}{(1+q^{s_k-s_j})^2}$$

- Provides non-perturbative completion



Comments (1)

- Well-defined for all q
- No first-principle derivation;
  a posteriori justification:
  - Correctly reproduces pert. expansions
  - Seiberg duality
  - Explicit checks for small N_1, N_2

Comments (2)

- $M = 0$ (ABJM) reduces to "mirror expression"

$$Z(N) = \frac{1}{2^N N!} \int \frac{d^N q}{(2\pi k)^N} \prod_i \frac{1}{2\cosh\frac{q_i}{2}} \prod_{i<j} \tanh^2 \frac{q_i - q_j}{2k}$$

— Starting point for Fermi gas approach

- Vanishes for $M > k$

$$(1-q)^{\frac{M(M-1)}{2}} G_2(M+1; q) = \prod_{j=1}^{M-1} (q)_j = 0 \quad \text{for} \quad q = e^{-\frac{2\pi i}{k}}$$

— Agrees with ABJ conjecture

Details / Example

Z_{lens}

$$Z_{\text{lens}}(N_1,N_2)_k = i^{-\frac{\kappa}{2}(N_1^2+N_2^2)}\left(\frac{g_s}{2\pi}\right)^{\frac{N}{2}} q^{-\frac{1}{6}N(N^2-1)}(1-q)^{\frac{1}{2}N(N-1)}G_2(N+1;q)\,S(N_1,N_2)$$

$$S(N_1,N_2) = \sum_{(\mathcal{N}_1,\mathcal{N}_2)}\prod_{C_j<D_a}\frac{q^{C_j}+q^{D_a}}{q^{C_j}-q^{D_a}}\prod_{D_a<C_j}\frac{q^{D_a}+q^{C_j}}{q^{D_a}-q^{C_j}}$$

$$q \equiv e^{-g_s} = e^{-\frac{2\pi i}{k}}, \qquad N \equiv N_1 + N_2$$

- $(\mathcal{N}_1, \mathcal{N}_2)$: partition of $\{1, \ldots, N\}$ into two sets

$$\mathcal{N}_1 = \{C_1, \ldots, C_{N_1}\}, \qquad \mathcal{N}_2 = \{D_1, \ldots, D_{N_2}\},$$
$$C_1 < \cdots < C_{N_1}, \qquad D_1 < \cdots < D_{N_2}$$

$N_1 = 1$

$$S(N_1, N_2) = \sum_{(\mathcal{N}_1, \mathcal{N}_2)} \prod_{C_j < D_a} \frac{q^{C_j} + q^{D_a}}{q^{C_j} - q^{D_a}} \prod_{D_a < C_j} \frac{q^{D_a} + q^{C_j}}{q^{D_a} - q^{C_j}}$$

⇓ $N_1 = 1$

$\mathcal{N}_1 = \{C\}, \quad \mathcal{N}_2 = \{1, \dots, C-1, C+1, \dots, N_2 + 1\},$

$$S(1, N_2) = \sum_{C=1}^{N_2+1} \prod_{C<a} \frac{q^C + q^a}{q^C - q^a} \prod_{a<C} \frac{q^a + q^C}{q^a - q^C}$$

$S(1, N_2)$

$$S(1,N_2) = \sum_{C=1}^{N_2+1} \prod_{C<a} \frac{q^C + q^a}{q^C - q^a} \prod_{a<C} \frac{q^a + q^C}{q^a - q^C} = \sum_{C=1}^{N_2+1} \prod_{j=1}^{N_2-C+1} \frac{1+q^j}{1-q^j} \prod_{j=1}^{C-1} \frac{1+q^j}{1-q^j}$$

$$= \sum_{C=1}^{N_2+1} \frac{(-q)_{N_2-C+1}}{(q)_{N_2-C+1}} \frac{(-q)_{C-1}}{(q)_{C-1}} = \sum_{n=0}^{N_2} \frac{(-q)_{N_2-n}}{(q)_{N_2-n}} \frac{(-q)_n}{(q)_n},$$

How do we continue this $N_2 \to -N_2$?

Obstacles:

1. What is $(a)_{-z}$?
2. Sum range depends on N_2

q-Pochhammer & anal. cont.

For $n = 1,2, \ldots,$

$$(a)_n \equiv (1-a)(1-aq)\cdots(1-aq^{n-1}) = \frac{(a)_\infty}{(aq^n)_\infty}.$$

For $z \in \mathbb{C}$,

$$(a)_z \equiv \frac{(a)_\infty}{(aq^z)_\infty}.$$

In particular,

$$(q)_{-n} \equiv \frac{(q)_\infty}{(q^{1-n})_\infty} = \frac{(1-q)(1-q^2)\cdots}{(1-q^{1-n})(1-q^{2-n})\cdots(1-q^0)\cdots} = \infty$$

$$\Rightarrow \quad S(1, N_2) = \sum_{n=0}^{\infty} \frac{(-q)_{N_2-n}}{(q)_{N_2-n}} \frac{(-q)_n}{(q)_n}$$

Can extend sum range

Good for any $N_2 \in \mathbb{C}$ ☺

Comments

- q-hypergeometric function

$$S(1, N_2) = \frac{(-q)_{N_2}}{(q)_{N_2}} \, {}_2\phi_1 \left(\begin{matrix} q^{-N_2}, -q \\ -q^{-N_2} \end{matrix} ; q, -1 \right)$$

$${}_r\phi_s \left(\begin{matrix} a_1, \ldots, a_r \\ b_1 \ldots, b_s \end{matrix} ; q, z \right) := \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_r)_n}{(q)_n (b_1)_n \cdots (b_s)_n} \left[(-1)^n q^{\binom{n}{2}} \right]^{1+s-r} z^n$$

- Normalization

$$Z_{\text{lens}}(1, N_2)_k = \frac{1}{N_2!} \int \ldots \quad \Rightarrow \quad Z_{\text{lens}}(1, -N_2)_k = \frac{1}{\Gamma(1-N_2)} \int \ldots$$

Need to continue “ungauged” MM partition function

Also need ϵ-prescription: $N_2 \to -N_2 + \epsilon$

$Z_{\mathrm{ABJ}}(1, N_2)_k$

$$Z_{ABJ}(1, N_2) = \frac{1}{2} i^{-\frac{\kappa}{2}(1+N_2^2)} \left(\frac{g_s}{2\pi}\right)^{\frac{1+N_2}{2}} \left[\prod_{j=1}^{N_2-2} (q)_j\right] \sum_{s=0}^{\infty} \frac{(q^{s+1})_{N_2-1}}{(-q^{s+1})_{N_2-1}} (-1)^s$$

$N_2 = 1$:

$$\sum_{s=0}^{\infty} (-1)^s = \frac{1}{2} \quad \Rightarrow \quad Z_{\mathrm{ABJ}}(1,1)_k = \frac{1}{4|k|} \quad ✓$$

$N_2 = 2$

$$\sum_{s=0}^{\infty} (-1)^s \frac{1 - q^{s+1}}{1 + q^{s+1}} = \sum_{s=0}^{\infty} (-1)^s \left[\frac{s+1}{2} g_s - \frac{(s+1)^3}{24} g_s^3 + \cdots\right]$$

$q = e^{-g_s}$

$$\mathrm{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}, \quad \mathrm{Li}_{-n}(-1) = \sum_{k=1}^{\infty} (-1)^s s^n$$

$$= \frac{1}{8} g_s + \frac{1}{192} g_s^3 + \cdots \quad ✓$$

Reproduces pert. expn. of ABJ MM

Perturbative expansion

$$
\begin{aligned}
F_{\text{lens}}(N_1,N_2)-F^{\text{tree}}_{\text{lens}}(N_1,N_2) &= g_s\left(\frac{N_1^3}{12}+\frac{N_1^2N_2}{4}+\frac{N_1N_2^2}{4}+\frac{N_2^3}{12}-\frac{N_1}{12}-\frac{N_2}{12}\right)\\
&+g_s^2\left(\frac{N_1^4}{288}+\frac{N_1^3N_2}{48}+\frac{N_2^2N_1^2}{16}+\frac{N_2^3N_1}{48}+\frac{N_2^4}{288}-\frac{N_1^2}{288}+\frac{N_1N_2}{48}-\frac{N_2^2}{288}\right)\\
&+g_s^4\Bigg(-\frac{N_1^6}{86400}-\frac{N_1^5N_2}{7680}-\frac{N_1^4N_2^2}{1536}-\frac{5N_1^3N_2^3}{1152}-\frac{N_1^2N_2^4}{1536}-\frac{N_1N_2^5}{7680}-\frac{N_2^6}{86400}\\
&+\frac{N_1^4}{34560}+\frac{7N_1^3N_2}{4608}-\frac{N_1^2N_2^2}{768}+\frac{7N_1N_2^3}{4608}+\frac{N_2^4}{34560}-\frac{N_1^2}{57600}-\frac{N_1N_2}{960}-\frac{N_2^2}{57600}\Bigg)\\
&+g_s^6\Bigg(\frac{N_1^8}{10160640}+\frac{N_1^7N_2}{645120}+\frac{N_1^6N_2^2}{92160}+\frac{N_1^5N_2^3}{92160}+\frac{7N_1^4N_2^4}{9216}+\frac{N_1^3N_2^5}{92160}+\frac{N_1^2N_2^6}{92160}+\frac{N_1N_2^7}{645120}+\frac{N_2^8}{10160640}\\
&-\frac{N_1^6}{2177280}+\frac{N_1^5N_2}{92160}-\frac{N_1^4N_2^2}{2304}+\frac{N_1^3N_2^3}{27648}-\frac{N_1^2N_2^4}{2304}+\frac{N_1N_2^5}{92160}-\frac{N_2^6}{2177280}\\
&+\frac{N_1^4}{1451520}+\frac{N_1^3N_2}{11520}+\frac{N_1^2N_2^2}{3840}+\frac{N_1N_2^3}{11520}+\frac{N_2^4}{1451520}-\frac{N_1^2}{3048192}-\frac{N_1N_2}{12096}-\frac{N_2^2}{3048192}\Bigg)\\
&+g_s^8\Bigg(-\frac{N_1^{10}}{870912000}-\frac{17N_1^9N_2}{743178240}-\frac{17N_1^8N_2^2}{82575360}-\frac{N_1^7N_2^3}{774144}+\frac{97N_1^6N_2^4}{4423680}-\frac{2821N_1^5N_2^5}{14745600}+\frac{97N_1^4N_2^6}{4423680}\\
&-\frac{N_1^3N_2^7}{774144}-\frac{17N_1^2N_2^8}{82575360}-\frac{17N_1N_2^9}{743178240}-\frac{N_2^{10}}{870912000}+\frac{N_1^8}{116121600}+\frac{29N_1^7N_2}{123863040}-\frac{259N_1^6N_2^2}{17694720}\\
&+\frac{937N_1^5N_2^3}{8847360}+\frac{53N_1^4N_2^4}{442368}+\frac{937N_1^3N_2^5}{8847360}-\frac{259N_1^2N_2^6}{17694720}+\frac{29N_1N_2^7}{123863040}+\frac{N_2^8}{116121600}-\frac{N_1^6}{41472000}\\
&+\frac{853N_1^5N_2}{58982400}-\frac{1487N_1^4N_2^2}{11796480}-\frac{83N_2^3N_1^3}{1769472}-\frac{1487N_1^2N_2^4}{11796480}+\frac{853N_1N_2^5}{58982400}-\frac{N_2^6}{41472000}+\frac{N_1^4}{34836480}\\
&-\frac{23N_1^3N_2}{37158912}+\frac{325N_1^2N_2^2}{3096576}-\frac{23N_1N_2^3}{37158912}+\frac{N_2^4}{34836480}-\frac{N_1^2}{82944000}-\frac{17N_1N_2}{1382400}-\frac{N_2^2}{82944000}\Bigg).
\end{aligned}
$$

Integral rep

A way to regularize the sum once and for all:

$$\sum_{s=0}^{\infty} \frac{(q^{s+1})_{N_2-1}}{(-q^{s+1})_{N_2-1}}(-1)^s \Rightarrow \frac{-1}{2\pi i}\int_{-i\infty-\epsilon}^{i\infty-\epsilon} \boxed{\frac{\pi\, ds}{\sin(\pi s)}} \frac{(q^{s+1})_{N_2-1}}{(-q^{s+1})_{N_2-1}}$$

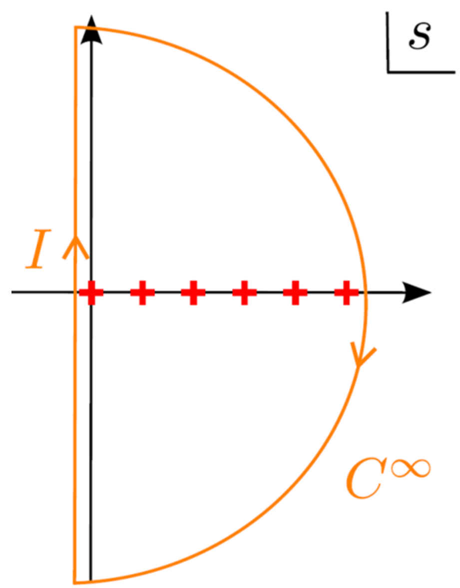
Poles at $s \in \mathbb{Z}$, residue $(-1)^s$

- Reproduces pert. expn.

$$\frac{-1}{2\pi i}\int_{-i\infty-\epsilon}^{i\infty-\epsilon} \frac{\pi ds}{\sin(\pi s)} s^n = \frac{1-2^{n+1}}{n+1} B_{n+1}$$

$$= \mathrm{Li}_{-n}(-1) = \text{“}\sum_{s=0}^{\infty}(-1)^n s^n\text{”}$$

- But more!

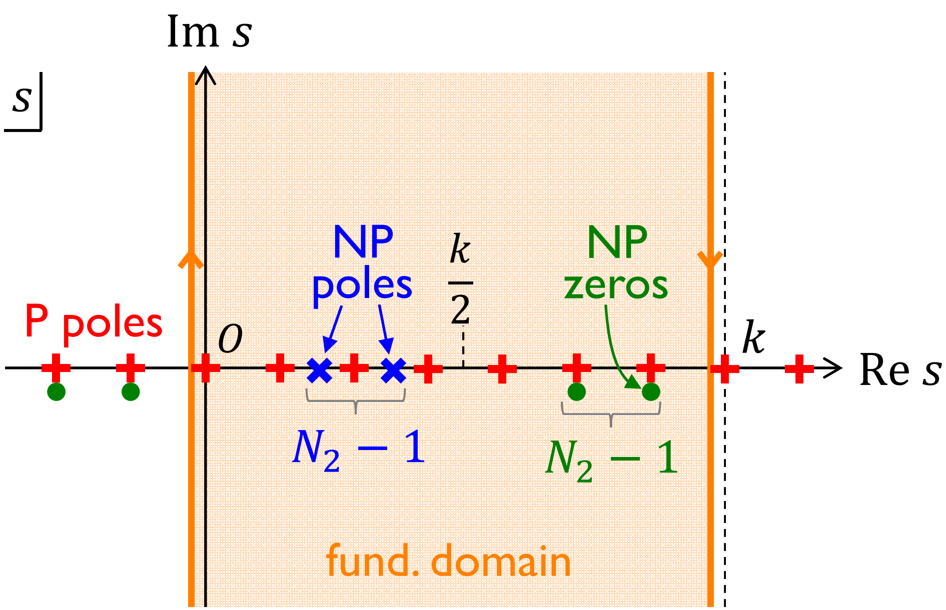


Non-perturbative completion

$$\frac{(q^{s+1})_{N_2-1}}{(-q^{s+1})_{N_2-1}} = \prod_{j=1}^{N_2-1} \tan\frac{\pi(s+j)}{k}$$

- poles at $s = \frac{k}{2} - j$
- zeros at $s = k - j$

$j = 1, \ldots, N_2 - 1$



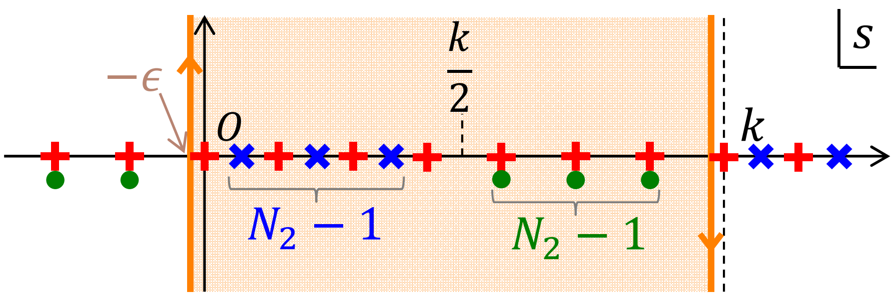
$s \sim k \sim \frac{1}{g_s}$: non-perturbative

- Integrand (anti)periodic
  $s \cong s + k$
- Integral = sum of pole residues in “fund. domain”
- Contrib. from both P and NP poles

Contour prescription

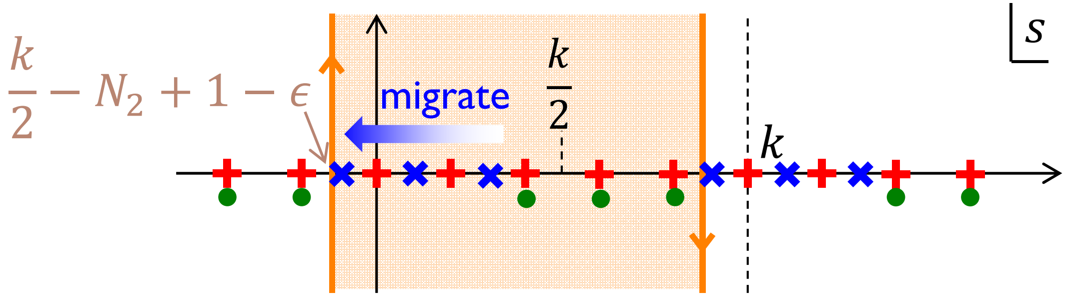
k large enough:

$$\frac{k}{2} - N_2 + 1 \geq 0$$



k smaller:

$$\frac{k}{2} - N_2 + 1 < 0$$



- Continuity in k fixes prescription
- Checked against original ABJ MM integral for $N_2 = 2,3$

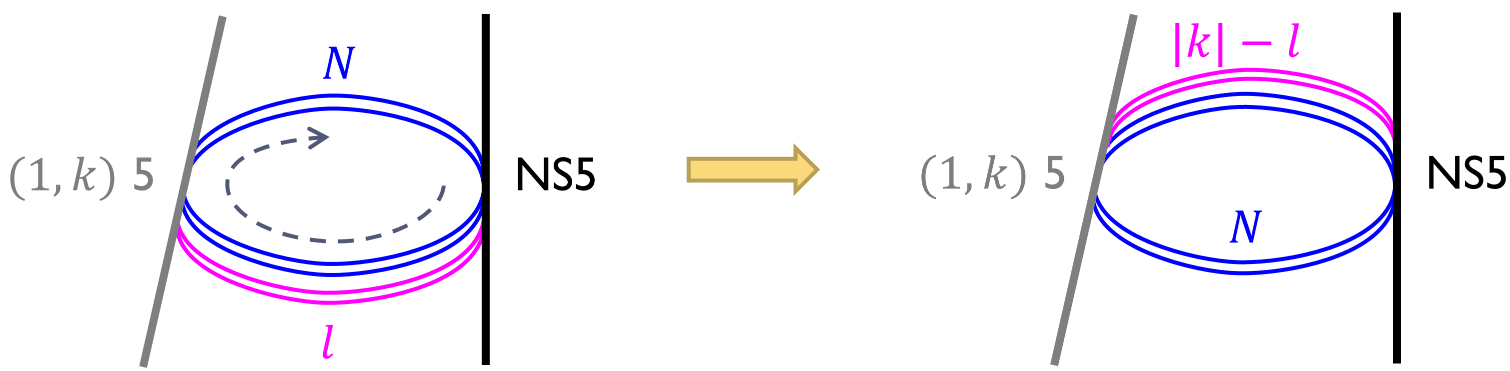
$N_1 \geq 2$

- Similar to $N_1 = 1$
- Guiding principle: reproduce pert. expn.
- Multiple q-hypergeometric functions appear

Seiberg duality

Passed perturbative test.
What about *non-perturbative* one?

$$U(N+l)_k \times U(N)_{-k} = U(N)_k \times U(N+|k|-l)_{-k}$$



Cf. [Kapustin-Willett-Yaakov]

Seiberg duality: $N_1 = 1$

$$U(1)_k \times U(N_2)_{-k} = U(1)_{-k} \times U(2 + |k| - N_2)_k$$

Partition function:

$$Z_{\rm ABJ}(1, N_2)_k = (2|k|)^{-1} Z^0_{\rm CS}(N_2 - 1)_k \, I(1, N_2)_k \, e^{i\theta(1,N_2)_k}$$

$$Z^0_{\rm CS}(M)_k = |k|^{-\frac{M}{2}} \prod_{j=1}^{M-1} \left(2 \sin \frac{\pi j}{|k|}\right)^{M-j}$$

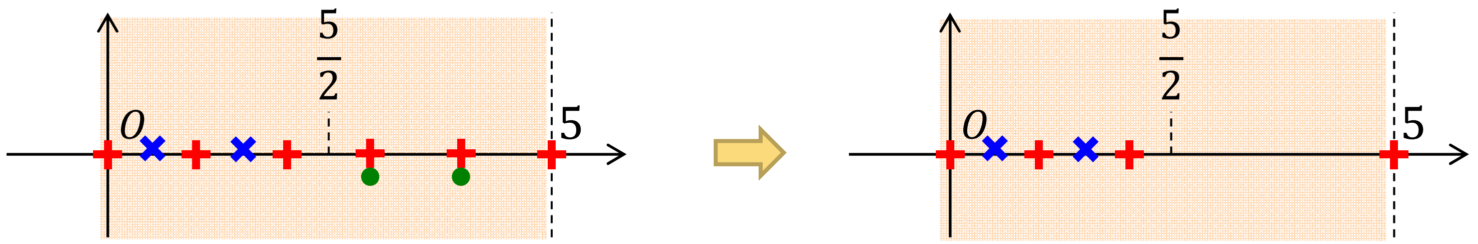
duality inv. by "level-rank duality"

$$I(1, N_2)_k := -\frac{1}{2\pi i} \int_I \frac{\pi \, ds}{\sin(\pi s)} \prod_{l=1}^{N_2-1} \tan\left(\frac{(s+l)\pi}{|k|}\right),$$

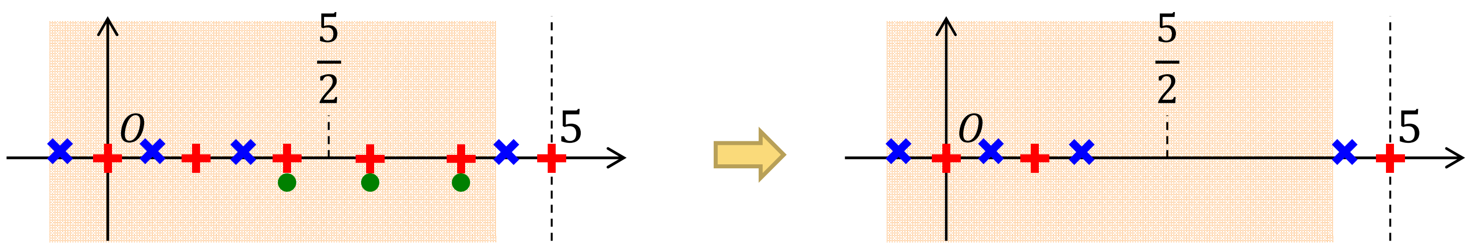
$$\theta(1, N_2)_k := -\frac{\pi}{6k} N_2(N_2 - 1)(N_2 - 2).$$

Odd k

$U(1)_5 \times U(3)_{-5}$ (original)



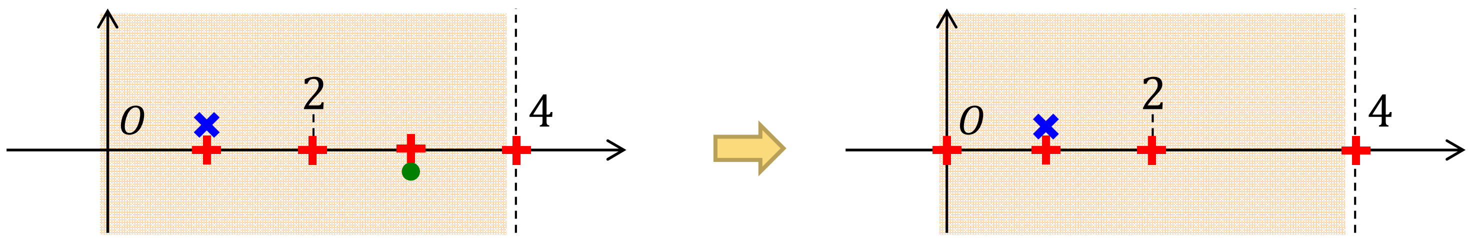
$U(1)_{-5} \times U(4)_5$ (dual)



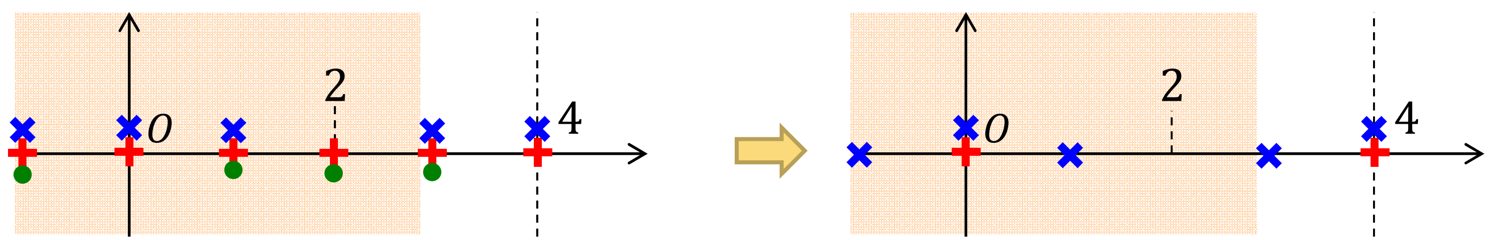
- Integrand identical (up to shift), and so is integral I ✓
- P and NP poles interchanged

Even k

$U(1)_4 \times U(2)_{-4}$ (original)



$U(1)_{-4} \times U(4)_4$ (dual)



- Integrand and thus integral are identical ✓
- P and NP poles interchanged, modulo ambiguity

Conclusions

Conclusions

- Exact expression for lens space MM
- Analytic continuation:

  $$Z_{\mathrm{lens}}(N_1, N_2) \rightarrow Z_{\mathrm{ABJ}}(N_1, N_2)$$

- Got expression i.t.o. $\min(N_1, N_2)$-dim integral, generalizing "mirror description" for ABJM
- Perturbative expansion agrees
- Passed non-perturbative check:
  Seiberg duality ($N_1 = 1$)

Future directions

- Make every step well-defined; understand q-hypergeom
- More general theory (necklace quiver)
- Wilson line [Awata+Hirano+Nii+MS]
- Fermi gas [Nagoya+Geneva]
- Relation to Vasiliev's higher spin theory
- Toward membrane dynamics?

Thanks!